CORRECTIONS TO
Introduction to Smooth Manifolds (Second Edition)
BY JOHN M. LEE
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(8/16/16) Page 6, just below the last displayed equation: Change \( \varphi([x]) \) to \( \varphi_{n+1}[x] \), and in the next line, change \( x^i \) to \( x^{n+1} \). After “(Fig. 1.4),” insert “with similar interpretations for the other charts.”

(8/16/16) Page 7, Fig. 1.4: Both occurrences of \( x^i \) should be \( x^{n+1} \).

(12/19/18) Page 9, proof of Theorem 1.15: In the second line of the proof, replace “For each \( j \)” with “For each \( j \geq 0 \).” Then in the fourth-to-last line, replace “positive integers” by “nonnegative integers.”

(1/15/21) Page 13, line 1: Delete the words “and injective.”

(1/18/21) Page 20, Example 1.31: There are multiple errors in this example. Replace everything after the first two sentences by the following: For each \( i = 1, \ldots, n+1 \), let \( (U_i^\pm \cap S^n, \varphi_i^\pm) \) denote the graph coordinate charts we constructed in Example 1.4. For any distinct indices \( i \) and \( j \), and any choices of \( \pm \) signs, the transition maps \( \varphi_i^\pm \circ (\varphi_j^\pm)^{-1} \) and \( \varphi_i^\pm \circ (\varphi_j^\pm)^{-1} \) are easily computed. For example, in the case \( i < j \), we get the following formula for all \( u \) in the domain of \( \varphi_i^\pm \circ (\varphi_j^\pm)^{-1} \):

\[
\varphi_i^\pm \circ (\varphi_j^\pm)^{-1}(u^1, \ldots, u^n) = (u^1, \ldots, u^i, \sqrt{1-|u|^2}, \ldots, u^n)
\]

(with \( u^i \) omitted and the square root replacing \( u^i \)), and similar formulas hold in the other cases. When \( i = j \), the domains of \( \varphi_i^\pm \) and \( \varphi_i^- \) are disjoint, so there is nothing to check. Thus, the collection of charts \( \{ (U_i^\pm \cap S^n, \varphi_i^\pm) \} \) is a smooth atlas, and so defines a smooth structure on \( S^n \). We call this its standard smooth structure.

(6/23/13) Page 23, two lines below the first displayed equation: Change “any subspace \( S \subseteq V \)” to “any \( k \)-dimensional subspace \( S \subseteq V \)”.

(9/15/19) Page 24, first full paragraph, fourth line: Change “any subspace \( S \)” to “any \( k \)-dimensional subspace \( S \).”

(12/19/18) Page 26, first line: Change \( U \cap \varphi^{-1}(\text{Int } \mathbb{H}^n) \) to \( \varphi^{-1}(\text{Int } \mathbb{H}^n) \).

(12/19/18) Page 27, last paragraph, sixth line: Change \( \bar{U} \cap \mathbb{H}^n \) to \( \bar{U} \cap U \).

(2/22/15) Page 29, proof of Theorem 1.46, second paragraph, line 4: Change \( \varphi(U \cap V) \) to \( \psi(U \cap V) \).

(10/8/15) Page 30, Problem 1-6: Interpret the formula for \( F_s \) to mean \( F_s(0) = 0 \) when \( s \leq 1 \).

(1/27/18) Page 31, Fig. 1.13: Change \( \{ x^n = 0 \} \) to \( \{ x^{n+1} = 0 \} \).

(3/12/18) Page 31, Problem 1-11, next-to-last line: Change \( S^n \) to \( S^n \setminus \{ N \} \).

(4/25/17) Page 45, second paragraph: Replace the last sentence of that paragraph with the following: “If \( N \) has empty boundary, we say that a map \( F : A \to N \) is smooth on \( A \) if it has a smooth extension in a neighborhood of each point: that is, if for every \( p \in A \) there exist an open subset \( W \subseteq M \) containing \( p \) and a smooth map \( \bar{F} : W \to N \) whose restriction to \( W \cap A \) agrees with \( F \). When \( \partial N \neq \emptyset \), we say \( F : A \to N \) is smooth on \( A \) if for every \( p \in A \) there exist an open subset \( W \subseteq M \) containing \( p \) and a smooth chart \( (V, \psi) \) for \( N \) whose domain contains \( F(p) \), such that \( F(W \cap A) \subseteq V \) and \( \psi \circ F \mid_{W \cap A} \) is smooth as a map into \( \mathbb{R}^n \) in the sense defined above (i.e., it has a smooth extension in a neighborhood of each point).”

(7/23/14) Page 45, last displayed equation: The first \( = \) sign should be \( \leq \).

