CORRECTIONS TO
Introduction to Smooth Manifolds (Second Edition)

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(8/8/16) **Page 6, just below the last displayed equation:** Change $\varphi([x])$ to $\varphi_{n+1}[x]$, and in the next line, change x^i to x^{n+1}. After "(Fig. 1.4)," insert "with similar interpretations for the other charts."

(8/8/16) **Page 7, Fig. 1.4:** Both occurrences of x^i should be x^{n+1}.

(12/19/18) **Page 9, proof of Theorem 1.15:** In the second line of the proof, replace "For each j" with "For each $j \geq 0$." Then in the fourth-to-last line, replace "positive integers" by "nonnegative integers."

(8/23/18) **Page 20, Example 1.31:** There are multiple errors in this example. Replace everything after the first two sentences by the following: For each $i = 1, \dots, n+1$, let $\left(U_i^{\pm} \cap \mathbb{S}^n, \varphi_i^{\pm}\right)$ denote the graph coordinate charts we constructed in Example 1.4. For any distinct indices i and j and any choices of $\pm$ signs, the transition maps $\varphi_i^{\pm} \circ \left(\varphi_j^{\pm}\right)^{-1}$ and $\varphi_i^{\pm} \circ \left(\varphi_j^{\mp}\right)^{-1}$ are easily computed. For example, in the case $i < j$, we get the following formula for all u in the domain of $\varphi_i^{+} \circ \left(\varphi_j^{+}\right)^{-1}$:

$$\varphi_i^{+} \circ \left(\varphi_j^{+}\right)^{-1}\left(u^1, \dots, u^n\right) = \left(u^1, \dots, \widehat{u^i}, \dots, \pm\sqrt{1-|u|^2}, \dots, u^n\right)$$

(with u^i omitted and the square root replacing u^j), and similar formulas hold in the other cases. When $i = j$, the domains of φ_i^{+} and φ_i^{-} are disjoint, so there is nothing to check. Thus, the collection of charts $\left\{\left(U_i^{\pm} \cap \mathbb{S}^n, \varphi_i^{\pm}\right)\right\}$ is a smooth atlas, and so defines a smooth structure on $\mathbb{S}^n$. We call this its ***standard smooth structure***. //

(6/23/13) **Page 23, two lines below the first displayed equation:** Change "any subspace $S \subseteq V$" to "any k-dimensional subspace $S \subseteq V$."

(12/19/18) **Page 26, first line:** Change $U \cap \varphi^{-1}(\operatorname{Int} \mathbb{H}^n)$ to $\varphi^{-1}(\operatorname{Int} \mathbb{H}^n)$.

(12/19/18) **Page 27, last paragraph, sixth line:** Change $\widetilde{U} \cap \mathbb{H}^n$ to $\widetilde{U} \cap U$.

(2/22/15) **Page 29, proof of Theorem 1.46, second paragraph, line 4:** Change $\varphi(U \cap V)$ to $\psi(U \cap V)$.

(10/8/15) **Page 30, Problem 1-6:** Interpret the formula for F_s to mean $F_s(0) = 0$ when $s \leq 1$.

(1/27/18) **Page 31, Fig. 1.13:** Change $\{x^n = 0\}$ to $\{x^{n+1} = 0\}$.

(3/12/18) **Page 31, Problem 1-11, next-to-last line:** Change $\mathbb{S}^n$ to $\mathbb{S}^n \smallsetminus \{N\}$.

(4/25/17) **Page 45, second paragraph:** Replace the last sentence of that paragraph with the following: "If N has empty boundary, we say that a map $F\colon A \to N$ is ***smooth on A*** if it has a smooth extension in a neighborhood of each point: that is, if for every $p \in A$ there exist an open subset $W \subseteq M$ containing p and a smooth map $\widetilde{F}\colon W \to N$ whose restriction to $W \cap A$ agrees with F. When $\partial N \neq \varnothing$, we say $F\colon A \to N$ is smooth on A if for every $p \in A$ there exist an open subset $W \subseteq M$ containing p and a smooth chart (V, ψ) for N whose domain contains $F(p)$, such that $F(W \cap A) \subseteq V$ and $\psi \circ F|_{W \cap A}$ is smooth as a map into $\mathbb{R}^n$ in the sense defined above (i.e., it has a smooth extension in a neighborhood of each point)."

(7/23/14) **Page 45, last displayed equation:** The first $=$ sign should be $\subseteq$.

(6/20/18) **Page 47, proof of Theorem 2.29, second paragraph:** Replace the first sentence of the paragraph by "Let $h\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth bump function that is positive in $B_1(0)$ and zero elsewhere."

(11/17/12) **Page 56, first displayed equation:** Change $d\iota(v)_p$ to $d\iota_p(v)$.

