The Real Numbers and the Integers

PRIMITIVE TERMS

To avoid circularity, we cannot give every term a rigorous mathematical definition; we have to accept some things as undefined terms. For this course, we will take the following fundamental notions as primitive undefined terms. You already know what these terms mean; but the only facts about them that can be used in proofs are the ones expressed in the axioms listed below (and any theorems that can be proved from the axioms).

- ***Real number:*** Intuitively, a real number represents a point on the number line, or a (signed) distance left or right from the origin, or any quantity that has a finite or infinite decimal representation. Real numbers include integers, positive and negative fractions, and irrational numbers like $\sqrt{2}$, π, and e.
- ***Integer:*** An integer is a whole number (positive, negative, or zero).
- ***Zero:*** The number zero is denoted by $\mathbf{0}$.
- ***One:*** The number one is denoted by $\mathbf{1}$.
- ***Addition:*** The result of adding two real numbers a and b is denoted by $\boldsymbol{a+b}$, and is called the ***sum of a and b***.
- ***Multiplication:*** The result of multiplying two real numbers a and b is denoted by $\boldsymbol{ab}$ or $\boldsymbol{a \cdot b}$ or $\boldsymbol{a \times b}$, and is called the ***product of a and b***.
- ***Less than:*** To say that a is less than b, denoted by $\boldsymbol{a < b}$, means intuitively that a is to the left of b on the number line.

DEFINITIONS

In all the definitions below, a and b represent arbitrary real numbers.

- The numbers **2** through **10** are defined by $2 = 1+1$, $3 = 2+1$, etc. The decimal representations for other numbers are defined by the usual rules of decimal notation: For example, 23 is defined to be $2 \cdot 10 + 3$, etc.
- The ***additive inverse*** or ***negative*** of a is the number $\boldsymbol{-a}$ that satisfies $a + (-a) = 0$, and whose existence and uniqueness are guaranteed by Axiom 9.
- The ***difference between a and b***, denoted by $\boldsymbol{a - b}$, is the real number defined by $a - b = a + (-b)$, and is said to be obtained by ***subtracting b from a***.
- If $a \neq 0$, the ***multiplicative inverse*** or ***reciprocal*** of a is the number $\boldsymbol{a^{-1}}$ that satisfies $a \cdot a^{-1} = 1$, and whose existence and uniqueness are guaranteed by Axiom 10.
- If $b \neq 0$, the ***quotient of a and b***, denoted by $\boldsymbol{a/b}$, is the real number defined by $a/b = ab^{-1}$, and is said to be obtained by ***dividing a by b***.
- A real number is said to be ***rational*** if it is equal to p/q for some integers p and q with $q \neq 0$.
- A real number is said to be ***irrational*** if it is not rational.
- The statement ***a is less than or equal to b***, denoted by $\boldsymbol{a \leq b}$, means $a < b$ or $a = b$.

- The statement ***a* *is greater than* *b***, denoted by $\boldsymbol{a > b}$, means $b < a$.
- The statement ***a* *is greater than or equal to* *b***, denoted by $\boldsymbol{a \geq b}$, means $a > b$ or $a = b$.
- A real number a is said to be ***positive*** if $a > 0$. The set of all positive real numbers is denoted by $\mathbb{R}^+$, and the set of all positive integers by $\mathbb{Z}^+$.
- A real number a is said to be ***negative*** if $a < 0$.
- A real number a is said to be ***nonnegative*** if $a \geq 0$.
- A real number a is said to be ***nonpositive*** if $a \leq 0$.
- If a and b are two distinct real numbers, a real number c is said to be ***between a and b*** if either $a < c < b$ or $a > c > b$.
- For any real number a, the ***absolute value of a***, denoted by $|a|$, is defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

