Some More Facts about Integers

An integer \( n \) is said to be \textbf{even} if it can be expressed in the form \( n = 2k \) for some integer \( k \), and \textbf{odd} if it can be expressed as \( n = 2l + 1 \) for some integer \( l \).

\textbf{Theorem 85.} Every integer is either even or odd, but not both.

\textit{Proof.} First we'll show by induction that every positive integer is even or odd. For the base case, just note that 1 is odd because it can be written \( 1 = 2 \cdot 0 + 1 \). For the inductive step, let \( n \) be a positive integer, and assume \( n \) is either even or odd. If \( n \) is even, then \( n = 2k \) for some integer \( k \), and thus \( n + 1 = 2k + 1 \), which is odd. If \( n \) is odd, then \( n = 2l + 1 \) for some integer \( l \), and then \( n + 1 = 2l + 2 = 2(l + 1) \), which is even. In either case, \( n + 1 \) is even or odd, and the induction is complete.

Now we'll show that all nonpositive integers are either even or odd. If \( n = 0 \), then \( n = 2 \cdot 0 \), which is even. And if \( n < 0 \), then \( -n \) is either even or odd by the argument above, so either \( -n = 2k \) or \( -n = 2l + 1 \). This implies \( n = 2(-k) \) in the first case, so \( n \) is even; and it implies \( n = -2l - 1 = 2(-l - 1) + 1 \) in the second, so \( n \) is odd. This completes the proof that every integer is even or odd.

To show that no integer can be both even and odd, suppose for contradiction that \( n \) is both. Then there are integers \( k \) and \( l \) such that \( n = 2k \) and \( n = 2l + 1 \). This means \( 2k = 2l + 1 \), which implies \( 1/2 = k + (-l) \), which is an integer. But this contradicts the fact that 1 is the smallest positive integer.

\textbf{Theorem 86.} [Properties of Even and Odd Integers] Suppose \( m \) and \( n \) are integers.

\begin{itemize}
  \item [(a)] \( m + n \) is even if and only if \( m \) and \( n \) are both even or both odd.
  \item [(b)] \( m + n \) is odd if and only if one of the summands is even and the other is odd.
  \item [(c)] \( mn \) is even if and only if \( m \) or \( n \) is even.
  \item [(d)] \( mn \) is odd if and only if \( m \) and \( n \) are both odd.
  \item [(e)] \( n^2 \) is even if and only if \( n \) is even, and odd if and only if \( n \) is odd.
\end{itemize}

\textit{Proof.} Exercise.

\textbf{Theorem 87.} The set \( \mathbb{Z}^+ \) of positive integers has no upper bound in \( \mathbb{R} \).

\textit{Proof.} Suppose for the sake of contradiction that \( \mathbb{Z}^+ \) has an upper bound. Since \( \mathbb{Z}^+ \) is not empty, it has a least upper bound; let’s call it \( M \). Since \( M \) is the least upper bound, it follows that \( M - 1 \) is not an upper bound for \( \mathbb{Z}^+ \); or in other words, there exists some integer \( k \) such that \( k > M - 1 \). But this implies \( k + 1 > M \), and since \( k + 1 \) is also a positive integer, this contradicts the fact that \( M \) is the least upper bound of \( \mathbb{Z}^+ \).

Rational Numbers

A real number is called a \textbf{rational number} if it can be expressed in the form \( p/q \), where \( p \) and \( q \) are integers and \( q \neq 0 \). The set of all rational numbers is denoted by \( \mathbb{Q} \). A real number is said to be \textbf{irrational} if it is not rational.

\textbf{Theorem 88.} 0 and 1 are rational numbers.

\textit{Proof.} Just note that 0 = 0/1 and 1 = 1/1.

\textbf{Theorem 89.} [Closure of \( \mathbb{Q} \)] If \( a \) and \( b \) are rational numbers than so are \( a + b, a - b, \) and \( ab \). If in addition \( b \neq 0 \), then \( a/b \) is a rational number.
Proof. These follow immediately from the formulas

\[ \frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs}, \quad \frac{p}{q} - \frac{r}{s} = \frac{ps - qr}{qs}, \quad \frac{p \cdot r}{q \cdot s} = \frac{pr}{qs}, \quad \frac{p}{q} / \frac{r}{s} = \frac{ps}{qs}. \]

together with the facts that sums, differences, and products of integers are integers. \(\square\)

A fraction \(p/q\) is said to be in **lowest terms** if the largest integer that evenly divides both \(p\) and \(q\) is 1.