(1/26/15) **Page 68, proof of Proposition 3.21:** Insert the following sentence at the beginning of the proof: "Let $n = \dim M$ and $m = \dim N$." Then in the second sentence, change (3.9) to (3.10). Finally, in the displayed equation, change F^n to F^m (twice).

(11/17/12) **Page 70, two lines above Corollary 3.25:** Change "Proposition 3.23" to "Proposition 3.24."

(3/5/15) **Page 76, Problem 3-8:** Add the following remark: "(For $p \in \partial M$, we need to allow curves with domain $[0,\varepsilon)$ or $(-\varepsilon, 0]$ and to interpret the derivatives as one-sided derivatives.)"

(10/23/18) **Page 78, proof of Prop. 4.1, third and fourth lines:** Change $m \times n$ to $n \times m$ (twice).

(11/9/16) **Page 79, proof of Theorem 4.5, fourth line:** Change $\widehat{F}(p)$ to $\widehat{F}(0)$.

(5/4/13) **Page 96, Problem 4-3:** This problem probably needs a better hint. First, to get a good result, you'll have to add the assumption that $\ker dF_p \not\subseteq T_p\partial M$. After choosing smooth coordinates, you can assume $M \subseteq \mathbb{H}^m$ and $N \subseteq \mathbb{R}^n$, and extend F to a smooth function $\widetilde{F}$ on an open subset of $\mathbb{R}^m$. If $\operatorname{rank} F = r$, show that there is a coordinate projection $\pi\colon \mathbb{R}^n \to \mathbb{R}^r$ such that $\pi \circ \widetilde{F}$ is a submersion, and apply the rank theorem to $\pi \circ \widetilde{F}$ to find new coordinates in which $\widetilde{F}$ has a coordinate representation of the form $\widehat{F}(x,y) = (x, R(x,y))$. Then use the rank condition to show that $R|_M$ is independent of y.

(9/8/15) **Page 100, proof of Proposition 5.4, next-to-last line:** Change "It a homeomorphism" to "It is a homeomorphism."

(11/9/16) **Page 112, Fig. 5.10:** Interchange the labels M and N on the figure, to be consistent with the notation in Theorem 5.29.

(7/15/15) **Page 120, proof of Proposition 5.46:** At the beginning of the proof, insert this sentence: "Let $F\colon D \hookrightarrow M$ denote the inclusion map."

(7/21/18) **Page 121, line 5:** Change x^m to x^n.

(12/19/18) **Page 129, proof of Sard's theorem, second paragraph:** Just before the last sentence of the paragraph, insert the following: "In the $\mathbb{H}^n$ case, extend F to a smooth map on an open subset of $\mathbb{R}^m$, and replace U by that open subset; if we can show that the set of critical values of the extended map has measure zero, then the same is true of the set of critical values of F."

(3/16/19) **Page 129, displayed equation near the bottom of the page:** Change "ith partial derivatives" to "ith-order partial derivatives."

(12/26/18) **Page 130, just below equation (6.2):** Right after the displayed equation, insert "(where the component functions $F^2, \dots, F^n$ might be different from the ones in the original coordinate chart)."

(1/8/18) **Page 131, four lines above Corollary 6.11:** Insert "at most" before "K^m balls."

(4/17/13) **Page 132, proof of Lemma 6.13, second paragraph:** This argument does not apply when $\partial M \neq \varnothing$, because in that case $M \times M$ is not a smooth manifold with boundary. Instead, we can consider the restrictions of κ to $(M \times \operatorname{Int} M) \smallsetminus \Delta_M$ and to $(M \times \partial M) \smallsetminus \Delta_M$ (both of which are smooth manifolds with boundary), and note that there is a point $[v] \in \mathbb{RP}^{N-1}$ that is not in the image of τ or either of these restrictions of κ. [Thanks to David Iglesias Ponte for suggesting this correction.]

(12/19/18) **Page 134, proof of Theorem 6.15, just after the fourth paragraph of the proof:** Insert the following: "The argument above still works when M is an arbitrary compact subset of a larger manifold $\widetilde{M}$ with or without boundary, by covering M with finitely many coordinate balls or half-balls for $\widetilde{M}$. The result is a smooth injective map $F\colon M \to \mathbb{R}^{nm+m}$ whose differential is injective at each point." [This is needed in the ensuing argument for the noncompact case, because the sets E_i might not be regular domains when $\partial M \neq \varnothing$.]

(7/3/15) **Page 134, displayed equations two-thirds of the way down the page:** In the definition of E_i, there's an "$i-i$" that should be "$i-1$." It should read $E_i = f^{-1}\big([b_{i-1}, a_{i+1}]\big)$.

(12/19/18) **Page 134, just below the displayed equations two-thirds of the way down the page:** Delete the sentence "By Proposition 5.47, each E_i is a compact regular domain."

