Chapter 1
# Smooth Manifolds

This book is about *smooth manifolds*. In the simplest terms, these are spaces that locally look like some Euclidean space $\mathbb{R}^n$, and on which one can do calculus. The most familiar examples, aside from Euclidean spaces themselves, are smooth plane curves such as circles and parabolas, and smooth surfaces such as spheres, tori, paraboloids, ellipsoids, and hyperboloids. Higher-dimensional examples include the set of points in $\mathbb{R}^{n+1}$ at a constant distance from the origin (an *n-sphere*) and graphs of smooth maps between Euclidean spaces.

The simplest manifolds are the topological manifolds, which are topological spaces with certain properties that encode what we mean when we say that they "locally look like" $\mathbb{R}^n$. Such spaces are studied intensively by topologists.

However, many (perhaps most) important applications of manifolds involve calculus. For example, applications of manifold theory to geometry involve such properties as volume and curvature. Typically, volumes are computed by integration, and curvatures are computed by differentiation, so to extend these ideas to manifolds would require some means of making sense of integration and differentiation on a manifold. Applications to classical mechanics involve solving systems of ordinary differential equations on manifolds, and the applications to general relativity (the theory of gravitation) involve solving a system of partial differential equations.

The first requirement for transferring the ideas of calculus to manifolds is some notion of "smoothness." For the simple examples of manifolds we described above, all of which are subsets of Euclidean spaces, it is fairly easy to describe the meaning of smoothness on an intuitive level. For example, we might want to call a curve "smooth" if it has a tangent line that varies continuously from point to point, and similarly a "smooth surface" should be one that has a tangent plane that varies continuously. But for more sophisticated applications it is an undue restriction to require smooth manifolds to be subsets of some ambient Euclidean space. The ambient coordinates and the vector space structure of $\mathbb{R}^n$ are superfluous data that often have nothing to do with the problem at hand. It is a tremendous advantage to be able to work with manifolds as abstract topological spaces, without the excess baggage of such an ambient space. For example, in general relativity, spacetime is modeled as a 4-dimensional smooth manifold that carries a certain geometric structure, called a
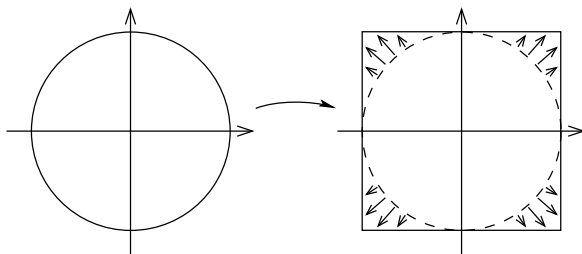
Fig. 1.1 A homeomorphism from a circle to a square

Lorentz metric, whose curvature results in gravitational phenomena. In such a model there is no physical meaning that can be assigned to any higher-dimensional ambient space in which the manifold lives, and including such a space in the model would complicate it needlessly. For such reasons, we need to think of smooth manifolds as abstract topological spaces, not necessarily as subsets of larger spaces.

It is not hard to see that there is no way to define a purely topological property that would serve as a criterion for "smoothness," because it cannot be invariant under homeomorphisms. For example, a circle and a square in the plane are homeomorphic topological spaces (Fig. 1.1), but we would probably all agree that the circle is "smooth," while the square is not. Thus, topological manifolds will not suffice for our purposes. Instead, we will think of a smooth manifold as a set with two layers of structure: first a topology, then a smooth structure.

In the first section of this chapter we describe the first of these structures. A *topological manifold* is a topological space with three special properties that express the notion of being locally like Euclidean space. These properties are shared by Euclidean spaces and by all of the familiar geometric objects that look locally like Euclidean spaces, such as curves and surfaces. We then prove some important topological properties of manifolds that we use throughout the book.

In the next section we introduce an additional structure, called a *smooth structure*, that can be added to a topological manifold to enable us to make sense of derivatives.

