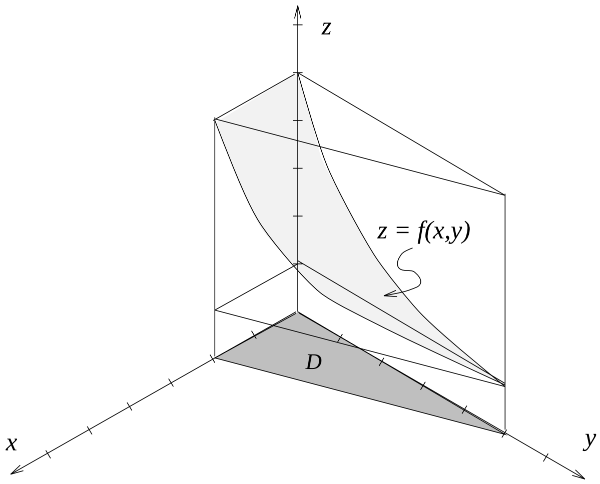
11.95

# SUPPLEMENT TO SECTION 24: RIEMANN SUMS

**NOTE:** *The following pages should be inserted after the end of Section 24 in [LN].*

**The basic estimate revisited: upper and lower sums.** The basic estimate discussed in Section 24 is a crude one, which typically gives us a broad range of values into which the integral of a function must fall. But if we want to get a more accurate estimate of an integral that we cannot compute explicitly, we need something more sophisticated. By cutting up the region of integration into smaller pieces, we can significantly improve upon the basic estimate.

Suppose $f(x, y)$ is a function that we wish to integrate on a region $D$, as in the figure below. In this picture, $D$ is the shaded triangle in the $xy$-plane, and the graph of $f$ is the shaded curved surface above $D$.




If we let $m_D$ denote the minimum value of $f$ on $D$ and $M_D$ its maximum value, the basic estimate reads

$$m_D \cdot A(D) \leq \iint_D f(x, y)\, dA \leq M_D \cdot A(D),$$

where $A(D)$ is the area of $D$. The lower estimate $m_D \cdot A(D)$ is just the volume of the small wedge-shaped figure under the graph of $f$, and the upper estimate is the volume of the large wedge-shaped figure that encloses the graph. As you can see in this picture, the volume under the graph of $f$ lies somewhere between these two volumes, but the estimate is still quite rough.

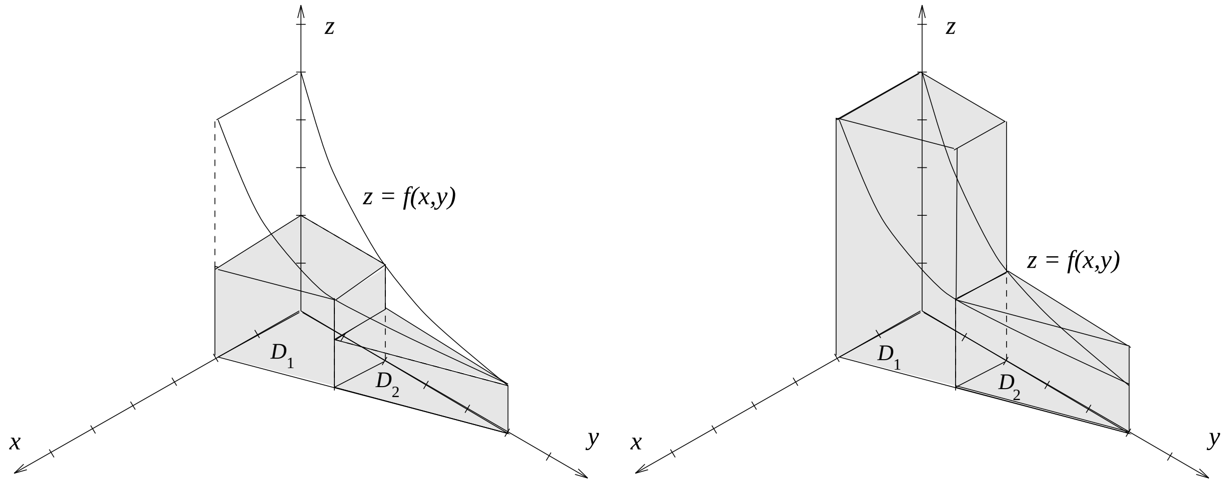
Now suppose we chop $D$ into two pieces, let's say $D_1$ and $D_2$. From property (iii) on page 221, it follows that