(7/2/18) **Page 137, first paragraph under the subheading "Tubular Neighborhoods," fifth line:** Change R^n to $\mathbb{R}^n$.

(7/27/18) **Page 138, proof of Theorem 6.23, end of the first paragraph:** Change "standard coordinate frame" to "standard coordinate basis."

(11/25/12) **Page 145, statement of Corollary 6.33:** After "immersed submanifold," insert "with $\dim S = \dim M$."

(12/5/16) **Page 145, paragraph above Prop. 6.34:** In the definition of ***smooth family of maps***, replace "$F\colon M \times S \to N$" by "$F\colon N \times S \to M$."

(11/25/12) **Page 148, Problem 6-13:** Delete part (c). [This statement is simply wrong. It is true with the added hypothesis that F' is an embedding, but then it's essentially just a restatement of part (b).]

(2/10/18) **Page 150, last line:** Change "Theorem 20.16" to "Theorem 20.22."

(12/30/17) **Page 160, first line:** Change $R_{hh_1^{-1}}$ to $R_{h_1^{-1}h}$.

(2/16/18) **Page 164, just above the subheading:** Replace the last line of the proof of Prop. 7.23 by "The action is smooth because each φ can be written locally as a composition of a smooth local section followed by π."

(8/26/14) **Page 169, first line:** Change $\widetilde{G}$ to G.

(3/18/19) **Page 171, third line from the end of the proof:** Change E_i to E_j, so the formula reads $\rho^i_j(g) = \pi^i(g \cdot E_j)$.

(9/17/14) **Page 173, Problem 7-21:** Replace the first sentence by "Prove that the groups in Problem 7-20 are isomorphic to direct products of the indicated groups in cases (a) and (c) if and only if n is odd, and in cases (b) and (d) if and only if $n = 1$."

(10/23/18) **Page 178, Example 8.10(d):** Change "Example 8.5" to "Example 8.4."

(11/17/12) **Page 196, proof of Proposition 8.45, next-to-last line:** Should read "$F_* \circ \big(F^{-1}\big)_* = \big(F \circ F^{-1}\big)_* = \mathrm{Id}_{\mathrm{Lie}(H)}$ and $\big(F^{-1}\big)_* \circ F_* = \mathrm{Id}_{\mathrm{Lie}(G)}$."

(5/27/17) **Page 208, first line:** Change to "This is just the existence and smoothness statements of Theorem D.1 ...."

(3/10/16) **Page 213, first sentence of the last paragraph:** The definition of t_0 should be $t_0 = \sup\{t \in \mathbb{R} : (t, p_0) \in W\}$.

(12/2/15) **Page 217, Fig. 9.7:** Both occurrences of φ should be Φ.

(12/2/15) **Page 219, second displayed equation:** Change "$V^j(0, p) = 0$" to "$\Phi^j(0, p) = 0$."

(12/2/15) **Page 219, two lines below (9.12):** Here and in the rest of the paragraph, change p_0 to p_1 (seven times) to avoid confusion with the prior unrelated use of p_0 in this proof.

(5/29/16) **Page 222, just below the section heading:** Insert the following sentence: "On a manifold with boundary, the definitions of ***flow domain***, ***flow***, and ***infinitesimal generator of a flow*** are exactly the same as on a manifold without boundary."

(2/15/19) **Page 223, line 2:** Change $\delta\colon M \to \mathbb{R}^+$ to $\delta\colon \partial M \to \mathbb{R}^+$.

(8/19/14) **Page 223, proof of Theorem 9.26:** There's a gap in this proof, because it is not necessarily the case that $M(a)$ is a regular domain in $\operatorname{Int} M$. To correct the problem, we have to choose our collar neighborhood more carefully. Replace the first sentence of the proof by the following paragraph:

"Theorem 9.25 shows that ∂M has a collar neighborhood C_0 in M, which is the image of a smooth embedding $E_0\colon [0,1)\times\partial M \to M$ satisfying $E_0(0,x) = x$ for all $x\in\partial M$. Let $f\colon M\to\mathbb{R}^+$ be a smooth positive exhaustion function. Note that $W = \{(t,x) : f(E_0(t,x)) > f(x) - 1\}$ is an open subset of $[0,1)\times\partial M$ containing $\{0\}\times\partial M$. Using a partition of unity as in the proof of Theorem 9.20, we may construct a smooth positive function $\delta\colon\partial M\to\mathbb{R}$ such that $(t,x)\in W$ whenever $0\le t<\delta(x)$. Define $E\colon[0,1)\times\partial M\to M$ by $E(t,x) = E_0(t\delta(x),x)$. Then E is a diffeomorphism onto a collar neighborhood C of ∂M, and by construction $f(E(t,x)) > f(x)-1$ for all $(t,x)\in[0,1)\times\partial M$. We will show that for each $a\in(0,1)$, the set $E([0,a]\times\partial M)$ is closed in M. Suppose p is a boundary point of $E([0,a]\times\partial M)$ in M; then there is a sequence $\{(t_i,x_i)\}$ in $[0,a]\times\partial M$ such that $E(t_i,x_i)\to p\in M$. Then $f(E(t_i,x_i))$ remains bounded, and thus $f(x_i) < f(E(t_i,x_i))+1$ also remains bounded. Since ∂M is closed in M, $f|_{\partial M}$ is also an exhaustion function, and therefore the sequence $\{x_i\}$ lies in some compact subset of ∂M. Passing to a subsequence, we may assume $(t_i,x_i)\to(t_0,x_0)$, and therefore $p = E(t_0,x_0)\in E([0,a]\times\partial M)$."