Following the basic definitions, we introduce a number of examples of manifolds, so you can have something concrete in mind as you read the general theory. At the end of the chapter we introduce the concept of a *smooth manifold with boundary*, an important generalization of smooth manifolds that will have numerous applications throughout the book, especially in our study of integration in Chapter 16.

Topological Manifolds

In this section we introduce topological manifolds, the most basic type of manifolds. We assume that the reader is familiar with the definition and basic properties of topological spaces, as summarized in Appendix A.

Suppose M is a topological space. We say that M is a ***topological manifold of dimension n*** or a ***topological n-manifold*** if it has the following properties:

- M is a ***Hausdorff space***: for every pair of distinct points $p, q \in M$, there are disjoint open subsets $U, V \subseteq M$ such that $p \in U$ and $q \in V$.
- M is ***second-countable***: there exists a countable basis for the topology of M.
- M is ***locally Euclidean of dimension n***: each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

The third property means, more specifically, that for each $p \in M$ we can find

- an open subset $U \subseteq M$ containing p,
- an open subset $\widehat{U} \subseteq \mathbb{R}^n$, and
- a homeomorphism $\varphi \colon U \to \widehat{U}$.

▶ **Exercise 1.1.** Show that equivalent definitions of manifolds are obtained if instead of allowing U to be homeomorphic to *any* open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.

If M is a topological manifold, we often abbreviate the dimension of M as $\dim M$. Informally, one sometimes writes "Let M^n be a manifold" as shorthand for "Let M be a manifold of dimension n." The superscript n is not part of the name of the manifold, and is usually not included in the notation after the first occurrence.

It is important to note that every topological manifold has, by definition, a specific, well-defined dimension. Thus, we do not consider spaces of mixed dimension, such as the disjoint union of a plane and a line, to be manifolds at all. In Chapter 17, we will use the theory of de Rham cohomology to prove the following theorem, which shows that the dimension of a (nonempty) topological manifold is in fact a topological invariant.

Theorem 1.2 (Topological Invariance of Dimension). *A nonempty n-dimensional topological manifold cannot be homeomorphic to an m-dimensional manifold unless $m = n$.*

For the proof, see Theorem 17.26. In Chapter 2, we will also prove a related but weaker theorem (diffeomorphism invariance of dimension, Theorem 2.17). See also [LeeTM, Chap. 13] for a different proof of Theorem 1.2 using singular homology theory.

The empty set satisfies the definition of a topological n-manifold for every n. For the most part, we will ignore this special case (sometimes without remembering to say so). But because it is useful in certain contexts to allow the empty manifold, we choose not to exclude it from the definition.

The basic example of a topological n-manifold is $\mathbb{R}^n$ itself. It is Hausdorff because it is a metric space, and it is second-countable because the set of all open balls with rational centers and rational radii is a countable basis for its topology.

Requiring that manifolds share these properties helps to ensure that manifolds behave in the ways we expect from our experience with Euclidean spaces. For example, it is easy to verify that in a Hausdorff space, finite subsets are closed and limits of convergent sequences are unique (see Exercise A.11 in Appendix A). The motivation for second-countability is a bit less evident, but it will have important
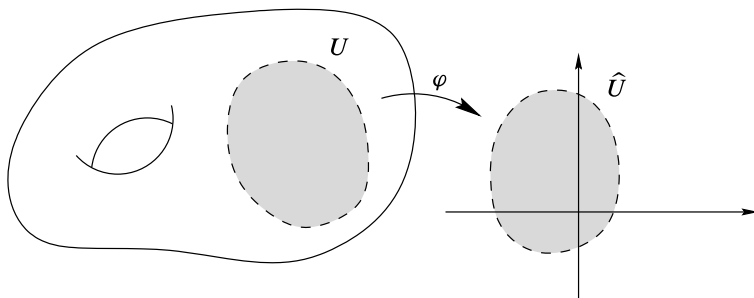
Fig. 1.2   A coordinate chart

consequences throughout the book, mostly based on the existence of partitions of unity (see Chapter 2).