- If a is a real number and n is a positive integer, the ***nth power of a***, denoted by a^n, is the product of n factors of a. The ***square of a*** is the number $a^2 = a \cdot a$.
- If a is a nonnegative real number, the ***square root of a***, denoted by $\sqrt{a}$, is the unique nonnegative real number whose square is a (see Theorem 9 below).
- If n and k are integers, we say that n is ***divisible by k*** if there is an integer m such that $n = km$.
- An integer n is said to be ***even*** if it is divisible by 2, and ***odd*** if not.
- If S is a set of real numbers, a real number b is said to be a ***maximum of S*** or a ***largest element of S*** if b is an element of S and, in addition, $b \geq x$ whenever x is any element of S. The terms ***minimum*** and ***smallest element*** are defined similarly.
- If S is a set of real numbers, a real number b (not necessarily in S) is said to be an ***upper bound for S*** if $b \geq x$ for every x in S. It is said to be a ***least upper bound for S*** if every other upper bound b' for S satisfies $b' \geq b$. The terms ***lower bound*** and ***greatest lower bound*** are defined similarly.

PROPERTIES OF EQUALITY

In modern mathematics, the relation “equals” can be used between any two “mathematical objects” of the same type, such as numbers, matrices, ordered pairs, sets, functions, etc. To say that $a = b$ is simply to say that the symbols a and b represent the very same object. Thus the concept of “equality” really belongs to mathematical logic rather than to any particular branch of mathematics.

Equality always has the following fundamental properties, no matter what kinds of objects it is applied to. In the following statements, a, b, and c can represent any mathematical objects whatsoever. (In our applications, they will usually be real numbers.)

General Properties of Equality

1. (Reflexivity) $a = a$.
2. (Symmetry) If $a = b$, then $b = a$.
3. (Transitivity) If $a = b$ and $b = c$, then $a = c$.
4. (Substitution) If $a = b$, then b may be substituted for a in any mathematical statement without affecting that statement's truth value.

In addition, for real numbers, we have the following properties. The first five statements say roughly that if you start with a true equation between two real numbers, you can "do the same thing to both sides" and still have a true equation. The last two say that if you start with two true equations, you will still have a true equation after adding them together, multiplying them together, subtracting one from the other, or dividing one by the other (provided you are not dividing by zero). All of these statements can be proved using only reflexivity of equality and substitution. In these statements, a, b, c, d represent arbitrary real numbers.

Properties of Equality of Real Numbers

1. If $a = b$, then $a + c = b + c$, $ac = bc$, and $a - c = b - c$.
2. If $a = b$ and c is nonzero, then $a/c = b/c$.
3. If $a = b$, then $-a = -b$.
4. If $a = b$, and a and b are both nonzero, then $a^{-1} = b^{-1}$.
5. If $a = b$, then $a^2 = b^2$.
6. If $a = b$ and $c = d$, then $a + c = b + d$, $ac = bd$, and $a - c = b - d$.
7. If $a = b$ and $c = d$, and c and d are both nonzero, then $a/c = b/d$.

AXIOMS FOR THE REAL NUMBERS AND INTEGERS

We assume that the following statements are true.

1. (Existence) There exists a set $\mathbb{R}$ consisting of all real numbers. It contains a subset $\mathbb{Z} \subseteq \mathbb{R}$ consisting of all integers.
2. (Closure of $\mathbb{Z}$) If a and b are integers, then so are $a + b$ and ab.
3. (Closure of $\mathbb{R}$) If a and b are real numbers, then so are $a + b$ and ab.
4. (Commutativity) $a + b = b + a$ and $ab = ba$ for all real numbers a and b.
5. (Associativity) $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$ for all real numbers a, b, and c.
6. (Distributivity) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all real numbers a, b, and c.
7. (Zero) 0 is an integer that satisfies $a + 0 = a = 0 + a$ for every real number a.
8. (One) 1 is an integer that is not equal to zero and satisfies $a \cdot 1 = a = 1 \cdot a$ for every real number a.
9. (Additive inverses) If a is any real number, there is a unique real number $-a$ such that $a + (-a) = 0$. If a is an integer, then so is $-a$.
10. (Multiplicative inverses) If a is any nonzero real number, there is a unique real number a^{-1} such that $a \cdot a^{-1} = 1$.
11. (Trichotomy law) If a and b are real numbers, then one and only one of the following three statements is true: $a < b$, $a = b$, or $a > b$.
12. (Closure of $\mathbb{R}^+$) If a and b are positive real numbers, then so are $a + b$ and ab.
13. (Addition law for inequalities) If a, b, and c are real numbers and $a < b$, then $a + c < b + c$.
14. (The well ordering axiom) Every nonempty set of positive integers contains a smallest integer.