Then at the end of the first paragraph of the proof, add the following sentences:

"To see that $M(a)$ is a regular domain, note first that it is closed in M because it is the complement of the open set $C(a)$. Let $p\in M(a)$ be arbitrary. If $p\notin E([0,a]\times\partial M)$, then p has a neighborhood in $\operatorname{Int} M$ contained in $M(a)$ by the argument above. If $p\in E([0,a]\times\partial M)$, then $p = E(a,x)$ for some $x\in\partial M$, and C is a neighborhood of p in which $M(a)\cap C$ is the diffeomorphic image of $[a,1)\times\partial M$."

(1/30/14) **Page 223, proof of Theorem 9.26, last line of the first paragraph:** Change $0\le t<a$ to $0\le s<a$.

(1/30/14) **Page 225, Example 9.31:** At the end of the example, insert the sentence "If $n\ge 2$, then $M_1 \# M_2$ is connected."

(7/25/16) **Page 226, Example 9.32, fifth line:** Replace the sentence beginning "It is a smooth manifold without boundary ..." by "It is a topological manifold without boundary, and can be given a smooth structure such that each of the natural maps $M\to D(M)$ (induced by inclusion into the left and right summands of the disjoint union) is a smooth embedding."

(4/23/13) **Page 230, second paragraph:** "from Case" should be "from Case 1."

(9/8/18) **Page 234, proof of Theorem 9.46, second paragraph:** Replace the two parenthesized sentences by the following: "(To see this, just choose $\varepsilon_1>0$ and $U_1\subseteq U$ such that θ_1 maps $(-\varepsilon_1,\varepsilon_1)\times U_1$ into U, and then inductively choose ε_i and U_i such that θ_i maps $(-\varepsilon_i,\varepsilon_i)\times U_i$ into U_{i-1}. Taking $\varepsilon=\min\{\varepsilon_i\}$ and $Y=U_k$ does the trick.)"

(2/26/18) **Page 230, fourth paragraph, last line:** Change $[X,Y]$ to $[V,W]$.

(5/29/16) **Page 241, Example 9.52:** At the end of the example, add the sentence "Note that u is smooth on the open set $\mathbb{R}^2\smallsetminus\{0\}$, which is a neighborhood of S."

(6/4/14) **Page 246, Problem 9-11:** Delete the second sentence of the hint. [Because N is inward-pointing along ∂M, no integral curve that starts on ∂M can hit the boundary again, because the vector field would have to be tangent to ∂M or outward-pointing at the first such point.]

(11/12/16) **Page 248, first displayed equation:** Should read

$$V(t,p) = \frac{\partial}{\partial t} H\big(H_t^{-1}(p),t\big).$$

(11/12/16) **Page 248, Problem 9-22(c):** Replace the problem statement by

(c) $\dfrac{\partial u}{\partial x} + u\dfrac{\partial u}{\partial y} = -y, \qquad u(0,y)=0.$

[Without this sign change, the third claim in Problem 9-23 is not true.]

(10/22/18) **Page 261, statement of Proposition 10.25, first line:** Change $\pi'\colon E \to M'$ to $\pi'\colon E' \to M'$.

(7/2/14) **Page 264, paragraph above the subheading, first sentence:** "homomorphism" should be "homomorphisms."

(6/29/15) **Page 278, Example 11.13, third line:** Change "every coordinate frame" to "every coordinate coframe."

(5/19/18) **Page 303, just below the commutative diagram:** Insert this sentence: "A natural transformation is called a ***natural isomorphism*** if each map λ_X is an isomorphism in D."

(5/19/18) **Page 303, Problem 11-18(b) and (c):** Change "natural transformation" to "natural isomorphism" in both parts.

(5/24/18) **Page 317, displayed equation just below the middle of the page:** Change $A^{i_1\dots i_k}_{j_1\dots i_l}$ to $A^{i_1\dots i_k}_{j_1\dots j_l}$ on the third line of the display, and again on the line below the display. [The last lower index should be j_l, not i_l.]

(4/18/17) **Page 320, statement of Proposition 12.25:** Change the domain and codomain of G: It should read $G\colon P \to M$.

