Math 134 **Honors Calculus** **Fall 2016**

Handout 5: The Integers (corrected 10/9/16)

THE NATURAL NUMBERS

A subset $S \subseteq \mathbb{R}$ is called an ***inductive set*** if it satisfies the following two properties:

- $1 \in S$.
- If $k \in S$, then $k + 1 \in S$.

There are many inductive sets; for example, it's easy to check that both $\mathbb{R}$ and $\mathbb{R}^+$ are inductive sets.

We define the set $\mathbb{N}$ to be the intersection of all inductive sets. In other words, a number n is in $\mathbb{N}$ if and only if n is in *every* inductive set. Elements of $\mathbb{N}$ are called ***natural numbers***.

The next theorem is the most important fact about the natural numbers.

Theorem 72. (The Principle of Mathematical Induction) *Suppose $S \subseteq \mathbb{N}$ is a set of natural numbers that satisfies the following two properties:*

(*a*) $1 \in S$.
(*b*) *If $k \in S$, then $k + 1 \in S$.*

Then $S = \mathbb{N}$.

Proof. The hypotheses imply that S is an inductive set. Since every element of $\mathbb{N}$ is contained in *every* inductive set by definition of $\mathbb{N}$, it follows that every element of $\mathbb{N}$ is in S. On the other hand, the hypothesis $S \subseteq \mathbb{N}$ shows that every element of S is in $\mathbb{N}$. Thus $S = \mathbb{N}$. $\square$

This leads to a powerful way of proving statements about the natural numbers, called ***proof by mathematical induction***. Basically, if you want to prove that a statement $P(n)$ is true for every natural number n, you first prove that $P(1)$ is true, and then you prove that for an arbitrary $k \in \mathbb{N}$, if we assume $P(k)$ is true, then $P(k+1)$ must be true. The first step (proving $P(1)$) is called the ***base case***, and the second step (assuming $P(k)$ and deducing $P(k + 1)$) is called the ***inductive step***. The assumption that $P(k)$ is true (which you get to assume when doing the inductive step) is called the ***inductive hypothesis***.

Here's an example.

Theorem 73. *If $n \in \mathbb{N}$, then $n \geq 1$.*

Proof. We'll prove this by induction. For the base case, just note that $1 \geq 1$. For the inductive step, let $k \in \mathbb{N}$ be arbitrary, and assume that $k \geq 1$. Then $k + 1 > k \geq 1$, where we have used the inductive hypothesis to prove the second inequality. Thus $k + 1 \geq 1$, which completes the inductive step and proves the theorem. $\square$

THE INTEGERS

A real number n is called an ***integer*** if any one of the following is true:

- $n \in \mathbb{N}$,
- $n = 0$, or
- $-n \in \mathbb{N}$.

The set of all integers is denoted by $\mathbb{Z}$.

Here are some basic theorems about the integers, which we will use all the time.

Theorem 74. *An integer n is positive if and only if $n \in \mathbb{N}$.*

Theorem 75. *If n is an integer, so is $-n$.*

Theorem 76. (Closure of Z) *If m and n are integers, then so are $m+n$ and mn.*

Theorem 77. *If m and n are integers and $m > n$, then $m \geq n+1$.*

Theorem 78. *There is no largest integer.*

Theorem 79. (The Well-Ordering Property) *Every nonempty set of positive integers contains a smallest integer.*

RECURSIVE DEFINITIONS

The principle of mathematical induction also gives us a way to define things for all positive integers. Basically, if we want to define what some expression means for all $n \in \mathbb{N}$, we just have to define what it means for $n = 1$, and then define the expression for $k+1$ in terms of the one for k. Such a definition is called a ***recursive definition***. Here are two examples.

For any $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we define the expression x^n (called the ***nth power of x***) recursively as follows:

- $x^1 = x$;
- For any $k \in \mathbb{N}$, $x^{k+1} = x^k \cdot x$.

We can extend this to all $n \in \mathbb{Z}$ by setting $x^0 = 1$ for any x, and $x^{-n} = (x^{-1})^n$ for any nonzero x and any $n \in \mathbb{N}$.

Similarly, suppose $a_1, a_2, \dots$ are real numbers, and $n \in \mathbb{N}$. The ***summation of a_i from 1 to n*** is denoted by

$$\sum_{i=1}^{n} a_i,$$

and is defined recursively as follows:

$$\sum_{i=1}^{1} a_i = a_1; \qquad \sum_{i=1}^{n+1} a_i = \left(\sum_{i=1}^{n} a_i\right) + a_{n+1}.$$

These definitions are perfectly suited to proving things by induction. Here's one example (which also appears in the textbook, but without using the summation notation):

Example 1. *For any positive integer n,*

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Proof. We'll prove this by induction. The base case $n = 1$ holds by definition of the summation notation:

$$\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}.$$

For the inductive step, let $k \in \mathbb{N}$ be arbitrary and assume that

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

Then, again using the definition of the summation notation together with the inductive hypothesis, we have

$$\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{n} i\right) + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+1+1)}{2}.$$

This completes the inductive step and thus the proof. □

For another example, see Example 2 on page 50 of the textbook.

Here are some important theorems about exponents. These theorems are typically proved for positive exponents by induction, and then for zero and negative exponents using their definition in terms of positive exponents. In these theorems, a and b are arbitrary real numbers, and m and n are arbitrary integers.

Theorem 80. $a^n b^n = (ab)^n$.

Theorem 81. $a^m a^n = a^{m+n}$.

Theorem 82. [deleted]

Theorem 83. $(a^m)^n = a^{mn}$.

Theorem 84. $a^n/b^n = (a/b)^n$ *if* $b \neq 0$.