

CORRECTIONS TO Introduction to Smooth Manifolds (Second Edition)

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MAY 5, 2022

- (8/8/16) **Page 6, just below the last displayed equation:** Change $\varphi([x])$ to $\varphi_{n+1}[x]$, and in the next line, change x^i to x^{n+1} . After “(Fig. 1.4),” insert “with similar interpretations for the other charts.”
- (8/8/16) **Page 7, Fig. 1.4:** Both occurrences of x^i should be x^{n+1} .
- (12/19/18) **Page 9, proof of Theorem 1.15:** In the second line of the proof, replace “For each j ” with “For each $j \geq 0$.” Then in the fourth-to-last line, replace “positive integers” by “nonnegative integers.”
- (1/15/21) **Page 13, line 1:** Delete the words “and injective.”
- (1/18/21) **Page 20, Example 1.31:** There are multiple errors in this example. Replace everything after the first two sentences by the following: For each $i = 1, \dots, n + 1$, let $(U_i^\pm \cap S^n, \varphi_i^\pm)$ denote the graph coordinate charts we constructed in Example 1.4. For any distinct indices i and j and any choices of \pm signs, the transition maps $\varphi_i^\pm \circ (\varphi_j^\pm)^{-1}$ and $\varphi_i^\pm \circ (\varphi_j^\mp)^{-1}$ are easily computed. For example, in the case $i < j$, we get the following formula for all u in the domain of $\varphi_i^+ \circ (\varphi_j^+)^{-1}$:
- $$\varphi_i^+ \circ (\varphi_j^+)^{-1}(u^1, \dots, u^n) = (u^1, \dots, \widehat{u^i}, \dots, \sqrt{1 - |u|^2}, \dots, u^n)$$
- (with u^i omitted and the square root replacing u^j), and similar formulas hold in the other cases. When $i = j$, the domains of φ_i^+ and φ_i^- are disjoint, so there is nothing to check. Thus, the collection of charts $\{(U_i^\pm \cap S^n, \varphi_i^\pm)\}$ is a smooth atlas, and so defines a smooth structure on S^n . We call this its *standard smooth structure*. //
- (6/23/13) **Page 23, two lines below the first displayed equation:** Change “any subspace $S \subseteq V$ ” to “any k -dimensional subspace $S \subseteq V$.”
- (9/15/19) **Page 24, first full paragraph, fourth line:** Change “any subspace S ” to “any k -dimensional subspace S .”
- (12/19/18) **Page 26, first line:** Change $U \cap \varphi^{-1}(\text{Int } \mathbb{H}^n)$ to $\varphi^{-1}(\text{Int } \mathbb{H}^n)$.
- (12/19/18) **Page 27, last paragraph, sixth line:** Change $\tilde{U} \cap \mathbb{H}^n$ to $\tilde{U} \cap U$.
- (2/22/15) **Page 29, proof of Theorem 1.46, second paragraph, line 4:** Change $\varphi(U \cap V)$ to $\psi(U \cap V)$.
- (10/8/15) **Page 30, Problem 1-6:** Interpret the formula for F_s to mean $F_s(0) = 0$ when $s \leq 1$.
- (1/27/18) **Page 31, Fig. 1.13:** Change $\{x^n = 0\}$ to $\{x^{n+1} = 0\}$.
- (3/12/18) **Page 31, Problem 1-11, next-to-last line:** Change S^n to $S^n \setminus \{N\}$.
- (4/25/17) **Page 45, second paragraph:** Replace the last sentence of that paragraph with the following: “If N has empty boundary, we say that a map $F : A \rightarrow N$ is *smooth on A* if it has a smooth extension in a neighborhood of each point: that is, if for every $p \in A$ there exist an open subset $W \subseteq M$ containing p and a smooth map $\tilde{F} : W \rightarrow N$ whose restriction to $W \cap A$ agrees with F . When $\partial N \neq \emptyset$, we say $F : A \rightarrow N$ is smooth on A if for every $p \in A$ there exist an open subset $W \subseteq M$ containing p and a smooth chart (V, ψ) for N whose domain contains $F(p)$, such that $F(W \cap A) \subseteq V$ and $\psi \circ F|_{W \cap A}$ is smooth as a map into \mathbb{R}^n in the sense defined above (i.e., it has a smooth extension in a neighborhood of each point).”
- (7/23/14) **Page 45, last displayed equation:** The first $=$ sign should be \subseteq .
- (9/15/19) **Page 46, line 9:** Change “on an open subset” to “on a nonempty open subset.”