(9/15/19) Page 46, line 9: Change “on an open subset” to “on a nonempty open subset.”
Page 47, proof of Theorem 2.29, second paragraph: Replace the first sentence of the paragraph by “Let $h : \mathbb{R}^n \to \mathbb{R}$ be a smooth bump function that is positive in $B_1(0)$ and zero elsewhere.”

Page 49, Problem 2-10(c): Change “an isomorphism” to “a bijection.”

Page 54, just after the first sentence: Insert “(The integral is a smooth function of $x$ by iterative application of Theorem C.14.)”

Page 56, first displayed equation: Change $d v_p$ to $d p_v$.

Page 56, just below the last displayed equation: Replace “the last two equalities follow” by “the last equality follows.”

Page 58, proof of Lemma 3.11, next-to-last line: Change $H^n$ to $\text{Int} H^n$.

Page 68, proof of Proposition 3.21: Insert the following sentence at the beginning of the proof: “Let $n \leq \dim M$ and $m \leq \dim N$.” Then in the second sentence, change (3.9) to (3.10). Finally, in the displayed equation, change $F^n$ to $F^m$ (twice).

Page 70, two lines above Corollary 3.25: Change “Proposition 3.23” to “Proposition 3.24.”

Page 76, Problem 3-8: Add the following remark: “(For $p \in \partial M$, we need to allow curves with domain $[0, \varepsilon]$ or $(-\varepsilon, 0]$ and to interpret the derivatives as one-sided derivatives.)”

Page 78, proof of Prop. 4.1, third and fourth lines: Change $m \times n$ to $n \times m$ (twice).

Page 79, proof of Theorem 4.5, fourth line: Change $\tilde{F}(p)$ to $\tilde{F}(0)$.

Page 82, line 4 from the bottom: Change “This is a diffeomorphism onto its image” to “This is an open map and a diffeomorphism onto its image.”

Page 100, first sentence: At the end of the sentence, change “smooth embeddings” to “smooth embeddings of smooth manifolds.”

Page 100, proof of Proposition 5.4, next-to-last line: Change “It a homeomorphism” to “It is a homeomorphism.”

Page 104, line below the proof of Theorem 5.11: Change “See Theorem 5.31” to “See Problem 5-24.” [Problem 5-24 is a new problem, described later in this list. Theorem 5.31 is not appropriate in this situation because it applies only to manifolds without boundary.]

Page 105, line 4 from the bottom: Change $F$ to $\Phi$.

Page 112, Fig. 5.10: Interchange the labels $M$ and $N$ on the figure, to be consistent with the notation in Theorem 5.29.
Page 113, line 6: Change the definition of $\bar{\psi}$ to $\bar{\psi} = \pi \circ \psi |_{V_0}$. After the end of that sentence, insert the following: “To see that $\bar{\psi}$ is a smooth coordinate map, let $i : V \hookrightarrow M$ be the inclusion map. Note first that for each $q \in V_0$, $x^{k+1}, \ldots, x^n$ are all constant on the image of $i$, so the image of $d i_q$ is contained in the span of $\partial / \partial x^1, \ldots, \partial / \partial x^k$. Since $d i_q$ is injective and its image has trivial intersection with $\text{Ker} \ d \bar{\psi}_q$, it follows that $d \bar{\psi}_q \circ d i_q$ is injective, so for dimensional reasons it is an isomorphism. Thus $\bar{\psi} \circ i$ is a local diffeomorphism by the inverse function theorem. Since it is bijective from $V_0$ to its image, it is a diffeomorphism and hence a smooth coordinate map for $V$."

Page 118, Fig. 5.13: Change $N$ to $v$.

Page 120, proof of Proposition 5.46: At the beginning of the proof, insert this sentence: “Let $F : D \hookrightarrow M$ denote the inclusion map.”

Page 121, line 5: Change $x^m$ to $x^n$.

Page 123, Problem 5-6: Add the assumption that $m > 0$.

Page 124: At the end of the page, add a new problem:
5-24. Suppose $M$ is a smooth manifold with boundary, $N$ is a smooth manifold, and $F : N \to M$ is a smooth map whose image is contained in $\partial M$. Show that $F$ is smooth as a map into $\partial M$, and use this to prove that $\partial M$ has a unique smooth structure making it an embedded submanifold of $M$.

Page 129, proof of Sard’s theorem, second paragraph: Just before the last sentence of the paragraph, insert the following: “In the $\mathbb{R}^n$ case, extend $F$ to a smooth map on an open subset of $\mathbb{R}^m$, and replace $U$ by that open subset; if we can show that the set of critical values of the extended map has measure zero, then the same is true of the set of critical values of $F$.”

Page 129, displayed equation near the bottom of the page: Change “$i$’th partial derivatives” to “$i$’th-order partial derivatives.”

Page 130, just below equation (6.2): Right after the displayed equation, insert “(where the component functions $F^2, \ldots, F^n$ might be different from the ones in the original coordinate chart).”