15. (The least upper bound axiom) Every nonempty set of real numbers that has an upper bound has a least upper bound.

SELECTED THEOREMS

These theorems can be proved from the axioms in the order listed below.

1. Properties of zero
   (a) $a - a = 0$.
   (b) $0 - a = -a$.
   (c) $0 \cdot a = 0$.
   (d) If $ab = 0$, then $a = 0$ or $b = 0$.
2. Properties of signs
   (a) $-0 = 0$.
   (b) $-(-a) = a$.
   (c) $(-a)b = -(ab) = a(-b)$.
   (d) $(-a)(-b) = ab$.
   (e) $-a = (-1)a$.
3. More distributive properties
   (a) $-(a + b) = (-a) + (-b) = -a - b$.
   (b) $-(a - b) = b - a$.
   (c) $-(-a - b) = a + b$.
   (d) $a + a = 2a$.
   (e) $a(b - c) = ab - ac = (b - c)a$.
   (f) $(a + b)(c + d) = ac + ad + bc + bd$.
   (g) $(a + b)(c - d) = ac - ad + bc - bd = (c - d)(a + b)$.
   (h) $(a - b)(c - d) = ac - ad - bc + bd$.
4. Properties of inverses
   (a) If a is nonzero, then so is a^{-1}.
   (b) $1^{-1} = 1$.
   (c) $(a^{-1})^{-1} = a$ if a is nonzero.
   (d) $(-a)^{-1} = -(a^{-1})$ if a is nonzero.
   (e) $(ab)^{-1} = a^{-1}b^{-1}$ if a and b are nonzero.
   (f) $(a/b)^{-1} = b/a$ if a and b are nonzero.
5. Properties of quotients
   (a) $a/1 = a$.
   (b) $1/a = a^{-1}$ if a is nonzero.
   (c) $a/a = 1$ if a is nonzero.
   (d) $(a/b)(c/d) = (ac)/(bd)$ if b and d are nonzero.
   (e) $(a/b)/(c/d) = (ad)/(bc)$ if b, c, and d are nonzero.
   (f) $(ac)/(bc) = a/b$ if b and c are nonzero.
   (g) $a(b/c) = (ab)/c$ if c is nonzero.
   (h) $(ab)/b = a$ if b is nonzero.
   (i) $(-a)/b = -(a/b) = a/(-b)$ if b is nonzero.
   (j) $(-a)/(-b) = a/b$ if b is nonzero.
   (k) $a/b + c/d = (ad + bc)/(bd)$ if b and d are nonzero.
   (l) $a/b - c/d = (ad - bc)/(bd)$ if b and d are nonzero.