- (6/20/18) **Page 47, proof of Theorem 2.29, second paragraph:** Replace the first sentence of the paragraph by “Let $h: \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth bump function that is positive in $B_1(0)$ and zero elsewhere.”
- (2/13/22) **Page 49, Problem 2-10(c):** Change “an isomorphism” to “a bijection.”
- (1/20/22) **Page 54, just after the first sentence:** Insert “(The integral is a smooth function of x by iterative application of Theorem C.14.)”
- (11/17/12) **Page 56, first displayed equation:** Change $d\iota(v)_p$ to $d\iota_p(v)$.
- (1/21/21) **Page 56, just below the last displayed equation:** Replace “the last two equalities follow” by “the last equality follows.”
- (6/9/19) **Page 58, proof of Lemma 3.11, next-to-last line:** Change \mathbb{H}^n to $\text{Int } \mathbb{H}^n$.
- (1/26/15) **Page 68, proof of Proposition 3.21:** Insert the following sentence at the beginning of the proof: “Let $n = \dim M$ and $m = \dim N$.” Then in the second sentence, change (3.9) to (3.10). Finally, in the displayed equation, change F^n to F^m (twice).
- (11/17/12) **Page 70, two lines above Corollary 3.25:** Change “Proposition 3.23” to “Proposition 3.24.”
- (3/5/15) **Page 76, Problem 3-8:** Add the following remark: “(For $p \in \partial M$, we need to allow curves with domain $[0, \varepsilon)$ or $(-\varepsilon, 0]$ and to interpret the derivatives as one-sided derivatives.)”
- (10/23/18) **Page 78, proof of Prop. 4.1, third and fourth lines:** Change $m \times n$ to $n \times m$ (twice).
- (11/9/16) **Page 79, proof of Theorem 4.5, fourth line:** Change $\hat{F}(p)$ to $\hat{F}(0)$.
- (12/12/21) **Page 82, line 4 from the bottom:** Change “This is a diffeomorphism onto its image” to “This is an open map and a diffeomorphism onto its image.”
- (12/12/21) **Page 83, proof of Theorem 4.14, line 8:** Change “no open subset” to “no nonempty open subset.”
- (5/4/13) **Page 96, Problem 4-3:** This problem probably needs a better hint. First, to get a good result, you’ll have to add the assumption that $\ker dF_p \not\subseteq T_p \partial M$. After choosing smooth coordinates, you can assume $M \subseteq \mathbb{H}^m$ and $N \subseteq \mathbb{R}^n$, and extend F to a smooth function \tilde{F} on an open subset of \mathbb{R}^m . If $\text{rank } F = r$, show that there is a coordinate projection $\pi: \mathbb{R}^n \rightarrow \mathbb{R}^r$ such that $\pi \circ \tilde{F}$ is a submersion, and apply the rank theorem to $\pi \circ \tilde{F}$ to find new coordinates in which \tilde{F} has a coordinate representation of the form $\tilde{F}(x, y) = (x, R(x, y))$. Then use the rank condition to show that $R|_M$ is independent of y .
- (12/22/21) **Page 100, first sentence:** At the end of the sentence, change “smooth embeddings” to “smooth embeddings of smooth manifolds.”
- (9/8/15) **Page 100, proof of Proposition 5.4, next-to-last line:** Change “It a homeomorphism” to “It is a homeomorphism.”
- (7/8/19) **Page 104, line below the proof of Theorem 5.11:** Change “See Theorem 5.31” to “See Problem 5-24.” [Problem 5-24 is a new problem, described later in this list. Theorem 5.31 is not appropriate in this situation because it applies only to manifolds without boundary.]
- (6/9/19) **Page 105, line 4 from the bottom:** Change F to Φ .
- (11/9/16) **Page 112, Fig. 5.10:** Interchange the labels M and N on the figure, to be consistent with the notation in Theorem 5.29.

- (5/5/22) **Page 113, line 6:** Change the definition of $\tilde{\psi}$ to $\tilde{\psi} = \pi \circ \psi|_{V_0}$. After the end of that sentence, insert the following: “To see that $\tilde{\psi}$ is a smooth coordinate map, let $i: V \hookrightarrow M$ be the inclusion map. Note first that for each $q \in V_0$, x^{k+1}, \dots, x^n are all constant on the image of i , so the image of di_q is contained in the span of $\partial/\partial x^1, \dots, \partial/\partial x^k$. Since di_q is injective and its image has trivial intersection with $\text{Ker } d\tilde{\psi}_q$, it follows that $d\tilde{\psi}_q \circ di_q$ is injective, so for dimensional reasons it is an isomorphism. Thus $\tilde{\psi} \circ i$ is a local diffeomorphism by the inverse function theorem. Since it is bijective from V_0 to its image, it is a diffeomorphism and hence a smooth coordinate map for V .”
- (9/15/19) **Page 118, Fig. 5.13:** Change N to v .
- (7/15/15) **Page 120, proof of Proposition 5.46:** At the beginning of the proof, insert this sentence: “Let $F: D \hookrightarrow M$ denote the inclusion map.”
- (7/21/18) **Page 121, line 5:** Change x^m to x^n .
- (9/15/19) **Page 123, Problem 5-6:** Add the assumption that $m > 0$.
- (7/8/19) **Page 124:** At the end of the page, add a new problem:
5-24. Suppose M is a smooth manifold with boundary, N is a smooth manifold, and $F: N \rightarrow M$ is a smooth map whose image is contained in ∂M . Show that F is smooth as a map into ∂M , and use this to prove that ∂M has a unique smooth structure making it an embedded submanifold of M .
- (12/19/18) **Page 129, proof of Sard’s theorem, second paragraph:** Just before the last sentence of the paragraph, insert the following: “In the \mathbb{H}^n case, extend F to a smooth map on an open subset of \mathbb{R}^m , and replace U by that open subset; if we can show that the set of critical values of the extended map has measure zero, then the same is true of the set of critical values of F .”
- (3/16/19) **Page 129, displayed equation near the bottom of the page:** Change “ i th partial derivatives” to “ i th-order partial derivatives.”
- (12/26/18) **Page 130, just below equation (6.2):** Right after the displayed equation, insert “(where the component functions F^2, \dots, F^n might be different from the ones in the original coordinate chart).”
- (3/28/20) **Page 131, two lines below the first displayed equation:** Change $A'(R/K)^{k+1}$ to $A'(R\sqrt{m}/K)^{k+1}$.
- (1/8/18) **Page 131, three lines below the first displayed equation:** Insert “at most” before “ K^m balls.”
- (3/28/20) **Page 131, second displayed equation:** Change the left-hand side to $K^m(A')^n(R\sqrt{m}/K)^{n(k+1)}$, and in the next line, change the definition of A'' to $A'' = (A')^n(R\sqrt{m})^{n(k+1)}$.
- (4/17/13) **Page 132, proof of Lemma 6.13, second paragraph:** This argument does not apply when $\partial M \neq \emptyset$, because in that case $M \times M$ is not a smooth manifold with boundary. Instead, we can consider the restrictions of κ to $(M \times \text{Int } M) \setminus \Delta_M$ and to $(M \times \partial M) \setminus \Delta_M$ (both of which are smooth manifolds with boundary), and note that there is a point $[v] \in \mathbb{R}\mathbb{P}^{N-1}$ that is not in the image of τ or either of these restrictions of κ . [Thanks to David Iglesias Ponte for suggesting this correction.]
- (10/25/21) **Page 134, proof of Theorem 6.15, just after the fourth paragraph of the proof:** Insert the following: “In case M is an arbitrary compact subset of a larger manifold \tilde{M} with or without boundary, we can adapt this argument to obtain an embedding of a neighborhood of M into \mathbb{R}^{nm+m} . After covering M with finitely many regular coordinate balls or half-balls for \tilde{M} , the argument above produces an injective immersion $F: \bigcup_i \bar{B}_i \rightarrow \mathbb{R}^{nm+m}$, which is an embedding because its domain is compact; the restriction of this map to the union of the sets B_i is the desired embedding.” [This is needed in the ensuing argument for the noncompact case, because the sets E_i might not be regular domains when $\partial M \neq \emptyset$.]
- (7/3/15) **Page 134, displayed equations two-thirds of the way down the page:** In the definition of E_i , there’s an “ $i - i$ ” that should be “ $i - 1$.” It should read $E_i = f^{-1}([b_{i-1}, a_{i+1}])$.