Page 131, two lines below the first displayed equation: Change $A'(R/K)^k+1$ to $A'(R\sqrt{m}/K)^{k+1}$.

Page 131, three lines below the first displayed equation: Insert “at most” before “$K^n$ balls.”

Page 131, second displayed equation: Change the left-hand side to $K^m(A')^n(R\sqrt{m}/K)^{n(k+1)}$, and in the next line, change the definition of $A''$ to $A'' = (A')^n(R\sqrt{m})^{n(k+1)}$.

Page 132, proof of Lemma 6.13, second paragraph: This argument does not apply when $\partial M \neq \emptyset$, because in that case $M \times M$ is not a smooth manifold with boundary. Instead, we can consider the restrictions of $\kappa$ to $(M \times \text{Int} M) \sim \Delta M$ and to $(M \times \partial M) \sim \Delta M$ (both of which are smooth manifolds with boundary), and note that there is a point $[v] \in \mathbb{R}P^{n-1}$ that is not in the image of $\tau$ or either of these restrictions of $\kappa$. [Thanks to David Iglesias Ponte for suggesting this correction.]

Page 134, proof of Theorem 6.15, just after the fourth paragraph of the proof: Insert the following: “In case $M$ is an arbitrary compact subset of a larger manifold $\tilde{M}$ with or without boundary, we can adapt this argument to obtain an embedding of a neighborhood of $M$ into $\mathbb{R}^{nm+m}$. After covering $M$ with finitely many regular coordinate balls or half-balls for $\tilde{M}$, the argument above produces an injective immersion $F : \bigcup B_i \to \mathbb{R}^{nm+m}$, which is an embedding because its domain is compact; the restriction of this map to the union of the sets $B_i$ is the desired embedding.” [This is needed in the ensuing argument for the noncompact case, because the sets $E_i$ might not be regular domains when $\partial M \neq \emptyset$.]

Page 134, displayed equations two-thirds of the way down the page: In the definition of $E_i$, there’s an “$i - i$” that should be “$i - 1$.” It should read $E_i = f^{-1}([b_{i-1}, a_{i+1}])$. 

(3/6/19)
Page 223, line 2: Change $\delta : M \to \mathbb{R}^+ \to \delta : \partial M \to \mathbb{R}^+$.

Page 228, proof of Theorem 9.26: There’s a gap in this proof, because it is not necessarily the case that $M(a)$ is a regular domain in $\text{Int} \, M$. To correct the problem, we have to choose our collar neighborhood more carefully. Replace the first sentence of the proof by the following paragraph:

“Theorem 9.25 shows that $\partial M$ has a collar neighborhood $C_0$ in $M$, which is the image of a smooth embedding $E_0 : [0,1] \times \partial M \to M$ satisfying $E_0(0,x) = x$ for all $x \in \partial M$. Let $f : M \to \mathbb{R}^+$ be a smooth positive exhaustion function. Note that $W = \{(t,x) : f(E_0(t,x)) > f(x) - 1\}$ is an open subset of $[0,1] \times \partial M$ containing $[0] \times \partial M$. Using a partition of unity as in the proof of Theorem 9.20, we may construct a smooth positive function $\delta : \partial M \to \mathbb{R}$ such that $(t,x) \in W$ whenever $0 \leq t < \delta(x)$. Define $E : [0,1] \times \partial M \to M$ by $E(t,x) = E_0(t\delta(x),x)$. Then $E$ is a diffeomorphism onto a collar neighborhood $C$ of $\partial M$, and by construction $f(E(t,x)) > f(x) - 1$ for all $(t,x) \in [0,1] \times \partial M$. We will show that for each $a \in (0,1)$, the set $E([0,a] \times \partial M)$ is closed in $M$. Suppose $p$ is a boundary point of $E([0,a] \times \partial M)$ in $M$; then there is a sequence $(t_i,x_i) \in [0,a] \times \partial M$ such that $E(t_i,x_i) \to p \in M$. Then $f(E(t_i,x_i))$ remains bounded, and thus $f(x_i) < f(E(t_i,x_i)) + 1$ also remains bounded. Since $\partial M$ is closed in $M$, $f|_{\partial M}$ is also an exhaustion function, and therefore the sequence $(x_i)$ lies in some compact subset of $\partial M$. Passing to a subsequence, we may assume $(t_i,x_i) \to (t_0,x_0)$, and therefore $p = E(t_0,x_0) \in E([0,a] \times \partial M)$.

Then at the end of the first paragraph of the proof, add the following sentences:

“To see that $M(a)$ is a regular domain, note first that it is closed in $M$ because it is the complement of the open set $C(a)$. Let $p \in M(a)$ be arbitrary. If $p \notin E([0,a] \times \partial M)$, then $p$ has a neighborhood in $\text{Int} \, M$ contained in $M(a)$ by the argument above. If $p \in E([0,a] \times \partial M)$, then $p = E(a,x)$ for some $x \in \partial M$, and $C$ is a neighborhood of $p$ in which $M(a) \cap C$ is the diffeomorphic image of $[a,1) \times \partial M$.”