In practice, both the Hausdorff and second-countability properties are usually easy to check, especially for spaces that are built out of other manifolds, because both properties are inherited by subspaces and finite products (Propositions A.17 and A.23). In particular, it follows that every open subset of a topological n-manifold is itself a topological n-manifold (with the subspace topology, of course).

We should note that some authors choose to omit the Hausdorff property or second-countability or both from the definition of manifolds. However, most of the interesting results about manifolds do in fact require these properties, and it is exceedingly rare to encounter a space "in nature" that would be a manifold except for the failure of one or the other of these hypotheses. For a couple of simple examples, see Problems 1-1 and 1-2; for a more involved example (a connected, locally Euclidean, Hausdorff space that is not second-countable), see [LeeTM, Problem 4-6].

Coordinate Charts

Let M be a topological n-manifold. A ***coordinate chart*** (or just a ***chart***) on M is a pair (U, φ), where U is an open subset of M and $\varphi \colon U \to \widehat{U}$ is a homeomorphism from U to an open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$ (Fig. 1.2). By definition of a topological manifold, each point $p \in M$ is contained in the domain of some chart (U, φ). If $\varphi(p) = 0$, we say that the chart is ***centered at p***. If (U, φ) is any chart whose domain contains p, it is easy to obtain a new chart centered at p by subtracting the constant vector $\varphi(p)$.

Given a chart (U, φ), we call the set U a ***coordinate domain***, or a ***coordinate neighborhood*** of each of its points. If, in addition, $\varphi(U)$ is an open ball in $\mathbb{R}^n$, then U is called a ***coordinate ball***; if $\varphi(U)$ is an open cube, U is a ***coordinate cube***. The map φ is called a ***(local) coordinate map***, and the component functions $(x^1, \ldots, x^n)$ of φ, defined by $\varphi(p) = \big(x^1(p), \ldots, x^n(p)\big)$, are called ***local coordinates*** on U. We sometimes write things such as "(U, φ) is a chart containing p" as shorthand for "(U, φ) is a chart whose domain U contains p." If we wish to emphasize the

coordinate functions $(x^1, \ldots, x^n)$ instead of the coordinate map φ, we sometimes denote the chart by $(U, (x^1, \ldots, x^n))$ or $(U, (x^i))$.

Examples of Topological Manifolds

Here are some simple examples.

Example 1.3 (Graphs of Continuous Functions). Let $U \subseteq \mathbb{R}^n$ be an open subset, and let $f : U \to \mathbb{R}^k$ be a continuous function. The **graph of f** is the subset of $\mathbb{R}^n \times \mathbb{R}^k$ defined by

$$\Gamma(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k : x \in U \text{ and } y = f(x)\},$$

with the subspace topology. Let $\pi_1 : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ denote the projection onto the first factor, and let $\varphi : \Gamma(f) \to U$ be the restriction of π_1 to $\Gamma(f)$:

$$\varphi(x, y) = x, \quad (x, y) \in \Gamma(f).$$

Because φ is the restriction of a continuous map, it is continuous; and it is a homeomorphism because it has a continuous inverse given by $\varphi^{-1}(x) = (x, f(x))$. Thus $\Gamma(f)$ is a topological manifold of dimension n. In fact, $\Gamma(f)$ is homeomorphic to U itself, and $(\Gamma(f), \varphi)$ is a global coordinate chart, called **graph coordinates**. The same observation applies to any subset of $\mathbb{R}^{n+k}$ defined by setting any k of the coordinates (not necessarily the last k) equal to some continuous function of the other n, which are restricted to lie in an open subset of $\mathbb{R}^n$. //

Example 1.4 (Spheres). For each integer $n \geq 0$, the unit n-sphere $\mathbb{S}^n$ is Hausdorff and second-countable because it is a topological subspace of $\mathbb{R}^{n+1}$. To show that it is locally Euclidean, for each index $i = 1, \ldots, n + 1$ let U_i^+ denote the subset of $\mathbb{R}^{n+1}$ where the ith coordinate is positive:

$$U_i^+ = \{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : x^i > 0\}.$$

(See Fig. 1.3.) Similarly, U_i^- is the set where $x^i < 0$.