- (10/24/21) **Page 134, just below the displayed equations two-thirds of the way down the page:** Delete the sentence “By Proposition 5.47, each E_i is a compact regular domain.” Two lines below that, replace “smooth embedding of E_i ” with “smooth embedding of a neighborhood of E_i .”
- (7/2/18) **Page 137, first paragraph under the subheading “Tubular Neighborhoods,” fifth line:** Change R^n to \mathbb{R}^n .
- (7/27/18) **Page 138, proof of Theorem 6.23, end of the first paragraph:** Change “standard coordinate frame” to “standard coordinate basis.”
- (11/25/12) **Page 145, statement of Corollary 6.33:** After “immersed submanifold,” insert “with $\dim S = \dim M$.”
- (12/5/16) **Page 145, paragraph above Prop. 6.34:** In the definition of *smooth family of maps*, replace “ $F : M \times S \rightarrow N$ ” by “ $F : N \times S \rightarrow M$.”
- (9/28/19) **Page 146, equation (6.9):** Should read $dF(T_{(p,s)}W) \subseteq T_q X$. [Change the equal sign to subset.]
- (9/28/19) **Page 146, line below the last displayed equation:** Change “ $= T_q X$ ” to “ $\subseteq T_q X$.”
- (11/25/12) **Page 148, Problem 6-13:** Delete part (c). [This statement is simply wrong. It is true with the added hypothesis that F' is an embedding, but then it’s essentially just a restatement of part (b).]
- (2/10/18) **Page 150, last line:** Change “Theorem 20.16” to “Theorem 20.22.”
- (12/30/17) **Page 160, first line:** Change $R_{hh^{-1}}$ to $R_{h^{-1}h}$.
- (2/16/18) **Page 164, just above the subheading:** Replace the last line of the proof of Prop. 7.23 by “The action is smooth because each φ can be written locally as a composition of a smooth local section followed by π .”
- (8/26/14) **Page 169, first line:** Change \tilde{G} to G .
- (6/21/20) **Page 169, statement of Theorem 7.35:** Replace the phrase “closed Lie subgroups such that N is normal” by “Lie subgroups such that N is normal and closed.” [In fact, using the result of Theorem 19.25 later in the book, the hypothesis that N is closed can also be omitted.]
- (3/18/19) **Page 171, third line from the end of the proof:** Change E_i to E_j , so the formula reads $\rho_j^i(g) = \pi^i(g \cdot E_j)$.
- (9/17/14) **Page 173, Problem 7-21:** Replace the first sentence by “Prove that the groups in Problem 7-20 are isomorphic to direct products of the indicated groups in cases (a) and (c) if and only if n is odd, and in cases (b) and (d) if and only if $n = 1$.”
- (1/18/21) **Page 178, Example 8.10(d):** Change “Example 8.4” to “Example 8.5.”
- (6/9/19) **Page 184, Example 8.20, next-to-last line:** Change $p = (u, v)$ to $q = (u, v)$.
- (3/19/21) **Page 184, proof of Proposition 8.22:** After “Proposition 5.37,” insert “in the case $\partial S = \emptyset$. When S has nonempty boundary, the proof of Proposition 5.37 still goes through using boundary slice coordinates for S .”
- (11/17/12) **Page 196, proof of Proposition 8.45, next-to-last line:** Should read “ $F_* \circ (F^{-1})_* = (F \circ F^{-1})_* = \text{Id}_{\text{Lie}(H)}$ and $(F^{-1})_* \circ F_* = \text{Id}_{\text{Lie}(G)}$.”
- (5/27/17) **Page 208, first line:** Change to “This is just the existence and smoothness statements of Theorem D.1”
- (3/10/16) **Page 213, first sentence of the last paragraph:** The definition of t_0 should be $t_0 = \sup\{t \in \mathbb{R} : (t, p_0) \in W\}$.
- (5/24/19) **Page 214, Fig. 9.6:** The shaded area should be labeled W , not \mathcal{D} .
- (12/2/15) **Page 217, Fig. 9.7:** Both occurrences of φ should be Φ .
- (12/2/15) **Page 219, second displayed equation:** Change “ $V^j(0, p) = 0$ ” to “ $\Phi^j(0, p) = 0$.”