Page 228, proof of Theorem 9.26, last line of the first paragraph: Change $0 \leq t < a$ to $0 \leq s < a$.

Page 225, Example 9.31: At the end of the example, insert the sentence “If $n \geq 2$, then $M_1 \# M_2$ is connected.”

Page 230, line 1 and first displayed equation: Change $\theta_j(x)$ to $\theta_j(u)$ (twice).

Page 230, second paragraph: “from Case” should be “from Case 1.”

Page 230, fourth paragraph, last line: Change $[X,Y]$ to $[V,W]$.

Page 231, Example 9.52: At the end of the example, add the sentence “Note that $u$ is smooth on the open set $\mathbb{R}^2 \setminus \{0\}$, which is a neighborhood of $S$.”

Page 246, Problem 9-11: Delete the second sentence of the hint. [Because $N$ is inward-pointing along $\partial M$, no integral curve that starts on $\partial M$ can hit the boundary again, because the vector field would have to be tangent to $\partial M$ or outward-pointing at the first such point.]
Page 248, first displayed equation: Should read
\[ V(t, p) = \frac{\partial}{\partial s} \bigg|_{s=t} H_s(H_t^{-1}(p)). \]

Page 248, Problem 9-22(c): Replace the problem statement by
\[ (c) \quad \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} = -y, \quad u(0, y) = 0. \]

[Without this sign change, the third claim in Problem 9-23 is not true.]

Page 254, paragraph beginning “With respect to,” third line:
Replace \( V_p \) with \( U_{\alpha} \).

Page 255, Example 10.8, line 5:
Replace the phrase “a bijective map \( \phi |_{\mathcal{S}} : (\pi|_{\mathcal{S}})^{-1}(U \cap \mathcal{S}) \to (U \cap \mathcal{S}) \times \mathbb{R}^k \)” with “a bijective map from \( (\pi|_{\mathcal{S}})^{-1}(U \cap \mathcal{S}) \) to \( (U \cap \mathcal{S}) \times \mathbb{R}^k \)” [The notation \( \phi |_{\mathcal{U}} \) is inappropriate here.]

Page 255, Example 10.8, lines 6–8:
Replace the sentence beginning with “If \( E \) is a smooth vector bundle” by the following: “If \( E \) is a smooth vector bundle and \( S \subseteq M \) is an embedded submanifold, it follows easily from the chart lemma that \( E|_{\mathcal{S}} \) is a smooth vector bundle. If \( S \) is merely immersed, we give \( E|_{\mathcal{S}} \) a topology and smooth structure making it into a smooth rank-\( k \) vector bundle over \( S \) as follows: For each \( p \in S \), choose a neighborhood \( U \) of \( p \) in \( M \) over which there is a local trivialization \( \phi \) of \( E \), and a neighborhood \( V \) of \( p \) in \( S \) that is embedded in \( M \) and contained in \( U \). Then the restriction of \( \phi \) to \( \pi^{-1}(V) \) is a bijection from \( \pi^{-1}(V) \) to \( V \times \mathbb{R}^k \), and we can apply the chart lemma to these bijections to yield the desired structure.”

Page 255, Example 10.8, last line:
Change “over \( M \)” to “over \( S \).”

Page 260, two lines above Proposition 10.22:
Change \( \tau^n(p) \) to \( \tau^k(p) \).

Page 261, statement of Proposition 10.25, first line:
Change \( \pi^*: E \to M' \) to \( \pi^*: E' \to M' \).

Page 263, first full paragraph:
In the first two lines of the paragraph, change \( \sigma_1, \sigma_2 \) to \( \tau_1, \tau_2 \) (twice).

Page 264, paragraph above the subheading, first sentence:
“homomorphism” should be “homomorphisms.”

Page 271, Problem 10-18(d):
Add the following: [Hint: For the “only if” direction, to show that \( F \) is compact, use a finite number of local trivializations to construct a closed set over which \( E \) is trivial.]

Page 276, proof of Proposition 11.9, first line:
Change “Theorem 10.4” to “Proposition 10.4.”

Page 278, Example 11.13, third line:
Change “every coordinate frame” to “every coordinate coframe.”

Page 296, line 6 from the bottom:
Change “closed forms” to “closed covector fields” (twice).

Page 301, Problem 11-10(c):
Change \( S^2 \) to \( S^2 \).

Page 301, Problem 11-13:
Add the assumption that \( n > 0 \).

Page 303, just below the commutative diagram:
Insert this sentence: “A natural transformation is called a \textit{natural isomorphism} if each map \( \lambda_X \) is an isomorphism in \( \mathcal{D} \).”