**Theorem 90.** Every rational number can be expressed as a fraction in lowest terms.

**Proof.** Suppose \(r\) is a rational number, and let \(S\) be the following set:

\[ S = \{ q \in \mathbb{Z}^+ : r \text{ has an expression of the form } p/q \text{ for some integer } p \}. \]

The fact that \(r\) is rational means that \(S\) is nonempty, so by the well-ordering property of \(\mathbb{Z}^+\) (Theorem 79 on Handout 5), \(S\) contains a smallest positive integer; call it \(q_0\). That means there is some integer \(p_0\) such that \(r = p_0/q_0\).

We wish to show that \(p_0/q_0\) is in lowest terms. Suppose not: then there is an integer \(k > 1\) that evenly divides both \(p_0\) and \(q_0\). This means there are integers \(p_1\) and \(q_1\) such that \(p_0 = kp_1\) and \(q_0 = kq_1\). Since \(k > 1\) and \(q_0\) is positive, this implies \(q_1 = q_0/k\) is positive and strictly smaller than \(q_0\). But then we have \(r = p_1/q_1\) with a denominator \(q_1\) smaller than \(q_0\), which is a contradiction. \(\square\)

**Theorem 91.** \(\sqrt{2}\) is irrational.

**Proof.** Suppose for contradiction that \(\sqrt{2}\) is rational. By Theorem 90, we can write \(\sqrt{2} = p/q\) in lowest terms. Now this means \(p^2/q^2 = 2\), or

\[ p^2 = 2q^2. \]

By Theorem 86(e), this implies \(p\) is even, so \(p = 2k\) for some integer \(k\). Substituting for \(p\) in the equation above, we conclude

\[ (2p)^2 = 2q^2, \quad \text{which implies } 2p^2 = q^2. \]

Using Theorem 86(e) once again, we conclude that \(q\) is also even. But this contradicts the fact that \(p/q\) is in lowest terms, so our assumption that \(\sqrt{2}\) is rational must have been false. \(\square\)

**Theorem 92.** [Density of Rational Numbers] If \(a\) and \(b\) are real numbers such that \(a < b\), then there exists a rational number \(c\) such that \(a < c < b\).

**Proof.** By Theorem 87, there is a positive integer \(q\) such that \(q > 1/(b - a)\). Now consider the set \(S\) of all positive integers \(n\) such that \(n > aq\). This set is nonempty by Theorem 87, so by the well-ordering property of \(\mathbb{Z}^+\), \(S\) contains a smallest integer \(p\). We will show that \(a < p/q < b\).

First, the fact that \(p \in S\) means by definition that \(p > aq\), and therefore \(a < p/q\). To show that \(p/q < b\), suppose for contradiction that \(p/q \geq b\). We chose \(q\) such that \(q > 1/(b - a)\), which implies \(1/q < b - a\) and therefore \(-1/q > a - b\). Now consider the rational number \((p - 1)/q:\)

\[ \frac{p - 1}{q} = \frac{p}{q} + \frac{-1}{q} > b + (a - b) = a. \]

This in turn implies \(p - 1 > aq\), so \(p - 1 \in S\). But this contradicts our choice of \(p\) as the smallest number in \(S\). This shows our assumption was false and therefore \(p/q < b\). \(\square\)

**Theorem 93.** [Density of Irrational Numbers] If \(a\) and \(b\) are real numbers such that \(a < b\), then there exists an irrational number \(c\) such that \(a < c < b\).

**Proof.** Since \(a/\sqrt{2} < b/\sqrt{2}\), the previous theorem implies that there is a rational number \(c\) such that \(a/\sqrt{2} < c < b/\sqrt{2}\). If \(c = 0\), then there is another rational \(c'\) such that \(a/\sqrt{2} < 0 < c' < b/\sqrt{2}\), so after replacing \(c\) by \(c'\) if necessary, we may assume \(c \neq 0\).

This in turn implies \(a < c\sqrt{2} < b\). If \(c\sqrt{2}\) were rational, then \(\sqrt{2} = (c\sqrt{2})/c\) would also be rational, which is a contradiction; so \(d = c\sqrt{2}\) is an irrational number between \(a\) and \(b\). \(\square\)