11.95

SUPPLEMENT TO SECTION 20: THE SECOND-DERIVATIVE TEST FOR LOCAL EXTREMA

NOTE: *The following pages should replace the text near the end of Section 20 in [LN], starting with the paragraph near the bottom of page 184 titled* **Is there a reliable test for local extrema?** *and continuing to the bottom of page 185.*

The second-derivative test. Remember the second-derivative test from one-variable calculus: if $x = a$ is a critical point of the function $y = f(x)$ (that is to say, if $f'(a) = 0$), and the second derivative $f''(a)$ is strictly positive, then you can conclude that the graph of $f(x)$ is concave up near $x = a$, and consequently $x = a$ is a local minimum for $f(x)$. Similarly, if $f''(a)$ is negative, then $x = a$ is a local maximum. If $f''(a) = 0$, no conclusion can be drawn.

For functions of two variables, the second partial derivatives also tell us something about the shape of the graph near a critical point $(x, y) = (a, b)$. There is a second-derivative test for local extrema of such functions, but it is more complicated. Here it is:

The second-derivative test for functions of two variables: First compute the following expression, called the *Hessian determinant* of $f(x, y)$:

$$H(x,y) = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = f_{xx}f_{yy} - (f_{xy})^2.$$

It will be a function of x and y. Then evaluate $H(a, b)$ for each critical point (a, b) of the function f and draw the following conclusions:

1. If $H(a, b) > 0$, then f has a local extremum at (a, b):
   - (1a) If $f_{xx}(a, b) > 0$ (or $f_{yy}(a, b) > 0$), it is a local minimum.
   - (1b) If $f_{xx}(a, b) < 0$ (or $f_{yy}(a, b) < 0$), it is a local maximum.
2. If $H(a, b) < 0$, then f has a saddle point at (a, b).
3. If $H(a, b) = 0$, the test is inconclusive.

We can get a good idea of why the test works by looking at some simple examples. Let us consider some quadratic polynomials of the form

$$f(x, y) = Ax^2 + Bxy + Cy^2 + D,$$

where A, B, C, D are constants. An easy computation shows that

$$grad\ f = (2Ax + By, Bx + 2Cy),$$

so any such function has a critical point at the origin $(x, y) = (0, 0)$.

For example, for the function $f(x, y) = x^2 - y^2 + 4$ shown in the left-hand illustration in the middle of page 184, the origin is a critical point, and at that point the second partial derivatives are $f_{xx} = 2$, $f_{yy} = -2$, and $f_{xy} = f_{yx} = 0$. Thus the Hessian determinant is

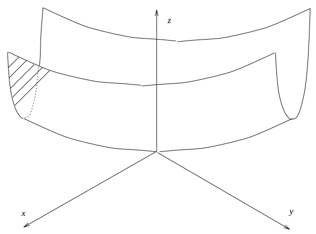
$$H(0,0) = \det \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = -4,$$

and the second-derivative test tells us that f has a saddle point at $(0, 0)$.

It is not hard to see why this is true geometrically: the fact that the derivatives f_{xx} and f_{yy} have opposite signs corresponds to the fact that, if we hold $y = 0$ constant and consider $f(x, 0)$ as a function of x, it has a local minimum at $x = 0$; while if we hold $x = 0$ constant and consider $f(0, y)$ as a function of y, it has a local maximum. It follows that the origin can be neither a local minimum nor a local maximum for f: it must be a saddle point. In fact, the same reasoning leads to the conclusion that *any* critical point for which the partial derivatives f_{xx} and f_{yy} have opposite signs must be a saddle point. (This follows automatically from the second-derivative test: if f_{xx} and f_{yy} have opposite signs, then $f_{xx}f_{yy}$ is negative and $-(f_{xy})^2$ is negative or zero, so $H(x, y) < 0$ and we are in case 2.)