(4/18/17) **Page 320, Proposition 12.25(e):** Should read $(F \circ G)^* B = G^*(F^* B)$.

(4/17/15) **Page 333, first line:** Change $U \subseteq M$ to $V \subseteq M$.

(7/1/14) **Page 345, Problem 13-10:** In the last line of the problem statement, change $L_{\overline{g}}(\widetilde{\gamma}) > L_{\overline{g}}(\gamma)$ to $L_{\overline{g}}(\widetilde{\gamma}) \geq L_{\overline{g}}(\gamma)$, and delete the phrase "unless $\widetilde{\gamma}$ is a reparametrization of γ." [Because the definition of reparametrization that I'm using requires a diffeomorphism of the parameter domain, the original problem statement was not true.]

(12/18/12) **Page 355, proof of Lemma 14.10:** At the beginning of the proof, insert "Let $(E_1, \dots, E_n)$ be the basis for V dual to $\left(\varepsilon^i\right)$."

(12/18/12) **Page 356, Case 4, second line:** Should read "brings us back to Case 3."

(7/3/15) **Page 368, second paragraph:** At the end of the first sentence of the paragraph, insert "(see pp. 341–343)."

(7/18/17) **Page 368, paragraph below equation (14.25):** Change TM to $T\mathbb{R}^3$ (twice).

(9/17/14) **Page 371, three lines above (14.31):** Change that sentence to "The only terms in this sum that can possibly be nonzero are those for which J has no repeated indices and m is equal to one of the indices in J, say $m = j_p$."

(3/9/16) **Page 386, just above Proposition 15.24:** After "determines an orientation on ∂M," insert "if M is oriented."

(7/20/17) **Page 389, Exercise 15.30:** Change "a local isometry" to "an orientation-preserving local isometry."

(6/2/16) **Page 397, Problem 15-4:** Change the first sentence to "Let θ be the flow of a smooth vector field on a smooth manifold." [The stated result is true also for manifolds with boundary and for nonmaximal flows, but to prove it, one must first do a little work to generalize some of the results of Theorem 9.12 to more general flows.]

(4/26/14) **Page 402, lines 2–3:** There should not be a paragraph break before "and."

(3/14/16) **Page 403, just after the last displayed equation:** Add "(In the $\mathbb{H}^n$ case, apply Theorem C.26 to the interiors of D and E considered as subsets of $\mathbb{R}^n$.)"

(5/28/18) **Page 409, line 2:** Change φ_i to φ.

(6/24/18) **Page 415, paragraph above Example 16.19:** Change "interior charts and charts with corners" to "interior charts, boundary charts, and charts with corners."

(6/2/16) **Page 416, line 3 from the bottom:** Change "$\gamma(0) = p$" to "$\gamma(0) = \psi(p)$."

(6/24/18) **Page 418, statement of Proposition 16.21:** Change "n-manifold" to "$(n+1)$-manifold."

(6/24/18) **Page 419, proof of Theorem 16.25, first paragraph:** Replace the second and third sentences of the paragraph by the following: "By means of smooth charts and a partition of unity, we may reduce the theorem to the cases in which $M = \mathbb{R}^n$, $M = \mathbb{H}^n$, or $M = \overline{\mathbb{R}}^n_+$. The $\mathbb{R}^n$ and $\mathbb{H}^n$ cases are treated just as before."

(7/22/15) **Page 424, second displayed equation:** Change $\iota_S^*\beta(X)$ to $\iota_{\partial M}^*\beta(X)$.

(2/18/13) **Page 426, three lines below the section heading:** "cam" should be "can."

(2/11/15) **Page 430, Proposition 16.38(c):** This statement is wrong. Change it to "If F is smooth, then $F^*\mu$ is a continuous density on M; and if F is a local diffeomorphism, $F^*\mu$ is smooth."

(7/27/16) **Page 439, Problem 16-23:** The formula for g should be

$$g = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}.$$

(2/19/13) **Page 444, two lines below equation (17.4):** Change $T_{(q,s)}M$ to $T_{(q,s)}(M \times \mathbb{R})$.

(6/6/18) **Page 447, Corollary 17.15:** Change "every closed form is exact" to "every closed p-form is exact for $p \geq 1$."

(5/15/15) **Page 450, proof of Theorem 17.21, line 5:** Change $H^1_{\mathrm{dR}}(\mathbb{S}^n)$ to $H^1_{\mathrm{dR}}(\mathbb{S}^1)$.

(8/14/17) **Page 451, proof of Corollary 17.25, next-to-last line:** Change $\mathrm{Id}_{H^{n-1}_{\mathrm{dR}}}(S)$ to $\mathrm{Id}_{H^{n-1}_{\mathrm{dR}}(S)}$.