6. Transitivity of inequalities
   (a) If $a < b$ and $b < c$, then $a < c$.
   (b) If $a \leq b$ and $b < c$, then $a < c$.
   (c) If $a < b$ and $b \leq c$, then $a < c$.
   (d) If $a \leq b$ and $b \leq c$, then $a \leq c$.
7. Other Properties of inequalities
   (a) If $a \leq b$ and $b \leq a$, then $a = b$.
   (b) If $a < b$, then $-a > -b$.
   (c) $0 < 1$.
   (d) If $a > 0$, then $a^{-1} > 0$.
   (e) If $a < 0$, then $a^{-1} < 0$.
   (f) If $a < b$ and a and b are both positive, then $a^{-1} > b^{-1}$.
   (g) If $a < b$ and $c < d$, then $a + c < b + d$.
   (h) If $a \leq b$ and $c < d$, then $a + c < b + d$.
   (i) If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$.
   (j) If $a < b$ and $c > 0$, then $ac < bc$.
   (k) If $a < b$ and $c < 0$, then $ac > bc$.
   (l) If $a \leq b$ and $c > 0$, then $ac \leq bc$.
   (m) If $a \leq b$ and $c < 0$, then $ac \geq bc$.
   (n) If $a < b$ and $c < d$, and a, b, c, d are nonnegative, then $ac < bd$.
   (o) If $a \leq b$ and $c \leq d$, and a, b, c, d are nonnegative, then $ac \leq bd$.
   (p) $ab > 0$ if and only if a and b are both positive or both negative.
   (q) $ab < 0$ if and only if one is positive and the other is negative.
   (r) There is no smallest positive real number.
   (s) (Density) If a and b are two distinct real numbers, then there are infinitely many rational numbers and infinitely many irrational numbers between a and b.
8. Properties of squares
   (a) For every a, $a^2 \geq 0$.
   (b) $a^2 = 0$ if and only if $a = 0$.
   (c) $a^2 > 0$ if and only if $a > 0$.
   (d) $(-a)^2 = a^2$.
   (e) $(a^{-1})^2 = 1/a^2$.
   (f) If $a^2 = b^2$, then $a = \pm b$.
   (g) If $a < b$ and a and b are both nonnegative, then $a^2 < b^2$.
   (h) If $a < b$ and a and b are both negative, then $a^2 > b^2$.
9. Properties of Square Roots
   (a) If a is any nonnegative real number, there is a unique nonnegative real number $\sqrt{a}$ such that $\left(\sqrt{a}\right)^2 = a$.
   (b) If $a = b$ and a and b are both nonnegative, then $\sqrt{a} = \sqrt{b}$.
   (c) If $a < b$ and a and b are both nonnegative, then $\sqrt{a} < \sqrt{b}$.
   (d) If $a^2 = b$ and b is nonnegative, then $a = \pm\sqrt{b}$.
10. Properties of Absolute Values
    (a) If a is any real number, then $|a| \geq 0$.
    (b) $|a| = 0$ if and only if $a = 0$.
    (c) $|a| > 0$ if and only if $a \neq 0$.
    (d) $|-a| = |a|$.
    (e) $|a| = \sqrt{a^2}$.

(f) $|a| = \max\{a, -a\}$.
(g) $|a^{-1}| = 1/|a|$ if $a \neq 0$.
(h) $|ab| = |a|\,|b|$.
(i) $|a/b| = |a|/|b|$ if $b \neq 0$.
(j) $|a| = |b|$ if and only if $a = \pm b$.
(k) If a and b are both nonnegative, then $|a| \geq |b|$ if and only if $a \geq b$.
(l) If a and b are both negative, then $|a| \geq |b|$ if and only if $a \leq b$.
(m) (The triangle inequality) $|a + b| \leq |a| + |b|$.
(n) (The reverse triangle inequality) $\big||a| - |b|\big| \leq |a - b|$.

11. Order properties of integers

(a) 1 is the smallest positive integer.
(b) If m and n are integers such that $m > n$, then $m \geq n + 1$.
(c) There is no largest or smallest integer.

12. Properties of Even and Odd Integers

In each of the following statements, m and n are assumed to be integers.

(a) n is even if and only if $n = 2k$ for some integer k, and odd if and only if $n = 2k + 1$ for some integer k.
(b) $m + n$ is even if and only if m and n are both odd or both even.
(c) $m + n$ is odd if and only if one of the summands is even and the other is odd.
(d) mn is even if and only if m or n is even.
(e) mn is odd if and only if m and n are both odd.
(f) n^2 is even if and only if n is even, and odd if and only if n is odd.

13. Properties of Exponents

In these statements, m and n are positive integers.

(a) $a^n b^n = (ab)^n$.
(b) $a^{m+n} = a^m b^n$.
(c) $(a^m)^n = a^{mn}$.
(d) $a^n/b^n = (a/b)^n$ if b is nonzero.