Page 303, Problem 11-18(b) and (c):
Change “natural transformation” to “natural isomorphism” in both parts.

Page 317, paragraph beginning “Any one”:
At the end of the paragraph, add this sentence: “If \( A \) and \( B \) are tensor fields, then \( A \otimes B \) denotes the tensor field defined by \( (A \otimes B)_p = A_p \otimes B_p \).”
(5/24/18) **Page 317, displayed equation just below the middle of the page:** Change $A_{j_1 \ldots j_k}^{i_1 \ldots i_l}$ to $A_{j_1 \ldots j_l}^{i_1 \ldots i_k}$ on the third line of the display, and again on the line below the display. [The last lower index should be $j_l$, not $i_l$.]

(4/18/17) **Page 320, statement of Proposition 12.25:** Change the domain and codomain of $G$: It should read $G: P \to M$.

(4/18/17) **Page 320, Proposition 12.25(e):** Should read $F \circ G$.

(4/17/15) **Page 333, first line:** Change $U_M$ to $V_M$.

(7/1/14) **Page 345, Problem 13-10:** In the last line of the problem statement, change $L x g z > L x g$ to $L x g z / L x g$, and delete the phrase “unless $z$ is a reparametrization of $x$.” [Because the definition of reparametrization that I’m using requires a diffeomorphism of the parameter domain, the original problem statement was not true.]

(12/18/12) **Page 355, proof of Lemma 14.10:** At the beginning of the proof, insert “Let $E_1, \ldots, E_n$ be the basis for $V$ dual to $(e^i)$.”

(12/18/12) **Page 356, Case 4, second line:** Should read “brings us back to Case 3.”

(7/3/15) **Page 368, second paragraph:** At the end of the first sentence of the paragraph, insert “(see pp. 341–343).”

(5/14/20) **Page 397, Problem 15-1:** At the end of the last sentence, add “when $n>1$.”

(5/28/22) **Page 397, Problem 15-4:** Change the first sentence to “Let $\theta$ be the flow of a smooth vector field on an oriented smooth manifold.” [The stated result is true also for manifolds with boundary and for nonmaximal flows, but to prove it, one must first do a little work to generalize some of the results of Theorem 9.12 to more general flows.]
(6/24/18) Page 419, proof of Theorem 16.25, first paragraph: Replace the second and third sentences of the paragraph by the following: “By means of smooth charts and a partition of unity, we may reduce the theorem to the cases in which \( M = \mathbb{R}^n \), \( M = \mathbb{H}^n \), or \( M = \mathbb{R}^n_+ \). The \( \mathbb{R}^n \) and \( \mathbb{H}^n \) cases are treated just as before.”

(7/22/15) Page 424, second displayed equation: Change \( \ell ^*_g \beta (X) \) to \( \ell ^*_M \beta (X) \).

(2/18/13) Page 426, three lines below the section heading: “can” should be “can.”

(2/11/15) Page 430, Proposition 16.38(c): This statement is wrong. Change it to “If \( F \) is smooth, then \( F^* \mu \) is a continuous density on \( M \); and if \( F \) is a local diffeomorphism, \( F^* \mu \) is smooth.”

(5/31/22) Page 435, Problem 16-4: Change “manifold with boundary” to “manifold with nonempty boundary.”

(7/27/16) Page 439, Problem 16-23: The formula for \( g \) should be
\[
g = \frac{dx^2 + dy^2}{(1-x^2-y^2)^2}.
\]

(2/19/13) Page 444, two lines below equation (17.4): Change \( T_{(q,s)}M \) to \( T_{(q,s)}(M \times \mathbb{R}) \).

(6/6/18) Page 447, Corollary 17.15: Change “every closed form is exact” to “every closed \( p \)-form is exact for \( p \geq 1 \).”

(5/15/15) Page 450, proof of Theorem 17.21, line 5: Change \( H^1_{\text{dr}}(S^n) \) to \( H^1_{\text{dr}}(S^1) \).

(8/14/17) Page 451, proof of Corollary 17.25, next-to-last line: Change \( \text{Id}_{H^1_{\text{dr}}(S)} \) to \( \text{Id}_{H^1_{\text{dr}}(S)} \).

(11/24/17) Pages 455–456, Proof of Theorem 17.32: The proof given in the book is incorrect, because the \( V_i \)’s might not be connected, so Theorem 17.30 does not apply to them. Here’s a correct proof.

**Lemma.** If \( M \) is a noncompact connected manifold, there is a countable, locally finite open cover \( \{V_i\}_{i=1}^\infty \) of \( M \) such that each \( V_i \) is connected and precompact, and for each \( j \), there exists \( k > j \) such that \( V_j \cap V_k \neq \emptyset \).