(11/24/17) **Pages 455–456, Proof of Theorem 17.32:** The proof given in the book is incorrect, because the V_i's might not be connected, so Theorem 17.30 does not apply to them. Here's a corrected proof.

Lemma. *If M is a noncompact connected manifold, there is a countable, locally finite open cover $\{V_j\}_{j=1}^\infty$ of M such that each V_i is connected and precompact, and for each j, there exists $k > j$ such that $V_j \cap V_k \neq \varnothing$.*

Proof. Let $\{W_j\}_{j=1}^\infty$ be a countably infinite, locally finite cover of M by precompact, connected open sets (such a cover exists by Prop. 1.19 and Thm. 1.15). By successively deleting unneeded sets and renumbering, we can ensure that no W_j is contained in the union of the other W_i's.

Let $Y_1 = \bigcup_{i=2}^\infty W_i$. Because M is connected, each component of Y_1 meets W_1, and by local finiteness of $\{W_j\}$, there are only finitely many such components. Such a component is precompact in M if and only if it is a union of finitely many W_i's. Let V_1 be the union of W_1 together with all of the precompact components of Y_1, and let X_1 be the union of all W_i's not contained in V_1. Then V_1 is connected and precompact, and X_1 has no precompact components. Proceeding by induction, suppose we have defined connected, precompact open sets $V_1, \dots, V_m$ whose union contains $W_1 \cup \dots \cup W_m$, and such that the union X_m of all the W_i's not contained in $V_1 \cup \dots \cup V_m$ has no precompact components. Let j_m be the smallest index such that W_{j_m} is not contained in $V_1 \cup \dots \cup V_m$, and let Y_{m+1} be the union of all W_i's other than W_{j_m} not contained in $V_1 \cup \dots \cup V_m$. Any precompact component of Y_{m+1} must meet W_{j_m}, because otherwise, it would be a precompact component of X_m. Let V_{m+1} be the union of W_{j_m} with all of the precompact components of Y_{m+1}. As before, V_{m+1} is precompact and connected, and the union X_{m+1} of the W_i's not contained in $V_1 \cup \dots \cup V_{m+1}$ has no precompact components. Then by construction, for each j, the set $X_j = \bigcup_{i>j} V_i$ has no precompact components. If some V_j does not meet V_k for any $k > j$, then V_j itself is a precompact component of X_{j-1}, which is a contradiction. Thus for each j, there is some $k > j$ such that $V_j \cap V_k \neq \varnothing$. □

Proof of Theorem 17.32. Choose an orientation on M. Let $\{V_j\}_{j=1}^\infty$ be an open cover of M satisfying the conclusions of the preceding lemma. For each j, let $K(j)$ denote the least integer $k > j$ such that $V_j \cap V_k \neq \varnothing$, and let θ_j be an n-form compactly supported in $V_j \cap V_{K(j)}$ whose integral is 1. Let $\{\psi_j\}_{j=1}^\infty$ be a smooth partition of unity subordinate to $\{V_j\}_{j=1}^\infty$.

Now suppose ω is any n-form on M, and let $\omega_j = \psi_j\omega$ for each j. Let $c_1 = \int_{V_1} \omega_1$, so that $\omega_1 - c_1\theta_1$ is compactly supported in V_1 and has zero integral. It follows from Theorem 17.30 that there exists $\eta_1 \in \Omega^{n-1}_c(V_1)$

such that $d\eta_1 = \omega_1 - c_1\theta_1$. Suppose by induction that we have found $\eta_1, \dots, \eta_m$ and constants $c_1, \dots, c_m$ such that for each $j = 1, \dots, m$, $\eta_j \in \Omega_c^{n-1}(V_j)$ and

$$d\eta_j = \left(\omega_j + \sum_{i:K(i)=j} c_i\theta_i\right) - c_j\theta_j. \qquad (*)$$

Let

$$c_{j+1} = \int_{V_{j+1}} \left(\omega_{j+1} + \sum_{i:K(i)=j+1} c_i\theta_i\right).$$

Then by Theorem 17.30, there exists $\eta_{j+1} \in \Omega_c^{n-1}(V_{j+1})$ satisfying the analog of $(*)$ with j replaced by $j+1$. Set $\eta = \sum_{j=1}^{\infty} \eta_j$, with each η_j extended to be zero on $M \smallsetminus V_j$. By local finiteness, this is a smooth $(n-1)$-form on M. It satisfies

$$d\eta = \omega + \sum_{j=1}^{\infty}\left(\sum_{i:K(i)=j} c_i\theta_i\right) - \sum_{j=1}^{\infty} c_j\theta_j.$$

Each term $c_i\theta_i$ appears exactly once in the first sum above, so the two sums cancel each other. □

(7/27/16) **Page 457, line below the second displayed equation:** Change “Theorem 17.31” to “Theorem 17.30.”