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$$

We can apply the basic estimate separately on $D_1$ and $D_2$ and add the resulting inequalities together to get a sharper estimate of the value of the integral:

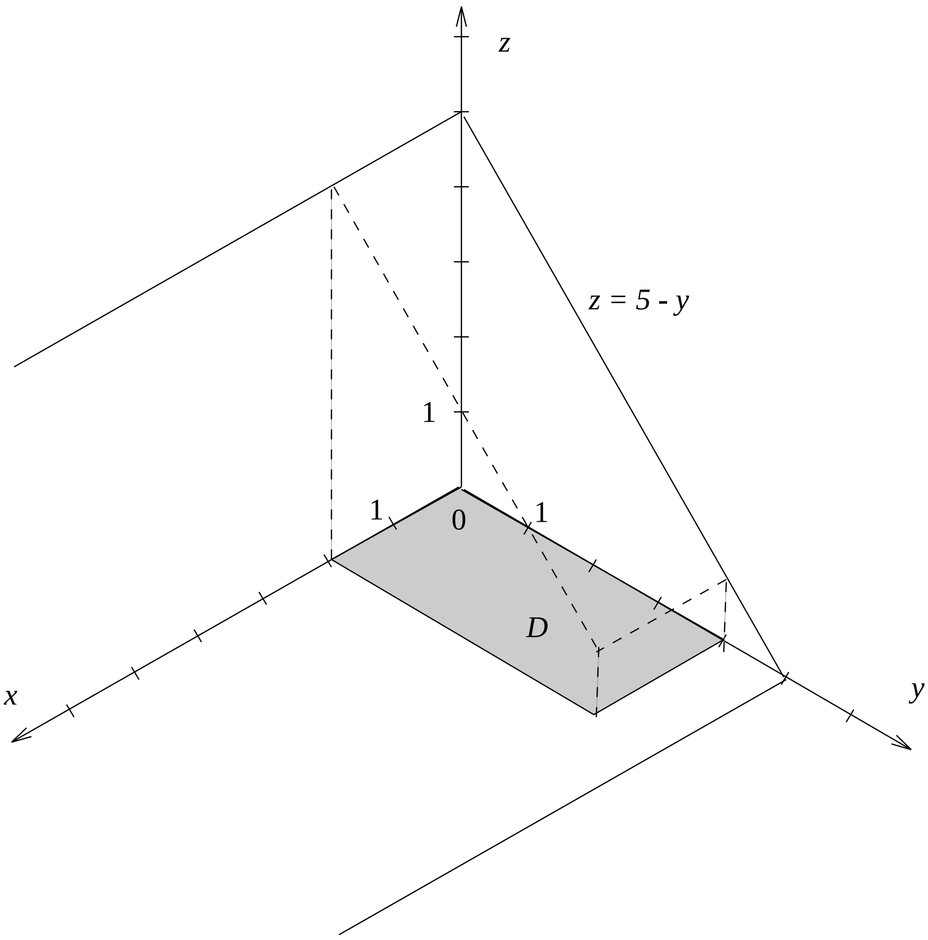
$$m_{D_1} \cdot A(D_1) + m_{D_2} \cdot A(D_2) \le \iint_D f(x,y)\,dA \le M_{D_1} \cdot A(D_1) + M_{D_2} \cdot A(D_2).$$

The sum on the left-hand side above is called the **lower sum** with respect to the subdivision of $D$ into $D_1$ and $D_2$, and the right-hand sum is called the **upper sum**. Geometrically, the lower sum is the volume of the shaded figure in the left-hand illustration below; the upper sum is the volume of the shaded figure on the right. You can see that both of these volumes are closer to the actual volume under the graph of $f$, which is the value of $\int_D f(x,y)\,dA$.




In order to explicitly compute the value of the upper and lower sums with respect to a particular subdivision, you must first find the maximum and minimum values of $f$ on each subregion of the subdivision.