- (12/2/15) **Page 219, two lines below (9.12):** Here and in the rest of the paragraph, change p_0 to p_1 (seven times) to avoid confusion with the prior unrelated use of p_0 in this proof.
- (5/29/16) **Page 222, just below the section heading:** Insert the following sentence: “On a manifold with boundary, the definitions of *flow domain*, *flow*, and *infinitesimal generator of a flow* are exactly the same as on a manifold without boundary.”
- (2/15/19) **Page 223, line 2:** Change $\delta: M \rightarrow \mathbb{R}^+$ to $\delta: \partial M \rightarrow \mathbb{R}^+$.
- (8/19/14) **Page 223, proof of Theorem 9.26:** There’s a gap in this proof, because it is not necessarily the case that $M(a)$ is a regular domain in $\text{Int } M$. To correct the problem, we have to choose our collar neighborhood more carefully. Replace the first sentence of the proof by the following paragraph:
 “Theorem 9.25 shows that ∂M has a collar neighborhood C_0 in M , which is the image of a smooth embedding $E_0: [0, 1) \times \partial M \rightarrow M$ satisfying $E_0(0, x) = x$ for all $x \in \partial M$. Let $f: M \rightarrow \mathbb{R}^+$ be a smooth positive exhaustion function. Note that $W = \{(t, x) : f(E_0(t, x)) > f(x) - 1\}$ is an open subset of $[0, 1) \times \partial M$ containing $\{0\} \times \partial M$. Using a partition of unity as in the proof of Theorem 9.20, we may construct a smooth positive function $\delta: \partial M \rightarrow \mathbb{R}$ such that $(t, x) \in W$ whenever $0 \leq t < \delta(x)$. Define $E: [0, 1) \times \partial M \rightarrow M$ by $E(t, x) = E_0(t\delta(x), x)$. Then E is a diffeomorphism onto a collar neighborhood C of ∂M , and by construction $f(E(t, x)) > f(x) - 1$ for all $(t, x) \in [0, 1) \times \partial M$. We will show that for each $a \in (0, 1)$, the set $E([0, a] \times \partial M)$ is closed in M . Suppose p is a boundary point of $E([0, a] \times \partial M)$ in M ; then there is a sequence $\{(t_i, x_i)\}$ in $[0, a] \times \partial M$ such that $E(t_i, x_i) \rightarrow p \in M$. Then $f(E(t_i, x_i))$ remains bounded, and thus $f(x_i) < f(E(t_i, x_i)) + 1$ also remains bounded. Since ∂M is closed in M , $f|_{\partial M}$ is also an exhaustion function, and therefore the sequence $\{x_i\}$ lies in some compact subset of ∂M . Passing to a subsequence, we may assume $(t_i, x_i) \rightarrow (t_0, x_0)$, and therefore $p = E(t_0, x_0) \in E([0, a] \times \partial M)$.”
 Then at the end of the first paragraph of the proof, add the following sentences:
 “To see that $M(a)$ is a regular domain, note first that it is closed in M because it is the complement of the open set $C(a)$. Let $p \in M(a)$ be arbitrary. If $p \notin E([0, a] \times \partial M)$, then p has a neighborhood in $\text{Int } M$ contained in $M(a)$ by the argument above. If $p \in E([0, a] \times \partial M)$, then $p = E(a, x)$ for some $x \in \partial M$, and C is a neighborhood of p in which $M(a) \cap C$ is the diffeomorphic image of $[a, 1) \times \partial M$.”
- (1/30/14) **Page 223, proof of Theorem 9.26, last line of the first paragraph:** Change $0 \leq t < a$ to $0 \leq s < a$.
- (1/30/14) **Page 225, Example 9.31:** At the end of the example, insert the sentence “If $n \geq 2$, then $M_1 \# M_2$ is connected.”
- (7/25/16) **Page 226, Example 9.32, fifth line:** Replace the sentence beginning “It is a smooth manifold without boundary . . .” by “It is a topological manifold without boundary, and can be given a smooth structure such that each of the natural maps $M \rightarrow D(M)$ (induced by inclusion into the left and right summands of the disjoint union) is a smooth embedding.”
- (3/2/21) **Page 230, line 1 and first displayed equation:** Change $\theta_t(x)$ to $\theta_t(u)$ (twice).
- (4/23/13) **Page 230, second paragraph:** “from Case” should be “from Case 1.”
- (2/26/18) **Page 230, fourth paragraph, last line:** Change $[X, Y]$ to $[V, W]$.
- (9/8/18) **Page 234, proof of Theorem 9.46, second paragraph:** Replace the two parenthesized sentences by the following: “(To see this, just choose $\varepsilon_1 > 0$ and $U_1 \subseteq U$ such that θ_1 maps $(-\varepsilon_1, \varepsilon_1) \times U_1$ into U , and then inductively choose ε_i and U_i such that θ_i maps $(-\varepsilon_i, \varepsilon_i) \times U_i$ into U_{i-1} . Taking $\varepsilon = \min\{\varepsilon_i\}$ and $Y = U_k$ does the trick.)”
- (5/29/16) **Page 241, Example 9.52:** At the end of the example, add the sentence “Note that u is smooth on the open set $\mathbb{R}^2 \setminus \{0\}$, which is a neighborhood of S .”
- (6/4/14) **Page 246, Problem 9-11:** Delete the second sentence of the hint. [Because N is inward-pointing along ∂M , no integral curve that starts on ∂M can hit the boundary again, because the vector field would have to be tangent to ∂M or outward-pointing at the first such point.]

(11/17/21) **Page 248, first displayed equation:** Should read

$$V(t, p) = \left. \frac{\partial}{\partial s} \right|_{s=t} H_s(H_t^{-1}(p)).$$

(11/12/16) **Page 248, Problem 9-22(c):** Replace the problem statement by

$$(c) \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} = -y, \quad u(0, y) = 0.$$

[Without this sign change, the third claim in Problem 9-23 is not true.]

(11/16/20) **Page 254, paragraph beginning “With respect to,” third line:** Replace $V_p \times \mathbb{R}^k$ with $U_\alpha \times \mathbb{R}^k$.

(11/4/21) **Page 255, Example 10.8, line 5:** Replace the phrase “a bijective map $\Phi|_U : (\pi|_S)^{-1}(U \cap S) \rightarrow (U \cap S) \times \mathbb{R}^k$ ” with “a bijective map from $(\pi|_S)^{-1}(U \cap S)$ to $(U \cap S) \times \mathbb{R}^k$.” [The notation $\Phi|_U$ is inappropriate here.]