(7/12/16) **Page 463, line above equation (17.15):** Insert missing space before “Similarly.”

(7/13/16) **Page 464, end of proof of Corollary 17.42:** Insert “Note that this construction produces a form σ whose support is contained in $U \cap V$.” [This might be useful for solving Problem 18-6.]

(7/12/16) **Page 471, last paragraph:** Replace the sentence starting “The hardest part …” with “The hardest part is showing that the singular chain complex of M can be replaced by a chain complex built out of simplices whose images lie in either U or V, without changing the homology.”

(9/12/17) **Page 487, Problem 18-1, first line:** Change “an oriented smooth manifold” to “a smooth manifold.”

(8/8/18) **Page 489, Problem 18-7(b):** Add to the hint: “In order to use Lemma 17.27, you’ll need to prove the following fact: *Every bounded convex open subset of $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$.* To prove this, let U be such a subset, and without loss of generality assume $0 \in U$. First show that there exists a smooth nonnegative function $f \in C^\infty(U)$ such that $f(0) = 0$ and $f(x) \geq 1/d(x)$ away from a small neighborhood of 0, where $d(x)$ is the distance from x to ∂U. Next, show that $g(x) = 1 + \int_0^1 t^{-1} f(tx)\,dt$ is a smooth positive exhaustion function on U that is nondecreasing along each ray starting at 0. Finally, show that the map $F\colon U \to \mathbb{R}^n$ given by $F(x) = g(x)x$ is a bijective local diffeomorphism. Also, you may use the fact that the conclusion of the five lemma is still true even if the appropriate diagram commutes only up to sign.”

(1/15/13) **Page 491, Example 19.1(c):** Delete the word “unit.”

(5/22/15) **Page 492, line above Proposition 19.2:** Change “lie” to “Lie.”

(12/17/15) **Page 492, proof of Proposition 19.2, fourth line:** Change “Given $p \in M$” to “Given $p \in U$.”

(9/12/16) **Page 506, Lemma 19.24, last line:** Before “left-invariant,” insert “smooth.”

(10/4/17) **Page 518, sentence before Prop. 20.3:** Change “one-parameter subgroups of $\mathrm{GL}(n,\mathbb{R})$” to “one-parameter subgroups of subgroups of $\mathrm{GL}(n,\mathbb{R})$.”

(5/23/16) **Page 521, first displayed equation:** Change $d\Phi_0$ to $d\Phi_e$ (twice).

(5/19/18) **Page 528, just below the displayed equation in the middle of the page:** The smoothness of the map σ_q is not quite immediate from the definition. Replace the three sentences beginning "It follows" with this: " Because S_p is a weakly embedded submanifold by Theorem 19.17, to show that σ_q is a smooth local section of S_p, it suffices to show that it is smooth into $G \times M$ and takes its values in S_p. The first component function is smooth as a map into G by smoothness of group multiplication. To show that the second component is smooth into M as a function of $\widehat{X}$ (and therefore of $\exp X$), you need to use the argument sketched out just below equation (20.10): as in the proof of Prop. 20.8, apply the fundamental theorem on flows to the vector field $\Xi_{(p,X)} = \big(\widehat{X}_g, 0\big)$ on $M \times \mathfrak{g}$. A straightforward computation shows that $\gamma(t) = \big(g \exp tX, \eta_{(\widehat{X})}(t,q)\big)$ is an integral curve of $\widetilde{X}$ starting at (g,q), from which it follows easily that $\sigma_q(g \exp X) = \gamma(1) \in S_p$."

(1/10/17) **Page 537, Problem 20-6(a):** Change $B \in \mathfrak{gl}(n,\mathbb{R})$ to $B \in \mathfrak{sl}(n,\mathbb{R})$.

(5/31/16) **Page 538, Problem 20-11(b):** Here's a better hint, which doesn't require proving part (a) first: "[Hint: Consider the graph of F as a subgroup of $G \times H$.]"

(10/18/17) **Page 542, middle of the paragraph before Example 21.3:** Change "the action of $\mathbb{R}^k$ on $\mathbb{R}^n$" to "the action of $\mathbb{R}^k$ on $\mathbb{R}^k \times \mathbb{R}^n$."

(2/25/18) **Page 548, last two lines:** Allen Hatcher's name is misspelled. (Sorry, Allen.)

(5/23/16) **Page 549, proof of Proposition 21.12, last sentence:** Change the first phrase of that sentence to "Second, if $p, p' \in E$ are in different orbits and $\pi(p) \neq \pi(p')$, ..." Then add the following sentences at the end of the proof: "If p and p' are in different orbits and $\pi(p) = \pi(p')$, let W be an evenly covered neighborhood of $\pi(p)$, and let V, V' be the components of $\pi^{-1}(W)$ containing p and p', respectively. For any $g \in \operatorname{Aut}_\pi(E)$, a simple connectedness argument shows that $g \cdot V$ is a component of $\pi^{-1}(W)$; if it had nontrivial intersection with V it would have to be equal to V, which would imply $g \cdot p = p'$, a contradiction."

