Math 134 **Honors Calculus** **Fall 2016**

Handout 7: The Sine and Cosine Functions

In most calculus courses, the trigonometric functions are "defined" geometrically, using some basic facts about angles, right triangles, similarity, area, and arc lengths. But the problem with that approach from the point of view of rigor is that none of those "basic facts" have been proved, and many of them (such as angles, areas, and arc lengths) can't even be defined rigorously at this stage.

For that reason, we are going to introduce the sine and cosine functions axiomatically, and prove everything we need from those axioms. Later in the course, we'll prove that there actually exist functions that satisfy these axioms.

First of all, we define the ***unit disk*** to be the set $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$. Later, we will rigorously develop the notion of area for such sets, but for now, we simply define a positive real number π to be the area of $\mathbb{D}$.

We assume that there exist two functions $\sin, \cos \colon \mathbb{R} \to \mathbb{R}$, satisfying the following axioms:

Axiom 1. (Pythagorean Identity) For all $\theta \in \mathbb{R}$,

$$\sin^2 \theta + \cos^2 \theta = 1.$$

Axiom 2. (Positivity) For $0 < \theta < \pi$,

$$\sin \theta > 0.$$

Axiom 3. (Special Values)

$$\begin{aligned} \sin 0 &= 0, & \cos 0 &= 1, \\ \sin \tfrac{\pi}{2} &= 1, & \cos \tfrac{\pi}{2} &= 0, \\ \sin \pi &= 0, & \cos \pi &= -1. \end{aligned}$$

Axiom 4. (Difference Formulas)

$$\begin{aligned} \sin(\theta - \varphi) &= \sin \theta \cos \varphi - \cos \theta \sin \varphi, \\ \cos(\theta - \varphi) &= \cos \theta \cos \varphi + \sin \theta \sin \varphi. \end{aligned}$$

Axiom 5. (Basic Inequalities) If $0 < \theta < \frac{\pi}{2}$, then

$$\sin \theta \cos \theta < \theta < \frac{\sin \theta}{\cos \theta}.$$

Here are some theorems that can be proved easily from these axioms. In all of these theorems, θ and φ represent arbitrary real numbers unless otherwise specified.

Theorem 94. $\sin(-\theta) = -\sin \theta$ *and* $\cos(-\theta) = \cos \theta$.

Theorem 95. (Complement Formulas) $\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$ *and* $\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$.

Theorem 96. (Addition Formulas)

$$\begin{aligned} \sin(\theta + \varphi) &= \sin \theta \cos \varphi + \cos \theta \sin \varphi, \\ \cos(\theta + \varphi) &= \cos \theta \cos \varphi - \sin \theta \sin \varphi. \end{aligned}$$

Theorem 97. (Double Angle Formulas)

$$\begin{aligned} \sin(2\theta) &= 2 \sin \theta \cos \theta, \\ \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta. \end{aligned}$$

Theorem 98. $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

Theorem 99. $\sin(2\pi) = 0$ *and* $\cos(2\pi) = 1$.

Theorem 100. (Periodicity) $\sin(\theta + 2\pi) = \sin\theta$ *and* $\cos(\theta + 2\pi) = \cos\theta$.

Theorem 101. $|\cos\theta| \le 1$ *and* $|\sin\theta| \le 1$.

Theorem 102. $\sin\theta < 0$ *for* $-\pi < \theta < 0$.

Theorem 103. $\cos\theta > 0$ *for* $|\theta| < \pi/2$.

The next one is a little less easy, so we will give a proof.

Theorem 104. (Monotonicity) *For* $0 < \theta < \pi/2$*, the cosine function is decreasing and the sine function is increasing: If* $0 < \theta_1 < \theta_2 < \pi/2$*, then* $\cos\theta_1 > \cos\theta_2$ *and* $\sin\theta_1 < \sin\theta_2$.

Proof. Suppose $0 < \theta_1 < \theta_2 < \pi/2$. It follows that $0 < \theta_1 - \theta_2 < \pi/2$, and therefore the numbers $\sin\theta_1$, $\cos\theta_1$, $\sin(\theta_2 - \theta_1)$, and $\cos(\theta_2 - \theta_1)$ are all positive. Moreover, $\cos(\theta_2 - \theta_1) \le |\cos(\theta_2 - \theta_1)| \le 1$ by Theorem 101, and multiplying both sides of this inequality by $\cos\theta_1$, we conclude that $\cos\theta_1 \cos(\theta_2 - \theta_1) \le \cos\theta_1$. Thus the addition formula yields

$$\begin{aligned}
\cos\theta_2 &= \cos\big(\theta_1 + (\theta_2 - \theta_1)\big) \\
&= \cos\theta_1 \cos(\theta_2 - \theta_1) - \sin\theta_1 \sin(\theta_2 - \theta_1) \\
&< \cos\theta_1 \cos(\theta_2 - \theta_1) \\
&\le \cos\theta_1.
\end{aligned}$$

The argument for sine is similar and left to the reader. $\square$

Theorem 105. $\lim\limits_{x\to 0} \sin x = 0$.

Proof. Let $\varepsilon > 0$ be given. Note that if $0 < \theta < \frac{\pi}{4}$, monotonicity implies $\cos\theta > \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, and one of the basic inequalities (Axiom 5) implies

$$\sin\theta < \frac{\theta}{\cos\theta} < \frac{\theta}{1/\sqrt{2}}.$$

Let $\delta = \min(\varepsilon/\sqrt{2}, \frac{\pi}{4})$, and assume that $0 < |\theta| < \delta$. Then either $0 < \theta < \delta$ or $-\delta < \theta < 0$. In the fisrt case,

$$|\sin\theta| = \sin\theta < \frac{\theta}{1/\sqrt{2}} < \frac{\varepsilon/\sqrt{2}}{1/\sqrt{2}} = \varepsilon.$$

On the other hand, if $-\delta < \theta < 0$, then $|\sin\theta| = |-\sin(-\theta)| = |\sin(-\theta)| < \varepsilon$ by the argument above. $\square$

Theorem 106. $\lim\limits_{x\to 0} \cos x = 1$.

Proof. Let $\varepsilon > 0$ be given. The previous theorem guarantees that there exists $\delta_1 > 0$ such that $0 < |\theta| < \delta$ implies $|\sin\theta| < \varepsilon$. Note that if $|\theta| < \frac{\pi}{2}$, then $\cos\theta > 0$ and so $\cos\theta + 1 > 1$. Let $\delta = \min(\delta_1, \frac{\pi}{2})$. If $0 < |\theta| < \delta$, then

$$|\cos\theta - 1| = \left|\frac{(\cos\theta - 1)(\cos\theta + 1)}{\cos\theta + 1}\right| = \frac{|-\sin^2\theta|}{\cos\theta + 1} \le |\sin^2\theta| \le |\sin\theta| < \varepsilon.$$

(In the next-to-last inequality we used the fact that $|\sin\theta| \le 1$, so $|\sin^2\theta| \le 1 \cdot |\sin\theta|$.) $\square$