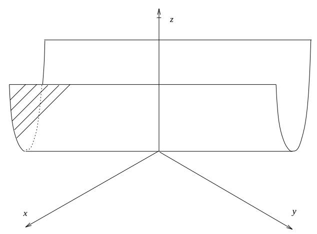
Now look at the right-hand illustration on page 184; this is the graph of the function $f(x, y) = x^2 + y^2 + 3$. The second partial derivatives of f are $f_{xx} = 2$, $f_{yy} = 2$, and $f_{xy} = 0$, so $H(0, 0) = 4$. Now the curves we get by setting $x = 0$ and $y = 0$ are both concave up, and reasoning by analogy with the one-variable case, we would be led to expect a strict local minimum at the origin. You can see from the graph that this is indeed true in this case, and the second-derivative test confirms this.

However, for more general functions we must be more careful, because the presence of a nonzero mixed second partial derivative f_{xy} can lead to unexpected results. Consider, for example, the three functions whose graphs are sketched below.



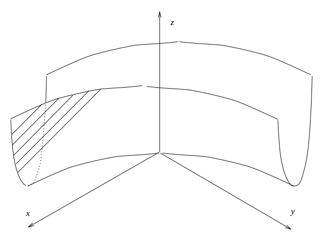
$f(x, y) = x^2 + y^2 + xy$

$H(0, 0) = 3$



$f(x, y) = x^2 + y^2 + 2xy$

$H(0, 0) = 0$



$f(x, y) = x^2 + y^2 + 3xy$

$H(0, 0) = -5$

Each of these is a quadratic polynomial of the type we have been considering, and each has a critical point at the origin. From the formulas we can compute that in each case $f_{xx} = f_{yy} = 2$. However, the Hessian determinant decreases as we go from left to right. In the first picture, $H(0, 0)$ is still positive; you can see that the graph is somewhat broader than the function in the right-hand illustration on page 184, but this function still has a local minimum at 0. In the second picture, $H(0,0) = 0$, and the graph has flattened out: the origin is a local minimum, but no longer a "strict" local minimum: the function takes on its minimum values at other nearby points in addition to the origin. In the third picture, $H(0,0)$ is negative, and the origin is no longer a local minimum at all: it is a saddle point.

In each of these cases, since $f_{xx} = f_{yy} = 2$, the curves you get by setting $x = 0$ or $y = 0$ are still concave up. But if you look at the curve where the graph intersects the plane $x = -y$ (the bottoms of the "troughs" in the pictures), you'll see that the graph is concave down in the third picture, which accounts for the fact that the origin is a saddle point there.

In advanced calculus courses, it is proved that the graph of $f(x,y)$ is well approximated near a critical point by the graph of a quadratic polynomial of the sort we have been considering. The exact shape of the polynomial depends on the Hessian determinant, which is why the second derivative test works.

Problem 20S.1 $f(x,y) = \frac{1}{2}x^2 - x - \frac{1}{3}y^3 + 4y$.

(a) Find the critical points of $f(x,y)$ (by computation). There will be two of them.

(b) Use the second-derivative test to classify each of the critical points you found in (a). ("Classifying" means deciding for each critical point, if possible, whether $f(x,y)$ has a local maximum, local minimum, or saddle point there.)

(c) In the illustration on page 186 you find some level curves of $f(x,y)$. Mark the critical points. How would you expect the level curves through the two critical points to look? Draw what you think they look like.

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20S.2 Find and classify the critical points of $f(x, y) = -x^3 - y^3 + 18xy$.

20S.3 Find and classify the critical points of $g(x, y) = xy + \frac{8}{x} + \frac{1}{y}$.

20S.4 We consider the function $f(x, y) = x^2 + y^3 - 3y$. At which of the following points does $f(x, y)$ have a local maximum, or local minimum, or saddle point:

$$(1, 0), \quad (-1, 0), \quad (0, 1), \quad (0, -1), \quad (1, 1), \quad (-1, -1)$$

Hint: "Local maximum/minimum" and "saddle point" make sense only for a critical point. Are all these points critical points?

20S.5 We have the following information on a function $f(x, y)$: *grad* $f = (x^2 - 9, y - 2)$. Is that enough information to find and classify the critical points? If yes, find and classify them; if no, explain.