**Proof.** Let \( \{W_i\}_{i=1}^\infty \) be a countably infinite, locally finite cover of \( M \) by precompact, connected open sets (such a cover exists by Prop. 1.19 and Thm. 1.15). By successively deleting unneeded sets and renumbering, we can ensure that no \( W_j \) is contained in the union of the other \( W_i \)’s.

Let \( Y_1 = \bigcup_{i=2}^\infty W_i \). Because \( M \) is connected, each component of \( Y_1 \) meets \( W_1 \), and by local finiteness of \( \{W_j\} \), there are only finitely many such components. Such a component is precompact in \( M \) if and only if it is a union of finitely many \( W_i \)’s. Let \( V_1 \) be the union of \( W_1 \) together with all of the precompact components of \( Y_1 \), and let \( X_1 \) be the union of all \( W_i \)’s not contained in \( V_1 \). Then \( V_1 \) is connected and precompact, and \( X_1 \) has no precompact components. Proceeding by induction, suppose we have defined connected, precompact open sets \( V_1, \ldots, V_m \) whose union contains \( V_1 \cup \cdots \cup V_m \), and such that the union \( X_m \) of all the \( W_i \)’s not contained in \( V_1 \cup \cdots \cup V_m \) has no precompact components. Let \( j_m \) be the smallest index such that \( W_{j_m} \) is not contained in \( V_1 \cup \cdots \cup V_m \), and let \( Y_{m+1} \) be the union of all \( W_i \)’s other than \( W_{j_m} \). Then \( Y_{m+1} \) is connected and precompact, and \( X_{m+1} \) has no precompact components. Then by construction, for each \( j \), the set \( X_j = \bigcup_{i>j} V_i \) has no precompact components. If some \( V_j \) does not meet \( V_k \) for any \( k > j \), then \( V_j \) itself is a precompact component of \( X_{j-1} \), which is a contradiction. Thus for each \( j \), there is some \( k > j \) such that \( V_j \cap V_k \neq \emptyset \).

**Proof of Theorem 17.32.** Choose an orientation on \( M \). Let \( \{V_j\}_{j=1}^\infty \) be an open cover of \( M \) satisfying the conclusions of the preceding lemma. For each \( j \), let \( K(j) \) denote the least integer \( k > j \) such that \( V_j \cap V_k \neq \emptyset \), and let \( \theta_j \) be an \( n \)-form compactly supported in \( V_j \cap V_{K(j)} \) whose integral is 1. Let \( \{\psi_j\}_{j=1}^\infty \) be a smooth partition of unity subordinate to \( \{V_j\}_{j=1}^\infty \). Now suppose \( \omega \) is any \( n \)-form on \( M \), and let \( \omega_j = \psi_j \omega \) for each \( j \). Let \( c_1 = \int_{V_1} \omega_1 \), so that \( \omega_1 - c_1 \theta_1 \) is compactly supported in \( V_1 \) and has zero integral. It follows from Theorem 17.30 that there exists \( \eta_1 \in \Omega^{n-1}_c(V_1) \)
such that $d\eta_1 = \omega_1 - c_1 \theta_1$. Suppose by induction that we have found $\eta_1, \ldots, \eta_m$ and constants $c_1, \ldots, c_m$ such that for each $j = 1, \ldots, m$, $\eta_j \in \Omega^{n-1}_c(V_j)$ and

$$d\eta_j = \left( \omega_j + \sum_{i:K(i)=j} c_i \theta_i \right) - c_j \theta_j. \quad (*)$$

Let

$$c_{j+1} = \int_{V_{j+1}} \left( \omega_{j+1} + \sum_{i:K(i)=j+1} c_i \theta_i \right).$$

Then by Theorem 17.30, there exists $\eta_{j+1} \in \Omega^{n-1}_c(V_{j+1})$ satisfying the analog of $(*)$ with $j$ replaced by $j + 1$. Set $\eta = \sum_{j=1}^\infty \eta_j$, with each $\eta_j$ extended to be zero on $M \setminus V_j$. By local finiteness, this is a smooth $(n-1)$-form on $M$. It satisfies

$$d\eta = \omega + \sum_{j=1}^\infty \left( \sum_{i:K(i)=j} c_i \theta_i \right) - \sum_{j=1}^\infty c_j \theta_j.$$

Each term $c_i \theta_i$ appears exactly once in the first sum above, so the two sums cancel each other. \hfill \square

(7/27/16) **Page 457, line below the second displayed equation:** Change “Theorem 17.31” to “Theorem 17.30.”

(7/12/16) **Page 463, line above equation (17.15):** Insert missing space before “Similarly.”

(7/13/16) **Page 464, end of proof of Corollary 17.42:** Insert “Note that this construction produces a form $\sigma$ whose support is contained in $U \cap V$.” [This might be useful for solving Problem 18-6.]

