Math 135 **Honors Calculus** **Winter 2017**

Handout 2: The Extreme Value Theorem in Two Variables

First, recall a couple of definitions for functions of two variables. Given two points (x_1, y_1) and (x_2, y_2) in the plane, the ***Euclidean distance*** between them is

$$d\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Now suppose $I, J \subseteq \mathbb{R}$ are intervals (they can be closed or open, bounded or unbounded), and $f\colon I \times J \to \mathbb{R}$ is a function. Given $(a, b) \in I \times J$, we say f is ***continuous at*** $\boldsymbol{(a, b)}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $(x, y) \in I \times J$ and $d\big((x, y), (a, b)\big) < \delta$, we have $|f(x, y) - f(a, b)| < \varepsilon$. If f is continuous at every point of $I \times J$, we simply say that f is ***continuous***.

Theorem 1. *Suppose $f\colon [a, b] \times [c, d] \to \mathbb{R}$ is continuous. Then it achieves its maximum and minimum values at points of $[a, b] \times [c, d]$.*

Proof. First, for simplicity, we'll rescale the rectangle on which f is defined. If we define

$$\widetilde{f}(x, y) = f(a + (b - a)x, c + (d - c)y),$$

then it is easy to check that $\widetilde{f}$ is continuous on the square $[0, 1] \times [0, 1]$, and $\widetilde{f}$ takes its maximum or minimum at a point (x, y) if and only if f takes its maximum or minimum at $(a + (b - a)x, c + (d - c)y)$. Let us simply replace f by $\widetilde{f}$, so henceforth we may assume f is continuous on $[0, 1] \times [0, 1]$. Let $S_0 = [0, 1] \times [0, 1]$.

We'll prove the theorem in two steps. The first step is to show that f is bounded: That is, there is a number C such that $|f(x, y)| \le C$ for all $(x, y) \in S_0$. Suppose for contradiction that it is not. Divide S_0 into four smaller squares of side length $\frac{1}{2}$ as follows:

$$[0, \tfrac{1}{2}] \times [0, \tfrac{1}{2}], \qquad [0, \tfrac{1}{2}] \times [\tfrac{1}{2}, 1], \qquad [\tfrac{1}{2}, 1] \times [0, \tfrac{1}{2}], \qquad [\tfrac{1}{2}, 1] \times [\tfrac{1}{2}, 1].$$

If f were bounded on each of these squares, it would be bounded on S_0, since we could just take the largest of the four bounds as a bound for $|f(x, y)|$ on the whole rectangle. Thus since we are assuming f is unbounded on S_0, it must be unbounded on at least one of these four smaller squares. Choose one; let's call that smaller square S_1, and label its boundaries as $[a_1, b_1] \times [c_1, d_1]$.

Now do it again: Divide S_1 into four squares of side length $\frac{1}{4}$. Since f is unbounded on S_1, it must be unbounded on one of the smaller squares; call it S_2, and label its boundaries as $[a_2, b_2] \times [c_2, d_2]$. Continuing by induction, we obtain a sequence of squares $S_0 \supseteq S_1 \supseteq S_2 \supseteq \dots$, with $S_n = [a_n, b_n] \times [c_n, d_n]$, such that f is unbounded on each S_n, and the side length of S_n is $1/2^n$. Notice also that because we chose S_{n+1} to be contained in S_n, for each n we have

$$0 \le a_1 \le a_2 \le a_3 \le \cdots \le a_n < b_n \le \cdots \le b_3 \le b_2 \le b_1 \le 1,$$

and similarly for the c_i's and d_i's. This means that the sequence $a_1, a_2, \dots$ is nondecreasing and bounded above, and therefore has a limit by the monotone sequences theorem; call the limit a_∞. Because $b_n - a_n = 1/2^n$ for each n, we have

$$a_n < b_n = a_n + \frac{1}{2^n},$$

so the pinching theorem implies that b_n also converges to the same limit. By similar reasoning, the sequences $c_1, c_2, \dots$ and $d_1, d_2, \dots$ both converge to a limit, which we call c_∞. Because $a_n \le a_\infty \le b_n$ and $c_n \le c_\infty \le d_n$ for each n, the point (a_∞, c_∞) actually lies in every square S_n.