**Problem 24S.1** $f(x,y)$ is the linear function $f(x,y) = 5 - y$ whose graph is the plane shown below, and $D$ is the shaded rectangle in the $xy$-plane with corners at $(0,0)$, $(2,0)$, $(0,4)$, and $(2,4)$.




(a) Determine what the basic estimate says about $\iint_D (5-y)\,dA$: that is, find the constants $k$ and $K$ such that

$$k \leq \iint_D (5-y)\,dA \leq K.$$

(b) Now subdivide $D$ into two subrectangles by cutting it along the line $y = 2$, and compute the upper and lower sums with respect to this subdivision. Verify that the new upper and lower bounds are closer together than those you found in (a).

(c) Compute the actual value of $\iint_D (5-y)\,dA$, and verify that it lies within the bounds you computed in (b). [Hint: you don't have to know how to compute a double integral; see Problem 23.1 for a suggestion.]

If we go on subdividing our region $D$ further, say into 512 pieces called $D_1$ through $D_{512}$, then we get a 512-term lower sum:

$$m_{D_1} \cdot A(D_1) + \cdots + m_{D_{512}} \cdot A(D_{512})$$

and a 512-term upper sum:

$$M_{D_1} \cdot A(D_1) + \cdots + M_{D_{512}} \cdot A(D_{512}),$$

and the value of the integral lies somewhere in between.

It is convenient to abbreviate sums like these using the following *summation notation*:

$$\textbf{Lower sum} = \sum_{i=1}^{N} m_{D_i} A(D_i);$$

$$\textbf{Upper sum} = \sum_{i=1}^{N} M_{D_i} A(D_i).$$

In this notation, the symbol $\sum$ (a capital Greek letter "sigma") is called a *summation sign*, and the expression $\sum_{i=1}^{N}$ is read "the sum as $i$ ranges from 1 to $N$."

What's the point of all this subdividing? It's this: as we continue to subdivide, and the length and width of the subregions get smaller and smaller, the maximum and minimum values of the the function $f$ will not be very far apart on any given subregion. Therefore the upper sum gets closer and closer to the lower sum, and the actual value of the integral gets trapped between ever closer upper and lower bounds. If we could compute the upper and lower sums for such a subdivision, we'd have a very accurate estimate of the value of the integral.

We can summarize the preceding discussion as follows.

BASIC PRINCIPLE OF UPPER AND LOWER SUMS: *If the region $D$ is subdivided into subregions $D_1$ through $D_N$, then the double integral of $f$ over $D$ lies between the lower sum and the upper sum, given by*

$$\textbf{Lower sum} = \sum_{i=1}^{N} m_{D_i} A(D_i),$$

$$\textbf{Upper sum} = \sum_{i=1}^{N} M_{D_i} A(D_i),$$

*where $m_{D_i}$ is the minimum value of $f$ on $D_i$ and $M_{D_i}$ is its maximum. If the number of subregions is increased in such a way that the size of the individual subregions (maximum length and width) gets very small, then the upper and lower sum can be made as close together as desired.*

**Riemann sums.** In practice, as the number of subregions gets large, computing the actual maximum and minimum values of $f$ on each subregion becomes quite impractical. Doing so, after all, would force us to do a global optimization problem on each of the many small subregions. However, once the size of the subregions gets sufficiently small, there's not much point in computing the maximum and minimum values of $f$ on each subregion. Since the maximum and minimum values of $f$ on any single subregion are so close to each other, we can just choose *any* point in each subregion, according to some arbitrary rule, and we'll still get a very good approximation to the integral.

Suppose, therefore, that we have subdivided $D$ into $N$ subregions, where $N$ is some very large number. If we let $(x_1, y_1)$ be an arbitrarily-chosen point in $D_1$, $(x_1, y_2)$ in $D_2$, and so forth, then we get an estimate for the value of the integral that looks like this:

$$\iint_D f(x,y)\, dA \approx \sum_{i=1}^{N} f(x_i, y_i) A(D_i).$$

This is almost the same as the lower and upper sums, except that the minimum and maximum values of $f$ have been replaced by the values of $f$ at the arbitrarily chosen points $(x_i, y_i)$.