(7/12/16) **Page 471, last paragraph:** Replace the sentence starting “The hardest part . . . ” with “The hardest part is showing that the singular chain complex of $M$ can be replaced by a chain complex built out of simplices whose images lie in either $U$ or $V$, without changing the homology.”

(9/12/17) **Page 487, Problem 18-1, first line:** Change “an oriented smooth manifold” to “a smooth manifold.”

(8/8/18) **Page 489, Problem 18-7(b):** Add to the hint: “In order to use Lemma 17.27, you’ll need to prove the following fact: Every bounded convex open subset of $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$. To prove this, let $U$ be such a subset, and without loss of generality assume $0 \in U$. First show that there exists a smooth nonnegative function $f \in C^\infty(U)$ such that $f(0) = 0$ and $f(x) \geq 1/d(x)$ away from a small neighborhood of 0, where $d(x)$ is the distance from $x$ to $\partial U$. Next, show that $g(x) = 1 + \int_0^1 f(tx) \, dt$ is a smooth positive exhaustion function on $U$ that is nondecreasing along each ray starting at 0. Finally, show that the map $F : U \to \mathbb{R}^n$ given by $F(x) = g(x)x$ is a bijective local diffeomorphism. Also, you may use the fact that the conclusion of the five lemma is still true even if the appropriate diagram commutes only up to sign.”

(1/15/13) **Page 491, Example 19.1(c):** Delete the word “unit.”

(5/22/15) **Page 492, line above Proposition 19.2:** Change “lie” to “Lie.”

(12/17/15) **Page 492, proof of Proposition 19.2, fourth line:** Change “Given $p \in M$” to “Given $p \in U$.”

(9/12/16) **Page 506, Lemma 19.24, last line:** Before “left-invariant,” insert “smooth.”

(6/1/20) **Page 512, Problem 19-4:** In the first line of the problem, change “all three coordinates are positive” to “$z$ is positive.” Then replace the last sentence by “Find an explicit global chart on $U$ in which $D$ is spanned by the first two coordinate vector fields.” [Technically it might not be a flat chart because its image need not be a cube in $\mathbb{R}^3$.]

(10/4/17) **Page 518, sentence before Prop. 20.3:** Change “one-parameter subgroups of $\text{GL}(n, \mathbb{R})$” to “one-parameter subgroups of subgroups of $\text{GL}(n, \mathbb{R})$.”
(5/23/16) Page 521, first displayed equation: Change $d\Phi_0$ to $d\Phi_e$ (twice).

(6/9/19) Page 528, line 9: Change two instances of $(g, p)$ in subscripts to $(g, q)$.

(5/19/18) Page 528, just below the displayed equation in the middle of the page: The smoothness of the map $\sigma_q$ is not quite immediate from the definition. Replace the three sentences beginning “It follows” with this: “Because $S_p$ is a weakly embedded submanifold by Theorem 19.17, to show that $\sigma_q$ is a smooth local section of $S_p$, it suffices to show that it is smooth into $G \times M$ and takes its values in $S_p$. The first component function is smooth as a map into $G$ by smoothness of group multiplication. To show that the second component is smooth into $M$ as a function of $\dot{X}$ (and therefore of $\exp X$), you need to use the argument sketched out just below equation (20.10): as in the proof of Prop. 20.8, apply the fundamental theorem on flows to the vector field $\mathcal{Z}_{(p, X)} = (\dot{X}_g, 0)$ on $M \times g$. A straightforward computation shows that $\gamma(t) = (g \exp t X, \eta(\dot{X})(t, q))$ is an integral curve of $\dot{X}$ starting at $(g, q)$, from which it follows easily that $\sigma_q(g \exp X) = \gamma(1) \in S_p$.”

(1/10/17) Page 537, Problem 20-6(a): Change $B \in \mathfrak{sl}(n, \mathbb{R})$ to $B \in \mathfrak{sl}(n, \mathbb{R})$.

(5/31/16) Page 538, Problem 20-11(b): Here’s a better hint, which doesn’t require proving part (a) first: “[Hint: Consider the graph of $F$ as a subgroup of $G \times H$.]”

(10/18/17) Page 542, middle of the paragraph before Example 21.3: Change “the action of $\mathbb{R}^k$ on $\mathbb{R}^n$” to “the action of $\mathbb{R}^k$ on $\mathbb{R}^k \times \mathbb{R}^n$."

(2/25/18) Page 548, last two lines: Allen Hatcher’s name is misspelled. (Sorry, Allen.)