Since f is continuous at the point (a_∞, c_∞), there is some $\delta > 0$ such that whenever (x, y) is in the domain of f and $d\big((x, y), (a_\infty, c_\infty)\big) < \delta$, we have $|f(x, y) - f(a_\infty, c_\infty)| < 1$. Thus for all such (x, y), we have the bounds

$$f(a_\infty, c_\infty) - 1 \le f(x, y) \le f(a_\infty, c_\infty) + 1. \tag{1}$$

Now choose n large enough that $1/2^n < \delta/2$. Since (a_∞, c_∞) lies in the square S_n with side length $1/2^n$, it follows that every point $(x, y) \in S_n$ satisfies $|x - a_\infty| < \delta/2$ and $|y - c_\infty| < \delta/2$ and therefore,

$$d\big((x, y), (a_\infty, c_\infty)\big) = \sqrt{(x - a_\infty)^2 + (y - c_\infty)^2} < \sqrt{(\delta/2)^2 + (\delta/2)^2} = \frac{\delta}{\sqrt{8}} < \delta.$$

Thus for every $(x, y) \in S_n$, we see that $f(x, y)$ satisfies the bound (1). But this is a contradiction, because f is unbounded on S_n. This completes the proof that f is bounded.

Now we have to show that f actually achieves its maximum and minumum. We'll prove it achieves its maximum; the proof for the minimum follows by considering the function $-f$.

Let $M = \text{lub}\{f(x, y) : (x, y) \in S_0\}$. The first part of the proof showed that M is a well-defined real number. We proceed as above: First divide S_0 into four squares of side length $\frac{1}{2}$. On each of those smaller squares, f is bounded above by M, so it has a least upper bound on each such square that is less than or equal to M. If the least upper bound were strictly less than M on all four squares, then it would be less than M on S_0, which is a contradiction; therefore at least one of the smaller squares must have the property that the restriction of f to that square still has least upper bound equal to M. Continue as above to produce a sequence of squares $S_0 \supseteq S_1 \supseteq S_2 \supseteq \dots$ and a point (a_∞, c_∞) such that

- (a_∞, c_∞) lies in each of the squares S_n;
- S_n has side length $1/2^n$;
- The least upper bound of f on S_n is equal to M.

Now let $L = f(a_\infty, c_\infty)$. Since M is an upper bound for f, it follows that $L \leq M$. If $L = M$, then f achieves its maximum at (a_∞, c_∞), and we are done. So assume for contradiction that $L < M$.

Let $\varepsilon = (M - L)/2$, so that $\varepsilon > 0$ and $L + \varepsilon = M - \varepsilon$. By continuity, there exists $\delta > 0$ such that whenever $(x, y) \in S_0$ and $d\big((x, y), (a_\infty, c_\infty)\big) < \delta$, we have $|f(x, y) - f(a_\infty, c_\infty)| < \varepsilon$. This means the following holds for all such (x, y):

$$f(x, y) < f(a_\infty, c_\infty) + \varepsilon = L + \varepsilon = M - \varepsilon.$$

If we choose n large enough that $1/2^n < \delta/2$, the same argument as above shows that every point $(x, y) \in S_n$ satsifies this inequality. However, because the least upper bound of f on S_n is equal to M, the basic property of least upper bounds (Theorem 11.1.2 in [SHE]) shows that there is some point $(x, y) \in S_n$ such that $f(x, y) > M - \varepsilon$. This is a contradiction, which shows that our assumption that $L < M$ was wrong, and f attains its maximum at (a_∞, c_∞). □