John M. Lee

Riemannian Manifolds:
An Introduction to Curvature
John M. Lee
University of Washington
Department of Mathematics
Seattle, WA 98195-4350
U.S.A.

June 27, 1997
©1997 by Springer-Verlag New York Inc.
To my family:
Phys., Nathan, and Jeremy Weizenbaum
Preface

This book is designed as a textbook for a one-quarter or one-semester graduate course on Riemannian geometry, for students who are familiar with topological and differentiable manifolds. It focuses on developing an intimate acquaintance with the geometric meaning of curvature. In so doing, it introduces and demonstrates the uses of all the main technical tools needed for a careful study of Riemannian manifolds.

I have selected a set of topics that can reasonably be covered in ten to fifteen weeks, instead of making any attempt to provide an encyclopedic treatment of the subject. The book begins with a careful treatment of the machinery of metrics, connections, and geodesics, without which one cannot claim to be doing Riemannian geometry. It then introduces the Riemann curvature tensor, and quickly moves on to submanifold theory in order to give the curvature tensor a concrete quantitative interpretation. From then on, all efforts are bent toward proving the four most fundamental theorems relating curvature and topology: the Gauss–Bonnet theorem (expressing the total curvature of a surface in terms of its topological type), the Cartan–Hadamard theorem (restricting the topology of manifolds of nonpositive curvature), Bonnet’s theorem (giving analogous restrictions on manifolds of strictly positive curvature), and a special case of the Cartan–Ambrose–Hicks theorem (characterizing manifolds of constant curvature).

Many other results and techniques might reasonably claim a place in an introductory Riemannian geometry course, but could not be included due to time constraints. In particular, I do not treat the Rauch comparison theorem, the Morse index theorem, Toponogov’s theorem, or their important applications such as the sphere theorem, except to mention some of them
in passing; and I do not touch on the Laplace–Beltrami operator or Hodge
theory, or indeed any of the multitude of deep and exciting applications
of partial differential equations to Riemannian geometry. These important
topics are for other, more advanced courses.

The libraries already contain a wealth of superb reference books on Rie-
nemannian geometry, which the interested reader can consult for a deeper
treatment of the topics introduced here, or can use to explore the more
esoteric aspects of the subject. Some of my favorites are the elegant intro-
duction to comparison theory by Jeff Cheeger and David Ebin [CE75]
(which has sadly been out of print for a number of years); Manfredo do
Carmo’s much more leisurely treatment of the same material and more
[DC92]; Barrett O’Neill’s beautifully integrated introduction to pseudo-
Riemannian and Riemannian geometry [ON83]; Isaac Chavel’s masterful
recent introductory text [Ch93], which starts with the foundations of the
subject and quickly takes the reader deep into research territory; Michael
Spivak’s classic tome [Sp79], which can be used as a textbook if plenty of
time is available, or can provide enjoyable bedtime reading; and, of course,
the “Encyclopaedia Britannica” of differential geometry books, *Founda-
tions of Differential Geometry* by Kobayashi and Nomizu [KN63]. At
the other end of the spectrum, Frank Morgan’s delightful little book [Mor93]
touches on most of the important ideas in an intuitive and informal way
with lots of pictures—I enthusiastically recommend it as a prelude to this
book.

It is not my purpose to replace any of these. Instead, it is my hope
that this book will fill a niche in the literature by presenting a selective
introduction to the main ideas of the subject in an easily accessible way.
The selection is small enough to fit into a single course, but broad enough,
I hope, to provide any novice with a firm foundation from which to pursue
research or develop applications in Riemannian geometry and other fields
that use its tools.

This book is written under the assumption that the student already
knows the fundamentals of the theory of topological and differential mani-

folds, as treated, for example, in [Mas67, chapters 1–5] and [Boo86, chapters
1–6]. In particular, the student should be conversant with the fundamental
group, covering spaces, the classification of compact surfaces, topological
and smooth manifolds, immersions and submersions, vector fields and flows,
Lie brackets and Lie derivatives, the Frobenius theorem, tensors, differential
forms, Stokes’s theorem, and elementary properties of Lie groups. On
the other hand, I do not assume any previous acquaintance with Riemann-
ian metrics, or even with the classical theory of curves and surfaces in \( \mathbb{R}^3 \).
(In this subject, anything proved before 1950 can be considered “classi-
cal.”) Although at one time it might have been reasonable to expect most
mathematics students to have studied surface theory as undergraduates,
few current North American undergraduate math majors see any differen-
tial geometry. Thus the fundamentals of the geometry of surfaces, including
a proof of the Gauss–Bonnet theorem, are worked out from scratch here.