(6/17/19) **Page 255, Example 10.8, lines 6–8:** Replace the sentence beginning with “If E is a smooth vector bundle” by the following: “If E is a smooth vector bundle and $S \subseteq M$ is an embedded submanifold, it follows easily from the chart lemma that $E|_S$ is a smooth vector bundle. If S is merely immersed, we give $E|_S$ a topology and smooth structure making it into a smooth rank- k vector bundle over S as follows: For each $p \in S$, choose a neighborhood U of p in M over which there is a local trivialization Φ of E , and a neighborhood V of p in S that is embedded in M and contained in U . Then the restriction of Φ to $\pi^{-1}(V)$ is a bijection from $\pi^{-1}(V)$ to $V \times \mathbb{R}^k$, and we can apply the chart lemma to these bijections to yield the desired structure.”

(3/30/21) **Page 255, Example 10.8, last line:** Change “over M ” to “over S .”

(11/27/20) **Page 260, two lines above Proposition 10.22:** Change $\tau^n(p)$ to $\tau^k(p)$.

(10/22/18) **Page 261, statement of Proposition 10.25, first line:** Change $\pi' : E \rightarrow M'$ to $\pi' : E' \rightarrow M'$.

(4/2/21) **Page 263, first full paragraph:** In the first two lines of the paragraph, change σ_1, σ_2 to τ_1, τ_2 (twice).

(7/2/14) **Page 264, paragraph above the subheading, first sentence:** “homomorphism” should be “homomorphisms.”

(4/2/21) **Page 271, Problem 10-18:** Change “a properly embedded” to “an embedded.”

(2/6/21) **Page 271, Problem 10-19(d):** Add the following: [Hint: For the “only if” direction, to show that F is compact, use a finite number of local trivializations to construct a closed set over which E is trivial.]

(2/6/22) **Page 276, proof of Proposition 11.9, first line:** Change “Theorem 10.4” to “Proposition 10.4.”

(6/29/15) **Page 278, Example 11.13, third line:** Change “every coordinate frame” to “every coordinate coframe.”

(6/11/19) **Page 296, line 6 from the bottom:** Change “closed forms” to “closed covector fields” (twice).

(4/18/20) **Page 301, Problem 11-10(c):** Change S^2 to \mathbb{S}^2 .

(4/20/20) **Page 301, Problem 11-13:** Add the assumption that $n > 0$.

(5/19/18) **Page 303, just below the commutative diagram:** Insert this sentence: “A natural transformation is called a *natural isomorphism* if each map λ_X is an isomorphism in \mathcal{D} .”

(5/19/18) **Page 303, Problem 11-18(b) and (c):** Change “natural transformation” to “natural isomorphism” in both parts.

(4/7/21) **Page 317, paragraph beginning “Any one”:** At the end of the paragraph, add this sentence: “If A and B are tensor fields, then $A \otimes B$ denotes the tensor field defined by $(A \otimes B)_p = A_p \otimes B_p$.”

- (5/24/18) **Page 317, displayed equation just below the middle of the page:** Change $A_{j_1 \dots j_l}^{i_1 \dots i_k}$ to $A_{j_1 \dots j_l}^{i_1 \dots i_k}$ on the third line of the display, and again on the line below the display. [The last lower index should be j_l , not i_l .]
- (4/18/17) **Page 320, statement of Proposition 12.25:** Change the domain and codomain of G : It should read $G: P \rightarrow M$.
- (4/18/17) **Page 320, Proposition 12.25(e):** Should read $(F \circ G)^* B = G^*(F^* B)$.
- (4/17/15) **Page 333, first line:** Change $U \subseteq M$ to $V \subseteq M$.
- (7/1/14) **Page 345, Problem 13-10:** In the last line of the problem statement, change $L_{\bar{g}}(\tilde{\gamma}) > L_{\bar{g}}(\gamma)$ to $L_{\bar{g}}(\tilde{\gamma}) \geq L_{\bar{g}}(\gamma)$, and delete the phrase “unless $\tilde{\gamma}$ is a reparametrization of γ .” [Because the definition of reparametrization that I’m using requires a diffeomorphism of the parameter domain, the original problem statement was not true.]
- (12/18/12) **Page 355, proof of Lemma 14.10:** At the beginning of the proof, insert “Let (E_1, \dots, E_n) be the basis for V dual to (e^i) .”
- (12/18/12) **Page 356, Case 4, second line:** Should read “brings us back to Case 3.”
- (7/3/15) **Page 368, second paragraph:** At the end of the first sentence of the paragraph, insert “(see pp. 341–343).”
- (7/18/17) **Page 368, paragraph below equation (14.25):** Change TM to $T\mathbb{R}^3$ (twice).
- (9/17/14) **Page 371, three lines above (14.31):** Change that sentence to “The only terms in this sum that can possibly be nonzero are those for which J has no repeated indices and m is equal to one of the indices in J , say $m = j_p$.”
- (5/14/20) **Page 374, Problem 14-2:** Add “[Hint: One way to approach this is to prove first that a k -covector ω is decomposable if and only if the map from \mathbb{R}^n to $\Lambda^{k-1}(\mathbb{R}^{n*})$ given by $v \mapsto v \lrcorner \omega$ has $(n-k)$ -dimensional kernel.]”
- (12/2/20) **Page 377, line 4:** Change “is a simply” to “is simply.”
- (10/17/21) **Page 382, proof of Proposition 15.6, second paragraph:** In the first sentence of the paragraph, after “smooth chart,” insert “with connected domain.”
- (3/9/16) **Page 386, just above Proposition 15.24:** After “determines an orientation on ∂M ,” insert “if M is oriented.”
- (4/24/22) **Page 388, last paragraph:** Change “Proposition 13.6” to “Corollary 13.8.”
- (7/20/17) **Page 389, Exercise 15.30:** Change “a local isometry” to “an orientation-preserving local isometry.”
- (5/9/20) **Page 397, Problem 15-1:** At the end of the last sentence, add “when $n > 1$.”
- (5/14/20) **Page 397, Problem 15-3:** Change $\bar{\mathbb{B}}^n$ to $\bar{\mathbb{B}}^{n+1}$ (twice).
- (6/2/16) **Page 397, Problem 15-4:** Change the first sentence to “Let θ be the flow of a smooth vector field on a smooth manifold.” [The stated result is true also for manifolds with boundary and for nonmaximal flows, but to prove it, one must first do a little work to generalize some of the results of Theorem 9.12 to more general flows.]
- (4/26/14) **Page 402, lines 2–3:** There should not be a paragraph break before “and.”
- (3/14/16) **Page 403, just after the last displayed equation:** Add “(In the \mathbb{H}^n case, apply Theorem C.26 to the interiors of D and E considered as subsets of \mathbb{R}^n .)”
- (5/28/18) **Page 409, line 2:** Change φ_i to φ .
- (6/24/18) **Page 415, paragraph above Example 16.19:** Change “interior charts and charts with corners” to “interior charts, boundary charts, and charts with corners.”
- (6/2/16) **Page 416, line 3 from the bottom:** Change “ $\gamma(0) = p$ ” to “ $\gamma(0) = \psi(p)$.”
- (9/25/19) **Page 418, statement of Proposition 16.21:** Delete “compact,” and change “ n -manifold” to “ $(n+1)$ -manifold.”