Let $f : \mathbb{B}^n \to \mathbb{R}$ be the continuous function

$$f(u) = \sqrt{1 - |u|^2}.$$

Then for each $i = 1, \ldots, n + 1$, it is easy to check that $U_i^+ \cap \mathbb{S}^n$ is the graph of the function

$$x^i = f(x^1, \ldots, \widehat{x^i}, \ldots, x^{n+1}),$$

where the hat indicates that x^i is omitted. Similarly, $U_i^- \cap \mathbb{S}^n$ is the graph of

$$x^i = -f(x^1, \ldots, \widehat{x^i}, \ldots, x^{n+1}).$$

Thus, each subset $U_i^\pm \cap \mathbb{S}^n$ is locally Euclidean of dimension n, and the maps $\varphi_i^\pm : U_i^\pm \cap \mathbb{S}^n \to \mathbb{B}^n$ given by

$$\varphi_i^\pm(x^1, \ldots, x^{n+1}) = (x^1, \ldots, \widehat{x^i}, \ldots, x^{n+1})$$
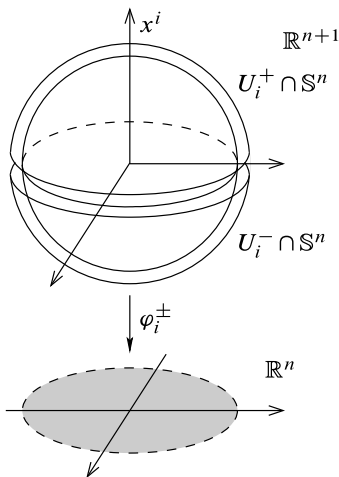
Fig. 1.3 Charts for $\mathbb{S}^n$

are graph coordinates for $\mathbb{S}^n$. Since each point of $\mathbb{S}^n$ is in the domain of at least one of these $2n + 2$ charts, $\mathbb{S}^n$ is a topological n-manifold.   //

Example 1.5 (Projective Spaces). The ***n-dimensional real projective space***, denoted by $\mathbb{RP}^n$ (or sometimes just $\mathbb{P}^n$), is defined as the set of 1-dimensional linear subspaces of $\mathbb{R}^{n+1}$, with the quotient topology determined by the natural map $\pi \colon \mathbb{R}^{n+1} \smallsetminus \{0\} \to \mathbb{RP}^n$ sending each point $x \in \mathbb{R}^{n+1} \smallsetminus \{0\}$ to the subspace spanned by x. The 2-dimensional projective space $\mathbb{RP}^2$ is called the ***projective plane***. For any point $x \in \mathbb{R}^{n+1} \smallsetminus \{0\}$, let $[x] = \pi(x) \in \mathbb{RP}^n$ denote the line spanned by x.

For each $i = 1, \ldots, n + 1$, let $\widetilde{U}_i \subseteq \mathbb{R}^{n+1} \smallsetminus \{0\}$ be the set where $x^i \neq 0$, and let $U_i = \pi(\widetilde{U}_i) \subseteq \mathbb{RP}^n$. Since $\widetilde{U}_i$ is a saturated open subset, U_i is open and $\pi|_{\widetilde{U}_i} \colon \widetilde{U}_i \to U_i$ is a quotient map (see Theorem A.27). Define a map $\varphi_i \colon U_i \to \mathbb{R}^n$ by

$$\varphi_i \left[x^1, \ldots, x^{n+1} \right] = \left(\frac{x^1}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^{n+1}}{x^i} \right).$$