The book begins with a nonrigorous overview of the subject in Chapter
1, designed to introduce some of the intuitions underlying the notion of
curvature and to link them with elementary geometric ideas the student
has seen before. This is followed in Chapter 2 by a brief review of some
background material on tensors, manifolds, and vector bundles, included
because these are the basic tools used throughout the book and because
often they are not covered in quite enough detail in elementary courses
on manifolds. Chapter 3 begins the course proper, with definitions of Rie-
mannian metrics and some of their attendant flora and fauna. The end of
the chapter describes the constant curvature “model spaces” of Riemannian
geometry, with a great deal of detailed computation. These models form a
sort of leitmotif throughout the text, and serve as illustrations and testbeds
for the abstract theory as it is developed. Other important classes of exam-
ples are developed in the problems at the ends of the chapters, particularly
invariant metrics on Lie groups and Riemannian submersions.

Chapter 4 introduces connections. In order to isolate the important prop-
eties of connections that are independent of the metric, as well as to lay
the groundwork for their further study in such arenas as the Chern–Weil theory
of characteristic classes and the Donaldson and Seiberg–Witten theories of
gauge fields, connections are defined first on arbitrary vector bundles. This
has the further advantage of making it easy to define the induced connec-
tions on tensor bundles. Chapter 5 investigates connections in the context
of Riemannian manifolds, developing the Riemannian connection, its geo-
desics, the exponential map, and normal coordinates. Chapter 6 continues
the study of geodesics, focusing on their distance-minimizing properties.
First, some elementary ideas from the calculus of variations are introduced
to prove that every distance-minimizing curve is a geodesic. Then the Gauss
lemma is used to prove the (partial) converse—that every geodesic is lo-
caley minimizing. Because the Gauss lemma also gives an easy proof that
minimizing curves are geodesics, the calculus-of-variations methods are not
strictly necessary at this point; they are included to facilitate their use later
in comparison theorems.

Chapter 7 unveils the first fully general definition of curvature. The cur-
vature tensor is motivated initially by the question of whether all Riemann-
ian metrics are locally equivalent, and by the failure of parallel translation
to be path-independent as an obstruction to local equivalence. This leads
naturally to a qualitative interpretation of curvature as the obstruction to
flatness (local equivalence to Euclidean space). Chapter 8 departs some-
what from the traditional order of presentation, by investigating subman-
ifold theory immediately after introducing the curvature tensor, so as to
define sectional curvatures and give the curvature a more quantitative geo-
metric interpretation.
The last three chapters are devoted to the most important elementary
global theorems relating geometry to topology. Chapter 9 gives a simple
moving-frames proof of the Gauss–Bonnet theorem, complete with a care-
ful treatment of Hopf’s rotation angle theorem (the *Umlaufsatz*). Chapter
10 is largely of a technical nature, covering Jacobi fields, conjugate points,
the second variation formula, and the index form for later use in com-
parison theorems. Finally in Chapter 11 comes the dénouement—proofs of
some of the “big” global theorems illustrating the ways in which curvature
and topology affect each other: the Cartan–Hadamard theorem, Bonnet’s
theorem (and its generalization, Myers’s theorem), and Cartan’s character-
ization of manifolds of constant curvature.

The book contains many questions for the reader, which deserve special
mention. They fall into two categories: “exercises,” which are integrated
into the text, and “problems,” grouped at the end of each chapter. Both are
essential to a full understanding of the material, but they are of somewhat
different character and serve different purposes.

The exercises include some background material that the student should
have seen already in an earlier course, some proofs that fill in the gaps from
the text, some simple but illuminating examples, and some intermediate
results that are used in the text or the problems. They are, in general,
elementary, but they are *not optional*—indeed, they are integral to the
continuity of the text. They are chosen and timed so as to give the reader
opportunities to pause and think over the material that has just been intro-
duced, to practice working with the definitions, and to develop skills that
are used later in the book. I recommend strongly that students stop and
do each exercise as it occurs in the text before going any further.

The problems that conclude the chapters are generally more difficult
than the exercises, some of them considerably so, and should be considered
a central part of the book by any student who is serious about learning the
subject. They not only introduce new material not covered in the body of
the text, but they also provide the student with indispensable practice in
using the techniques explained in the text, both for doing computations and
for proving theorems. If more than a semester is available, the instructor
might want to present some of these problems in class.

*Acknowledgments:* I owe an unpayable debt to the authors of the many
Riemannian geometry books I have used and cherished over the years,
especially the ones mentioned above—I have done little more than rear-
range their ideas into a form that seems handy for teaching. Beyond that,
I would like to thank my Ph.D. advisor, Richard Melrose, who many years
ago introduced me to differential geometry in his eccentric but thoroughly
enlightening way; Judith Arms, who, as a fellow teacher of Riemannian
geometry at the University of Washington, helped brainstorm about the
“ideal contents” of this course; all my graduate students at the University
of Washington who have suffered with amazing grace through the flawed
early drafts of this book, especially Jed Mihalisin, who gave the manuscript
a meticulous reading from a user’s viewpoint and came up with numerous
valuable suggestions; and Ina Lindemann of Springer-Verlag, who encour-
age me to turn my lecture notes into a book and gave me free rein in de-
ciding on its shape and contents. And of course my wife, Ph Weizenbaum,
who contributed professional editing help as well as the loving support and
encouragement I need to keep at this day after day.
xii Preface
Contents