Any sum of this form used to approximate an integral is called a **Riemann sum**. Since, on each subregion $D_i$, the value of $f(x_i, y_i)$ at the chosen point must lie between the minimum value $m_{D_i}$ and the maximum value $M_{D_i}$ of $f$ on that subregion, we conclude that

$$m_{D_i} \cdot A(D_i) \leq f(x_i, y_i) \cdot A(D_i) \leq M_{D_i} \cdot A(D_i).$$

Summing over $i$, we conclude that any Riemann sum must lie between the lower and upper sums for that subdivision. Therefore, by choosing a subdivision of the region $D$ such that the lengths and widths of all the subregions $D_i$ are very small, we can make the value of the Riemann sum as close to the actual value of the integral as we wish. To summarize:

BASIC PRINCIPLE OF RIEMANN SUMS: *If the region $D$ is subdivided into subregions $D_1$ through $D_N$, then the double integral of $f$ over $D$ can be approximated by any Riemann sum, which is a sum of the form*

$$\textbf{Riemann sum} = \sum_{i=1}^{N} f(x_i, y_i) A(D_i),$$

*where the points $(x_i, y_i)$ are chosen arbitrarily within each subregion $D_i$. Any Riemann sum defined with respect to a given subdivision lies between the lower and upper sums for that subdivision, and hence the Riemann sums approach the actual value of the integral as the size of the subregions gets small.*

Riemann sums have both theoretical and practical value. Theoretically, in many treatments of multivariable calculus, double integrals are *defined* as the limit of Riemann sums as the size of the subregions approaches zero. Practically, this is how integrals are usually calculated numerically: A computer would approximate $\iint_D f(x,y)\, dA$ by subdividing $D$ into a very large number of square subregions, evaluating $f(x,y)$ at, say, the lower-left corner of each square, and then computing the corresponding Riemann sum.

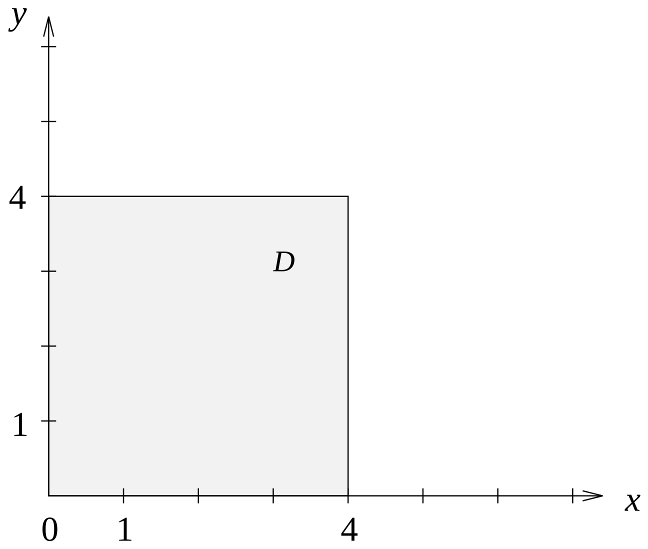
## 24S. RIEMANN SUMS — PROBLEMS

**24S.2** EMVC 7.1 (p. 55).

**24S.3** EMVC 7.2 (p. 56).

**24S.4** EMVC 7.5 (p. 57).

**24S.5** $f$ is the function $f(x, y) = 2x^2 - 4x + y^2 - 6y$, and $D$ is the square shown below.




(a) By finding the maximum and minimum values of $f$ on $D$, compute the upper and lower bounds for $\iint_D f(x, y)\, dA$ given by the basic estimate.

(b) Now subdivide $D$ into four subregions by cutting it along the lines $x = 2$, $y = 2$, and compute the values of the upper and lower sums with respect to this subdivision. [Hint: you will have to do four global optimization problems. Begin by writing down *all* the candidates for extrema of $f$; you can use the work you did in (a).]

(c) Now compute a four-term Riemann sum for $\iint_D f(x, y)\, dA$ using the same subdivision as in (b), and for each subregion $D_i$ choose the point $(x_i, y_i)$ to be the center point. Verify that the value of your Riemann sum lies between the upper and lower sums you computed in (b).

(d) Same as (c), but now choose the lower-left corner of each subregion.