This map is well defined because its value is unchanged by multiplying x by a nonzero constant. Because $\varphi_i \circ \pi$ is continuous, φ_i is continuous by the characteristic property of quotient maps (Theorem A.27). In fact, φ_i is a homeomorphism, because it has a continuous inverse given by

$$\varphi_i^{-1} \left(u^1, \ldots, u^n \right) = \left[u^1, \ldots, u^{i-1}, 1, u^i, \ldots, u^n \right],$$

as you can check. Geometrically, $\varphi([x]) = u$ means $(u, 1)$ is the point in $\mathbb{R}^{n+1}$ where the line $[x]$ intersects the affine hyperplane where $x^i = 1$ (Fig. 1.4). Because the sets $U_1, \ldots, U_{n+1}$ cover $\mathbb{RP}^n$, this shows that $\mathbb{RP}^n$ is locally Euclidean of dimension n. The Hausdorff and second-countability properties are left as exercises.   //
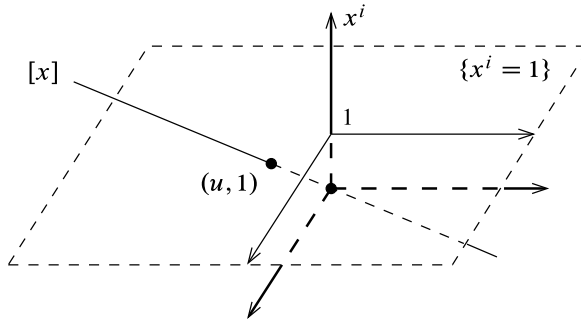
Fig. 1.4 A chart for $\mathbb{RP}^n$

▶ **Exercise 1.6.** Show that $\mathbb{RP}^n$ is Hausdorff and second-countable, and is therefore a topological n-manifold.

▶ **Exercise 1.7.** Show that $\mathbb{RP}^n$ is compact. [Hint: show that the restriction of π to $\mathbb{S}^n$ is surjective.]

Example 1.8 (Product Manifolds). Suppose $M_1, \ldots, M_k$ are topological manifolds of dimensions $n_1, \ldots, n_k$, respectively. The product space $M_1 \times \cdots \times M_k$ is shown to be a topological manifold of dimension $n_1 + \cdots + n_k$ as follows. It is Hausdorff and second-countable by Propositions A.17 and A.23, so only the locally Euclidean property needs to be checked. Given any point $(p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$, we can choose a coordinate chart (U_i, φ_i) for each M_i with $p_i \in U_i$. The product map

$$\varphi_1 \times \cdots \times \varphi_k \colon U_1 \times \cdots \times U_k \to \mathbb{R}^{n_1 + \cdots + n_k}$$

is a homeomorphism onto its image, which is a product open subset of $\mathbb{R}^{n_1 + \cdots + n_k}$. Thus, $M_1 \times \cdots \times M_k$ is a topological manifold of dimension $n_1 + \cdots + n_k$, with charts of the form $(U_1 \times \cdots \times U_k, \varphi_1 \times \cdots \times \varphi_k)$.   //

Example 1.9 (Tori). For a positive integer n, the ***n-torus*** (plural: ***tori***) is the product space $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. By the discussion above, it is a topological n-manifold. (The 2-torus is usually called simply ***the torus***.)   //

Topological Properties of Manifolds

As topological spaces go, manifolds are quite special, because they share so many important properties with Euclidean spaces. Here we discuss a few such properties that will be of use to us throughout the book.

Most of the properties we discuss in this section depend on the fact that every manifold possesses a particularly well-behaved basis for its topology.

Lemma 1.10. *Every topological manifold has a countable basis of precompact coordinate balls.*

Proof. Let M be a topological n-manifold. First we consider the special case in which M can be covered by a single chart. Suppose $\varphi\colon M \to \widehat{U} \subseteq \mathbb{R}^n$ is a global coordinate map, and let $\mathcal{B}$ be the collection of all open balls $B_r(x) \subseteq \mathbb{R}^n$ such that r is rational, x has rational coordinates, and $B_{r'}(x) \subseteq \widehat{U}$ for some $r' > r$. Each such ball is precompact in $\widehat{U}$, and it is easy to check that $\mathcal{B}$ is a countable basis for the topology of $\widehat{U}$. Because φ is a homeomorphism, it follows that the collection of sets of the form $\varphi^{-1}(B)$ for $B \in \mathcal{B}$ is a countable basis for the topology of M, consisting of precompact coordinate balls, with the restrictions of φ as coordinate maps.