Preface vii

1 What Is Curvature? 1
   The Euclidean Plane .............................................. 2
   Surfaces in Space .................................................. 4
   Curvature in Higher Dimensions ................................. 8

2 Review of Tensors, Manifolds, and Vector Bundles 11
   Tensors on a Vector Space ........................................ 11
   Manifolds ................................................................ 14
   Vector Bundles ....................................................... 16
   Tensor Bundles and Tensor Fields .............................. 19

3 Definitions and Examples of Riemannian Metrics 23
   Riemannian Metrics .................................................. 23
   Elementary Constructions Associated with Riemannian Metrics 27
   Generalizations of Riemannian Metrics ........................ 30
   The Model Spaces of Riemannian Geometry .................. 33
   Problems ............................................................. 43

4 Connections 47
   The Problem of Differentiating Vector Fields ............... 48
   Connections ........................................................ 49
   Vector Fields Along Curves ..................................... 55
## Contents

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>Riemannian Geodesics</td>
<td>58</td>
</tr>
<tr>
<td></td>
<td>Geodesics</td>
<td>65</td>
</tr>
<tr>
<td></td>
<td>Problems</td>
<td>63</td>
</tr>
<tr>
<td>6</td>
<td>Geodesics and Distance</td>
<td>91</td>
</tr>
<tr>
<td></td>
<td>Geodesics and Distance on Riemannian Manifolds</td>
<td>91</td>
</tr>
<tr>
<td></td>
<td>Geodesics and Minimizing Curves</td>
<td>96</td>
</tr>
<tr>
<td></td>
<td>Completeness</td>
<td>108</td>
</tr>
<tr>
<td></td>
<td>Problems</td>
<td>112</td>
</tr>
<tr>
<td>7</td>
<td>Curvature</td>
<td>115</td>
</tr>
<tr>
<td></td>
<td>Local Invariants</td>
<td>115</td>
</tr>
<tr>
<td></td>
<td>Flat Manifolds</td>
<td>119</td>
</tr>
<tr>
<td></td>
<td>Symmetries of the Curvature Tensor</td>
<td>121</td>
</tr>
<tr>
<td></td>
<td>Ricci and Scalar Curvatures</td>
<td>124</td>
</tr>
<tr>
<td></td>
<td>Problems</td>
<td>128</td>
</tr>
<tr>
<td>8</td>
<td>Riemannian Submanifolds</td>
<td>131</td>
</tr>
<tr>
<td></td>
<td>Riemannian Submanifolds and the Second Fundamental Form</td>
<td>132</td>
</tr>
<tr>
<td></td>
<td>Hypersurfaces in Euclidean Space</td>
<td>139</td>
</tr>
<tr>
<td></td>
<td>Geometric Interpretation of Curvature in Higher Dimensions</td>
<td>145</td>
</tr>
<tr>
<td></td>
<td>Problems</td>
<td>150</td>
</tr>
<tr>
<td>9</td>
<td>The Gauss–Bonnet Theorem</td>
<td>155</td>
</tr>
<tr>
<td></td>
<td>Some Plane Geometry</td>
<td>156</td>
</tr>
<tr>
<td></td>
<td>The Gauss–Bonnet Formula</td>
<td>162</td>
</tr>
<tr>
<td></td>
<td>The Gauss–Bonnet Theorem</td>
<td>166</td>
</tr>
<tr>
<td></td>
<td>Problems</td>
<td>171</td>
</tr>
<tr>
<td>10</td>
<td>Jacobi Fields</td>
<td>173</td>
</tr>
<tr>
<td></td>
<td>The Jacobi Equation</td>
<td>174</td>
</tr>
<tr>
<td></td>
<td>Computations of Jacobi Fields</td>
<td>178</td>
</tr>
<tr>
<td></td>
<td>Conjugate Points</td>
<td>181</td>
</tr>
<tr>
<td></td>
<td>The Second Variation Formula</td>
<td>185</td>
</tr>
<tr>
<td></td>
<td>Geodesics Do Not Minimize Past Conjugate Points</td>
<td>187</td>
</tr>
<tr>
<td></td>
<td>Problems</td>
<td>191</td>
</tr>
<tr>
<td>11</td>
<td>Curvature and Topology</td>
<td>193</td>
</tr>
<tr>
<td></td>
<td>Some Comparison Theorems</td>
<td>194</td>
</tr>
<tr>
<td></td>
<td>Manifolds of Negative Curvature</td>
<td>196</td>
</tr>
</tbody>
</table>
Contents

Manifolds of Positive Curvature ........................................... 199
Manifolds of Constant Curvature ........................................... 204
Problems ............................................................................. 208

References ............................................................................ 209

Index ..................................................................................... 212
xvi Contents