(6/24/18) **Page 419, proof of Theorem 16.25, first paragraph:** Replace the second and third sentences of the paragraph by the following: “By means of smooth charts and a partition of unity, we may reduce the theorem to the cases in which $M = \mathbb{R}^n$, $M = \mathbb{H}^n$, or $M = \overline{\mathbb{R}}_+^n$. The \mathbb{R}^n and \mathbb{H}^n cases are treated just as before.”

(7/22/15) **Page 424, second displayed equation:** Change $\iota_S^* \beta(X)$ to $\iota_{\partial M}^* \beta(X)$.

(2/18/13) **Page 426, three lines below the section heading:** “cam” should be “can.”

(2/11/15) **Page 430, Proposition 16.38(c):** This statement is wrong. Change it to “If F is smooth, then $F^* \mu$ is a continuous density on M ; and if F is a local diffeomorphism, $F^* \mu$ is smooth.”

(7/27/16) **Page 439, Problem 16-23:** The formula for g should be

$$g = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}.$$

(2/19/13) **Page 444, two lines below equation (17.4):** Change $T_{(q,s)} M$ to $T_{(q,s)}(M \times \mathbb{R})$.

(6/6/18) **Page 447, Corollary 17.15:** Change “every closed form is exact” to “every closed p -form is exact for $p \geq 1$.”

(5/15/15) **Page 450, proof of Theorem 17.21, line 5:** Change $H_{\text{dR}}^1(\mathbb{S}^n)$ to $H_{\text{dR}}^1(\mathbb{S}^1)$.

(8/14/17) **Page 451, proof of Corollary 17.25, next-to-last line:** Change $\text{Id}_{H_{\text{dR}}^{n-1}(S)}$ to $\text{Id}_{H_{\text{dR}}^{n-1}(S)}$.

(11/24/17) **Pages 455–456, Proof of Theorem 17.32:** The proof given in the book is incorrect, because the V_i ’s might not be connected, so Theorem 17.30 does not apply to them. Here’s a corrected proof.

Lemma. *If M is a noncompact connected manifold, there is a countable, locally finite open cover $\{V_j\}_{j=1}^\infty$ of M such that each V_i is connected and precompact, and for each j , there exists $k > j$ such that $V_j \cap V_k \neq \emptyset$.*

Proof. Let $\{W_j\}_{j=1}^\infty$ be a countably infinite, locally finite cover of M by precompact, connected open sets (such a cover exists by Prop. 1.19 and Thm. 1.15). By successively deleting unneeded sets and renumbering, we can ensure that no W_j is contained in the union of the other W_i ’s.

Let $Y_1 = \bigcup_{i=2}^\infty W_i$. Because M is connected, each component of Y_1 meets W_1 , and by local finiteness of $\{W_j\}$, there are only finitely many such components. Such a component is precompact in M if and only if it is a union of finitely many W_i ’s. Let V_1 be the union of W_1 together with all of the precompact components of Y_1 , and let X_1 be the union of all W_i ’s not contained in V_1 . Then V_1 is connected and precompact, and X_1 has no precompact components. Proceeding by induction, suppose we have defined connected, precompact open sets V_1, \dots, V_m whose union contains $W_1 \cup \dots \cup W_m$, and such that the union X_m of all the W_i ’s not contained in $V_1 \cup \dots \cup V_m$ has no precompact components. Let j_m be the smallest index such that W_{j_m} is not contained in $V_1 \cup \dots \cup V_m$, and let Y_{m+1} be the union of all W_i ’s other than W_{j_m} not contained in $V_1 \cup \dots \cup V_m$. Any precompact component of Y_{m+1} must meet W_{j_m} , because otherwise, it would be a precompact component of X_m . Let V_{m+1} be the union of W_{j_m} with all of the precompact components of Y_{m+1} . As before, V_{m+1} is precompact and connected, and the union X_{m+1} of the W_i ’s not contained in $V_1 \cup \dots \cup V_{m+1}$ has no precompact components. Then by construction, for each j , the set $X_j = \bigcup_{i>j} V_i$ has no precompact components. If some V_j does not meet V_k for any $k > j$, then V_j itself is a precompact component of X_{j-1} , which is a contradiction. Thus for each j , there is some $k > j$ such that $V_j \cap V_k \neq \emptyset$. \square

Proof of Theorem 17.32. Choose an orientation on M . Let $\{V_j\}_{j=1}^\infty$ be an open cover of M satisfying the conclusions of the preceding lemma. For each j , let $K(j)$ denote the least integer $k > j$ such that $V_j \cap V_k \neq \emptyset$, and let θ_j be an n -form compactly supported in $V_j \cap V_{K(j)}$ whose integral is 1. Let $\{\psi_j\}_{j=1}^\infty$ be a smooth partition of unity subordinate to $\{V_j\}_{j=1}^\infty$.