Now let M be an arbitrary n-manifold. By definition, each point of M is in the domain of a chart. Because every open cover of a second-countable space has a countable subcover (Proposition A.16), M is covered by countably many charts $\{(U_i, \varphi_i)\}$. By the argument in the preceding paragraph, each coordinate domain U_i has a countable basis of coordinate balls that are precompact in U_i, and the union of all these countable bases is a countable basis for the topology of M. If $V \subseteq U_i$ is one of these balls, then the closure of V in U_i is compact, and because M is Hausdorff, it is closed in M. It follows that the closure of V in M is the same as its closure in U_i, so V is precompact in M as well. $\qquad\square$

Connectivity

The existence of a basis of coordinate balls has important consequences for the connectivity properties of manifolds. Recall that a topological space X is

- *connected* if there do not exist two disjoint, nonempty, open subsets of X whose union is X;
- *path-connected* if every pair of points in X can be joined by a path in X; and
- *locally path-connected* if X has a basis of path-connected open subsets.

(See Appendix A.) The following proposition shows that connectivity and path connectivity coincide for manifolds.

Proposition 1.11. *Let M be a topological manifold.*

(a) *M is locally path-connected.*
(b) *M is connected if and only if it is path-connected.*
(c) *The components of M are the same as its path components.*
(d) *M has countably many components, each of which is an open subset of M and a connected topological manifold.*

Proof. Since each coordinate ball is path-connected, (a) follows from the fact that M has a basis of coordinate balls. Parts (b) and (c) are immediate consequences of (a) and Proposition A.43. To prove (d), note that each component is open in M by Proposition A.43, so the collection of components is an open cover of M. Because M is second-countable, this cover must have a countable subcover. But since the components are all disjoint, the cover must have been countable to begin with, which is to say that M has only countably many components. Because the components are open, they are connected topological manifolds in the subspace topology. $\qquad\square$

Local Compactness and Paracompactness

The next topological property of manifolds that we need is local compactness (see Appendix A for the definition).

Proposition 1.12 (Manifolds Are Locally Compact). *Every topological manifold is locally compact.*

Proof. Lemma 1.10 showed that every manifold has a basis of precompact open subsets. ☐

Another key topological property possessed by manifolds is called *paracompactness*. It is a consequence of local compactness and second-countability, and in fact is one of the main reasons why second-countability is included in the definition of manifolds.

Let M be a topological space. A collection $\mathfrak{X}$ of subsets of M is said to be **locally finite** if each point of M has a neighborhood that intersects at most finitely many of the sets in $\mathfrak{X}$. Given a cover $\mathcal{U}$ of M, another cover $\mathcal{V}$ is called a **refinement of** $\mathcal{U}$ if for each $V \in \mathcal{V}$ there exists some $U \in \mathcal{U}$ such that $V \subseteq U$. We say that M is **paracompact** if every open cover of M admits an open, locally finite refinement.

Lemma 1.13. *Suppose $\mathfrak{X}$ is a locally finite collection of subsets of a topological space M.*

(a) *The collection $\{\bar{X} : X \in \mathfrak{X}\}$ is also locally finite.*
(b) $\overline{\bigcup_{X \in \mathfrak{X}} X} = \bigcup_{X \in \mathfrak{X}} \bar{X}$.

▶ **Exercise 1.14.** Prove the preceding lemma.