(7/26/16) **Page 567, two lines above Proposition 22.8:** Insert "a" before "2-covector."

(10/9/15) **Page 568, Example 22.9(a), first line:** The coordinates should be $\big(x^1, \dots, x^n, y^1, \dots, y^n\big)$. (The last coordinate is y^n, not x^n.)

(3/27/19) **Page 572, middle of the page:** Replace the sentence starting "On the other hand" by this: "On the other hand, the left-hand side is just the ordinary t-derivative of a time-dependent tensor on a fixed vector space, and expanding in terms of a basis shows that it satisfies a similar product rule:"

(10/5/17) **Page 573, statement of Proposition 22.15, second line:** Change "$V\colon J \times M$" to "$V\colon J \times M \to TM$"; and change ψ to θ.

(11/18/17) **Page 583, line 4:** Change $\mathbb{R}^{2n+1} \smallsetminus \{0\}$ to $\mathbb{R}^{2n+2} \smallsetminus \{0\}$.

(7/26/16) **Page 583, third displayed equation:** Should read

$$T \lrcorner\, d\Theta = -2 \sum_{i=1}^{n+1} \big(x^i\, dx^i + y^i\, dy^i\big) = -d\big(|x|^2 + |y|^2\big).$$

(7/26/16) **Page 583, two lines below the third displayed equation:** The formula for $d\Theta(N,T)$ should be $d\Theta(N,T) = 2\big(|x|^2 + |y|^2\big)$.

(11/28/12) **Page 584, Exercise 22.29:** Part (b) should read

$$\text{(b)}\ \ T = \frac{\partial}{\partial z};$$

(8/14/14) **Page 584, paragraph above Theorem 22.33:** Change all occurrences of θ in this paragraph to ψ, to avoid confusion with the use of θ for a contact form elsewhere in this section.

(11/24/17) **Page 585, statement of Theorem 22.34, last line:** Change H to F.

(11/17/12) **Page 587, equation (22.27):** Change both occurrences of $\sigma(s)$ to $\sigma(x)$.

(11/18/17) **Page 592, Problem 22-15:** Add the hypothesis that M is connected.

(9/22/15) **Page 608, Proposition A.41(a):** Insert the following phrase at the beginning of this statement: *With the exception of the word "closed" in part* (*d*).

(7/22/13) **Page 616, Proposition A.77(b), last line:** Change $\widetilde{f}(0)$ to $\widetilde{f}_e(0)$.

(12/19/18) **Page 619, proof of Lemma B.2, fourth line:** Replace "By Exercise B.1(b)" with "If w_1 is equal to one of the v_i's, then the ordered $(n+1)$-tuple $(w_1, v_1, \dots, v_n)$ is linearly dependent; if not, then by Exercise B.1(b), ..."

(9/1/16) **Page 632, Exercise B.29:** Change "by a matrix" to "by a certain matrix" (twice).

(12/19/18) **Page 637, Exercise B.42:** Delete the words "is a homeomorphism that." [Checking that it's a homeomorphism requires the norm topology, which is not defined until later on that page.]

(9/6/16) **Page 637, Exercise B.44:** Change "basis map" to "basis isomorphism."

(12/19/18) **Page 653, proof of Proposition C.21, second paragraph, second line:** Change f to f_D.

(2/25/18) **Page 658, two lines above (C.15):** Change $B_\delta(0)$ to $\overline{B}_\delta(0)$.

(2/25/18) **Page 660, display (C.20):** Change $F^{-1}(x)$ to $F^{-1}(y)$.

(12/2/15) **Page 666, just below the fifth display:** After the sentence ending "by our choice of δ and ε," insert "(If $t < t_0$, interchange t and t_0 in the second line above.)"

(12/2/15) **Page 667, proof of Theorem D.5, second paragraph:** In the first sentence of the paragraph, after "$\overline{J}_1 \subseteq J_0$," insert "and $\theta\big(\overline{J}_1 \times \{x_0\}\big) \subseteq U_0$." Then in the fourth line of that paragraph, change "$\overline{B}_{2c}(y) \subseteq U$" to "$\overline{B}_{2c}(y) \subseteq U_0$." [This is to ensure that $J_1 \times W$ will be contained in the domain of θ.]

(3/19/14) **Page 668, second line:** Change W to $\overline{W}$.

(3/19/14) **Page 668, paragraph below equation (D.10):** In the fourth line of the paragraph, change $\overline{W}$ to W, and in the fifth line, change W to $\overline{W}$.

(2/25/18) **Page 693:** The index entry for "Hatcher, Allen" is misspelled.