Now suppose ω is any n -form on M , and let $\omega_j = \psi_j \omega$ for each j . Let $c_1 = \int_{V_1} \omega_1$, so that $\omega_1 - c_1 \theta_1$ is compactly supported in V_1 and has zero integral. It follows from Theorem 17.30 that there exists $\eta_1 \in \Omega_c^{n-1}(V_1)$

such that $d\eta_1 = \omega_1 - c_1\theta_1$. Suppose by induction that we have found η_1, \dots, η_m and constants c_1, \dots, c_m such that for each $j = 1, \dots, m$, $\eta_j \in \Omega_c^{n-1}(V_j)$ and

$$d\eta_j = \left(\omega_j + \sum_{i:K(i)=j} c_i\theta_i \right) - c_j\theta_j. \quad (*)$$

Let

$$c_{j+1} = \int_{V_{j+1}} \left(\omega_{j+1} + \sum_{i:K(i)=j+1} c_i\theta_i \right).$$

Then by Theorem 17.30, there exists $\eta_{j+1} \in \Omega_c^{n-1}(V_{j+1})$ satisfying the analog of (*) with j replaced by $j+1$. Set $\eta = \sum_{j=1}^{\infty} \eta_j$, with each η_j extended to be zero on $M \setminus V_j$. By local finiteness, this is a smooth $(n-1)$ -form on M . It satisfies

$$d\eta = \omega + \sum_{j=1}^{\infty} \left(\sum_{i:K(i)=j} c_i\theta_i \right) - \sum_{j=1}^{\infty} c_j\theta_j.$$

Each term $c_i\theta_i$ appears exactly once in the first sum above, so the two sums cancel each other. □

(7/27/16) **Page 457, line below the second displayed equation:** Change “Theorem 17.31” to “Theorem 17.30.”

(7/12/16) **Page 463, line above equation (17.15):** Insert missing space before “Similarly.”

(7/13/16) **Page 464, end of proof of Corollary 17.42:** Insert “Note that this construction produces a form σ whose support is contained in $U \cap V$.” [This might be useful for solving Problem 18-6.]

(7/12/16) **Page 471, last paragraph:** Replace the sentence starting “The hardest part ...” with “The hardest part is showing that the singular chain complex of M can be replaced by a chain complex built out of simplices whose images lie in either U or V , without changing the homology.”

(9/12/17) **Page 487, Problem 18-1, first line:** Change “an oriented smooth manifold” to “a smooth manifold.”

(8/8/18) **Page 489, Problem 18-7(b):** Add to the hint: “In order to use Lemma 17.27, you’ll need to prove the following fact: *Every bounded convex open subset of \mathbb{R}^n is diffeomorphic to \mathbb{R}^n .* To prove this, let U be such a subset, and without loss of generality assume $0 \in U$. First show that there exists a smooth nonnegative function $f \in C^\infty(U)$ such that $f(0) = 0$ and $f(x) \geq 1/d(x)$ away from a small neighborhood of 0, where $d(x)$ is the distance from x to ∂U . Next, show that $g(x) = 1 + \int_0^1 t^{-1} f(tx) dt$ is a smooth positive exhaustion function on U that is nondecreasing along each ray starting at 0. Finally, show that the map $F : U \rightarrow \mathbb{R}^n$ given by $F(x) = g(x)x$ is a bijective local diffeomorphism. Also, you may use the fact that the conclusion of the five lemma is still true even if the appropriate diagram commutes only up to sign.”

(1/15/13) **Page 491, Example 19.1(c):** Delete the word “unit.”

(5/22/15) **Page 492, line above Proposition 19.2:** Change “lie” to “Lie.”

(12/17/15) **Page 492, proof of Proposition 19.2, fourth line:** Change “Given $p \in M$ ” to “Given $p \in U$.”

(9/12/16) **Page 506, Lemma 19.24, last line:** Before “left-invariant,” insert “smooth.”

(6/1/20) **Page 512, Problem 19-4:** In the first line of the problem, change “all three coordinates are positive” to “ z is positive.” Then replace the last sentence by “Find an explicit global chart on U in which D is spanned by the first two coordinate vector fields.” [Technically it might not be a flat chart because its image need not be a cube in \mathbb{R}^3 .]

(10/4/17) **Page 518, sentence before Prop. 20.3:** Change “one-parameter subgroups of $GL(n, \mathbb{R})$ ” to “one-parameter subgroups of subgroups of $GL(n, \mathbb{R})$.”