Theorem 1.15 (Manifolds Are Paracompact). *Every topological manifold is paracompact. In fact, given a topological manifold M, an open cover $\mathfrak{X}$ of M, and any basis $\mathcal{B}$ for the topology of M, there exists a countable, locally finite open refinement of $\mathfrak{X}$ consisting of elements of $\mathcal{B}$.*

Proof. Given M, $\mathfrak{X}$, and $\mathcal{B}$ as in the hypothesis of the theorem, let $(K_j)_{j=1}^{\infty}$ be an exhaustion of M by compact sets (Proposition A.60). For each j, let $V_j = K_{j+1} \smallsetminus \operatorname{Int} K_j$ and $W_j = \operatorname{Int} K_{j+2} \smallsetminus K_{j-1}$ (where we interpret K_j as $\varnothing$ if $j < 1$). Then V_j is a compact set contained in the open subset W_j. For each $x \in V_j$, there is some $X_x \in \mathfrak{X}$ containing x, and because $\mathcal{B}$ is a basis, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq X_x \cap W_j$. The collection of all such sets B_x as x ranges over V_j is an open cover of V_j, and thus has a finite subcover. The union of all such finite subcovers as j ranges over the positive integers is a countable open cover of M that refines $\mathfrak{X}$. Because the finite subcover of V_j consists of sets contained in W_j, and $W_j \cap W_{j'} = \varnothing$ except when $j - 2 \le j' \le j + 2$, the resulting cover is locally finite. ☐

Problem 1-5 shows that, at least for connected spaces, paracompactness can be used as a substitute for second-countability in the definition of manifolds.

Fundamental Groups of Manifolds

The following result about fundamental groups of manifolds will be important in our study of covering manifolds in Chapter 4. For a brief review of the fundamental group, see Appendix A.

Proposition 1.16. *The fundamental group of a topological manifold is countable.*

Proof. Let M be a topological manifold. By Lemma 1.10, there is a countable collection $\mathcal{B}$ of coordinate balls covering M. For any pair of coordinate balls $B, B' \in \mathcal{B}$, the intersection $B \cap B'$ has at most countably many components, each of which is path-connected. Let $\mathcal{X}$ be a countable set containing a point from each component of $B \cap B'$ for each $B, B' \in \mathcal{B}$ (including $B = B'$). For each $B \in \mathcal{B}$ and each $x, x' \in \mathcal{X}$ such that $x, x' \in B$, let $h^B_{x,x'}$ be some path from x to x' in B.

Since the fundamental groups based at any two points in the same component of M are isomorphic, and $\mathcal{X}$ contains at least one point in each component of M, we may as well choose a point $p \in \mathcal{X}$ as base point. Define a ***special loop*** to be a loop based at p that is equal to a finite product of paths of the form $h^B_{x,x'}$. Clearly, the set of special loops is countable, and each special loop determines an element of $\pi_1(M, p)$. To show that $\pi_1(M, p)$ is countable, therefore, it suffices to show that each element of $\pi_1(M, p)$ is represented by a special loop.

Suppose $f : [0, 1] \to M$ is a loop based at p. The collection of components of sets of the form $f^{-1}(B)$ as B ranges over $\mathcal{B}$ is an open cover of $[0, 1]$, so by compactness it has a finite subcover. Thus, there are finitely many numbers $0 = a_0 < a_1 < \cdots < a_k = 1$ such that $[a_{i-1}, a_i] \subseteq f^{-1}(B)$ for some $B \subseteq \mathcal{B}$. For each i, let f_i be the restriction of f to the interval $[a_{i-1}, a_i]$, reparametrized so that its domain is $[0, 1]$, and let $B_i \in \mathcal{B}$ be a coordinate ball containing the image of f_i. For each i, we have $f(a_i) \in B_i \cap B_{i+1}$, and there is some $x_i \in \mathcal{X}$ that lies in the same component of $B_i \cap B_{i+1}$ as $f(a_i)$. Let g_i be a path in $B_i \cap B_{i+1}$ from x_i to $f(a_i)$ (Fig. 1.5), with the understanding that $x_0 = x_k = p$, and g_0 and g_k are both equal to the constant path c_p based at p. Then, because $\overline{g}_i \cdot g_i$ is path-homotopic to a constant path (where $\overline{g}_i(t) = g_i(1 - t)$ is the reverse path of g_i),