(5/23/16) Page 549, proof of Proposition 21.12, last sentence: Change the first phrase of that sentence to “Second, if $p, p' \in E$ are in different orbits and $\pi(p) \neq \pi(p')$, . . . .” Then add the following sentences at the end of the proof: “If $p$ and $p'$ are in different orbits and $\pi(p) = \pi(p')$, let $W$ be an evenly covered neighborhood of $\pi(p)$, and let $V, V'$ be the components of $\pi^{-1}(W)$ containing $p$ and $p'$, respectively. For any $g \in \text{Aut}_q(E)$, a simple connectedness argument shows that $g \cdot V$ is a component of $\pi^{-1}(W)$; if it had nontrivial intersection with $V$ it would have to be equal to $V$, which would imply $g \cdot p = p'$, a contradiction.”

(7/26/16) Page 567, two lines above Proposition 22.8: Insert “a” before “2-covector.”

(10/9/15) Page 568, Example 22.9(a), first line: The coordinates should be $(x^1, \ldots, x^n, y^1, \ldots, y^n)$. (The last coordinate is $y^n$, not $x^n$.)

(11/17/21) Page 571, line below equation (22.5): Delete the spurious word “theorem” at the end of the line.

(3/27/19) Page 572, middle of the page: Replace the sentence starting “On the other hand” by this: “On the other hand, the left-hand side is just the ordinary $t$-derivative of a time-dependent tensor on a fixed vector space, and expanding in terms of a basis shows that it satisfies a similar product rule:”

(10/5/17) Page 573, statement of Proposition 22.15, second line: Change “$V : J \times M$” to “$V : J \times M \to TM$”; and change $\psi$ to $\theta$.

(11/18/17) Page 583, line 4: Change $\mathbb{R}^{2n+1} \sim \{0\}$ to $\mathbb{R}^{2n+2} \sim \{0\}$.

(7/26/16) Page 583, third displayed equation: Should read

$$T \cdot d\Theta = -2 \sum_{i=1}^{n+1} (x^i \, dx^i + y^i \, dy^i) = -d(|x|^2 + |y|^2).$$

(7/26/16) Page 583, two lines below the third displayed equation: The formula for $d\Theta(N, T)$ should be $d\Theta(N, T) = 2(|x|^2 + |y|^2)$. 


Page 584, Exercise 22.29: Part (b) should read

\[(b) \quad T = \frac{\partial}{\partial z};\]

Page 584, paragraph above Theorem 22.33: Change all occurrences of \( \theta \) in this paragraph to \( \psi \), to avoid confusion with the use of \( \theta \) for a contact form elsewhere in this section.

Page 585, statement of Theorem 22.34, last line: Change \( H \) to \( F \).

Page 587, equation (22.27): Change both occurrences of \( s \) to \( x \).

Page 591, Problem 22-5: Add the hypothesis \( n > 0 \).

Page 592, Problem 22-15: Add the hypothesis that \( M \) is connected.

Page 608, Proposition A.41(a): Insert the following phrase at the beginning of this statement: With the exception of the word “closed” in part (d).

Page 616, Proposition A.77(b), last line: Change \( z \) to \( e^{z} \).

Page 653, proof of Lemma B.2, fourth line: Replace “By Exercise B.1(b)” with “If \( w_1 \) is equal to one of the \( v_i \)'s, then the ordered \( (n+1) \)-tuple \( (w_1, v_1, \ldots, v_n) \) is linearly dependent; if not, then by Exercise B.1(b), . . . .”

Page 658, two lines above (C.15): Change \( F \) to \( F_0 \).

Page 660, display (C.20): Change \( F^{-1}(x) \) to \( F^{-1}(y) \).

Page 664, statement of Theorem D.1(b): After the phrase “Any two differentiable solutions to (D.3)–(D.4),” insert “defined on intervals containing \( t_0 \).”

Page 666, just below the fifth display: After the sentence ending “by our choice of \( \varepsilon \) and \( \delta \),” insert “(If \( t < t_0 \), interchange \( t \) and \( t_0 \) in the second line above.)”

Page 664, statement of Theorem D.4: After the phrase “any two differentiable solutions to (D.3)–(D.4),” insert “defined on intervals containing \( t_0 \).”

Page 668, paragraph below equation (D.10): In the fourth line of the paragraph, change \( \bar{W} \) to \( W \).

Page 670, displayed inequality between (D.17) and (D.18): Change \( n \) to \( n^2 \).

Page 670, last line: Change \( n \) to \( n^2 \) in the definition of \( B \).

Page 671, inequality (D.19): Change \( n \) to \( n^2 \) (twice).

Page 671, just below (D.19): Replace the sentence “Since the expression on the right can be made as small as desired by choosing \( h \) and \( \tilde{h} \) sufficiently small, this shows . . . .” by the following: “Thus the expression on the left can be made as small as desired by choosing \( h \) and \( \tilde{h} \) sufficiently small. This shows . . . .”

Page 692: Under the entry for “Form,” delete the references to page 294 for “closed” and page 292 for “exact.”

Page 693: The index entry for “Hatcher, Allen” is misspelled.