- (5/23/16) **Page 521, first displayed equation:** Change $d\Phi_0$ to $d\Phi_e$ (twice).
- (6/9/19) **Page 528, line 9:** Change two instances of (g, p) in subscripts to (g, q) .
- (5/19/18) **Page 528, just below the displayed equation in the middle of the page:** The smoothness of the map σ_q is not quite immediate from the definition. Replace the three sentences beginning “It follows” with this: “Because S_p is a weakly embedded submanifold by Theorem 19.17, to show that σ_q is a smooth local section of S_p , it suffices to show that it is smooth into $G \times M$ and takes its values in S_p . The first component function is smooth as a map into G by smoothness of group multiplication. To show that the second component is smooth into M as a function of \hat{X} (and therefore of $\exp X$), you need to use the argument sketched out just below equation (20.10): as in the proof of Prop. 20.8, apply the fundamental theorem on flows to the vector field $\mathcal{E}_{(p,X)} = (\hat{X}_g, 0)$ on $M \times \mathfrak{g}$. A straightforward computation shows that $\gamma(t) = (g \exp tX, \eta_{(\hat{X})}(t, q))$ is an integral curve of \hat{X} starting at (g, q) , from which it follows easily that $\sigma_q(g \exp X) = \gamma(1) \in S_p$.”
- (1/10/17) **Page 537, Problem 20-6(a):** Change $B \in \mathfrak{gl}(n, \mathbb{R})$ to $B \in \mathfrak{sl}(n, \mathbb{R})$.
- (5/31/16) **Page 538, Problem 20-11(b):** Here’s a better hint, which doesn’t require proving part (a) first: “[Hint: Consider the graph of F as a subgroup of $G \times H$.]”
- (10/18/17) **Page 542, middle of the paragraph before Example 21.3:** Change “the action of \mathbb{R}^k on \mathbb{R}^n ” to “the action of \mathbb{R}^k on $\mathbb{R}^k \times \mathbb{R}^n$.”
- (2/25/18) **Page 548, last two lines:** Allen Hatcher’s name is misspelled. (Sorry, Allen.)
- (5/23/16) **Page 549, proof of Proposition 21.12, last sentence:** Change the first phrase of that sentence to “Second, if $p, p' \in E$ are in different orbits and $\pi(p) \neq \pi(p'), \dots$ ” Then add the following sentences at the end of the proof: “If p and p' are in different orbits and $\pi(p) = \pi(p')$, let W be an evenly covered neighborhood of $\pi(p)$, and let V, V' be the components of $\pi^{-1}(W)$ containing p and p' , respectively. For any $g \in \text{Aut}_\pi(E)$, a simple connectedness argument shows that $g \cdot V$ is a component of $\pi^{-1}(W)$; if it had nontrivial intersection with V it would have to be equal to V , which would imply $g \cdot p = p'$, a contradiction.”
- (7/26/16) **Page 567, two lines above Proposition 22.8:** Insert “a” before “2-covector.”
- (10/9/15) **Page 568, Example 22.9(a), first line:** The coordinates should be $(x^1, \dots, x^n, y^1, \dots, y^n)$. (The last coordinate is y^n , not x^n .)
- (11/17/21) **Page 571, line below equation (22.5):** Delete the spurious word “theorem” at the end of the line.
- (3/27/19) **Page 572, middle of the page:** Replace the sentence starting “On the other hand” by this: “On the other hand, the left-hand side is just the ordinary t -derivative of a time-dependent tensor on a fixed vector space, and expanding in terms of a basis shows that it satisfies a similar product rule.”
- (10/5/17) **Page 573, statement of Proposition 22.15, second line:** Change “ $V : J \times M$ ” to “ $V : J \times M \rightarrow TM$ ”; and change ψ to θ .
- (11/18/17) **Page 583, line 4:** Change $\mathbb{R}^{2n+1} \setminus \{0\}$ to $\mathbb{R}^{2n+2} \setminus \{0\}$.
- (7/26/16) **Page 583, third displayed equation:** Should read
- $$T \lrcorner d\Theta = -2 \sum_{i=1}^{n+1} (x^i dx^i + y^i dy^i) = -d(|x|^2 + |y|^2).$$
- (7/26/16) **Page 583, two lines below the third displayed equation:** The formula for $d\Theta(N, T)$ should be $d\Theta(N, T) = 2(|x|^2 + |y|^2)$.

(11/28/12) **Page 584, Exercise 22.29:** Part (b) should read

$$(b) T = \frac{\partial}{\partial z};$$

- (8/14/14) **Page 584, paragraph above Theorem 22.33:** Change all occurrences of θ in this paragraph to ψ , to avoid confusion with the use of θ for a contact form elsewhere in this section.
- (11/24/17) **Page 585, statement of Theorem 22.34, last line:** Change H to F .
- (11/17/12) **Page 587, equation (22.27):** Change both occurrences of $\sigma(s)$ to $\sigma(x)$.
- (11/18/17) **Page 592, Problem 22-15:** Add the hypothesis that M is connected.
- (9/22/15) **Page 608, Proposition A.41(a):** Insert the following phrase at the beginning of this statement: *With the exception of the word “closed” in part (d).*
- (7/22/13) **Page 616, Proposition A.77(b), last line:** Change $\tilde{f}(0)$ to $\tilde{f}_e(0)$.
- (12/19/18) **Page 619, proof of Lemma B.2, fourth line:** Replace “By Exercise B.1(b)” with “If w_1 is equal to one of the v_i ’s, then the ordered $(n + 1)$ -tuple (w_1, v_1, \dots, v_n) is linearly dependent; if not, then by Exercise B.1(b), . . .”
- (9/1/16) **Page 632, Exercise B.29:** Change “by a matrix” to “by a certain matrix” (twice).
- (12/19/18) **Page 637, Exercise B.42:** Delete the words “is a homeomorphism that.” [Checking that it’s a homeomorphism requires the norm topology, which is not defined until later on that page.]
- (9/6/16) **Page 637, Exercise B.44:** Change “basis map” to “basis isomorphism.”
- (12/19/18) **Page 653, proof of Proposition C.21, second paragraph, second line:** Change f to f_D .
- (2/25/18) **Page 658, two lines above (C.15):** Change $B_\delta(0)$ to $\bar{B}_\delta(0)$.
- (2/25/18) **Page 660, display (C.20):** Change $F^{-1}(x)$ to $F^{-1}(y)$.
- (1/18/21) **Page 664, statement of Theorem D.1(b):** After the phrase “Any two differentiable solutions to (D.3)–(D.4),” insert “defined on intervals containing t_0 .”
- (12/2/15) **Page 666, just below the fifth display:** After the sentence ending “by our choice of δ and ε ,” insert “(If $t < t_0$, interchange t and t_0 in the second line above.)”
- (1/18/21) **Page 664, statement of Theorem D.4:** After the phrase “any two differentiable solutions to (D.3)–(D.4),” insert “defined on intervals containing t_0 .”
- (1/18/21) **Page 668, paragraph below equation (D.10):** In the fourth line of the paragraph, change \bar{W} to W .
- (1/18/21) **Page 670, displayed inequality between (D.17) and (D.18):** Change n to n^2 .
- (1/18/21) **Page 670, last line:** Change n to n^2 in the definition of B .
- (1/18/21) **Page 671, inequality (D.19):** Change n to n^2 (twice).
- (12/15/20) **Page 671, just below (D.19):** Replace the sentence “Since the expression on the right can be made as small as desired by choosing h and \tilde{h} sufficiently small, this shows . . .” by the following: “Thus the expression on the left can be made as small as desired by choosing h and \tilde{h} sufficiently small. This shows . . .”
- (6/11/19) **Page 692:** Under the entry for “Form,” delete the references to page 294 for “closed” and page 292 for “exact.”
- (2/25/18) **Page 693:** The index entry for “Hatcher, Allen” is misspelled.