$$f \sim f_1 \cdot \cdots \cdot f_k$$
$$\sim g_0 \cdot f_1 \cdot \overline{g}_1 \cdot g_1 \cdot f_2 \cdot \overline{g}_2 \cdot \cdots \cdot \overline{g}_{k-1} \cdot g_{k-1} \cdot f_k \cdot \overline{g}_k$$
$$\sim \widetilde{f}_1 \cdot \widetilde{f}_2 \cdot \cdots \cdot \widetilde{f}_k,$$

where $\widetilde{f}_i = g_{i-1} \cdot f_i \cdot \overline{g}_i$. For each i, $\widetilde{f}_i$ is a path in B_i from x_{i-1} to x_i. Since B_i is simply connected, $\widetilde{f}_i$ is path-homotopic to $h^{B_i}_{x_{i-1}, x_i}$. It follows that f is path-homotopic to a special loop, as claimed. $\qquad \square$

Smooth Structures

The definition of manifolds that we gave in the preceding section is sufficient for studying topological properties of manifolds, such as compactness, connectedness,
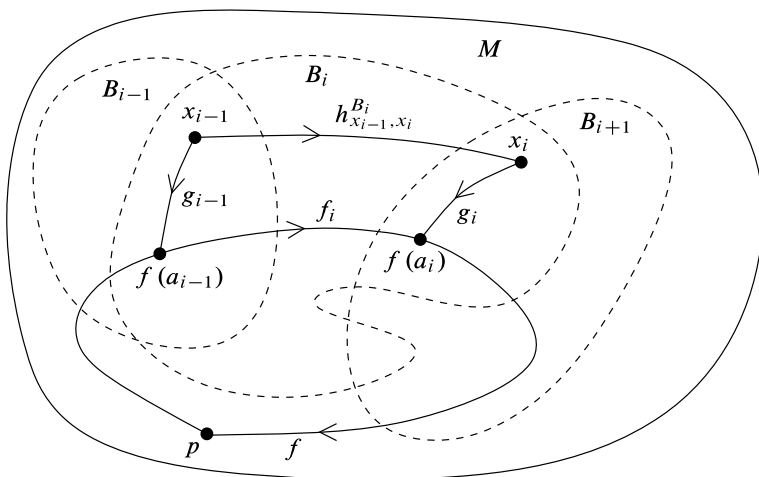
**Fig. 1.5** The fundamental group of a manifold is countable

simple connectivity, and the problem of classifying manifolds up to homeomorphism. However, in the entire theory of topological manifolds there is no mention of calculus. There is a good reason for this: however we might try to make sense of derivatives of functions on a manifold, such derivatives cannot be invariant under homeomorphisms. For example, the map $\varphi \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $\varphi(u, v) = \left(u^{1/3}, v^{1/3}\right)$ is a homeomorphism, and it is easy to construct differentiable functions $f \colon \mathbb{R}^2 \to \mathbb{R}$ such that $f \circ \varphi$ is not differentiable at the origin. (The function $f(x, y) = x$ is one such.)

To make sense of derivatives of real-valued functions, curves, or maps between manifolds, we need to introduce a new kind of manifold called a *smooth manifold*. It will be a topological manifold with some extra structure in addition to its topology, which will allow us to decide which functions to or from the manifold are smooth.

The definition will be based on the calculus of maps between Euclidean spaces, so let us begin by reviewing some basic terminology about such maps. If U and V are open subsets of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, a function $F \colon U \to V$ is said to be ***smooth*** (or ***C^∞***, or ***infinitely differentiable***) if each of its component functions has continuous partial derivatives of all orders. If in addition F is bijective and has a smooth inverse map, it is called a ***diffeomorphism***. A diffeomorphism is, in particular, a homeomorphism.

A review of some important properties of smooth maps is given in Appendix C. You should be aware that some authors define the word *smooth* differently—for example, to mean continuously differentiable or merely differentiable. On the other hand, some use the word *differentiable* to mean what we call *smooth*. Throughout this book, *smooth* is synonymous with C^∞.

To see what additional structure on a topological manifold might be appropriate for discerning which maps are smooth, consider an arbitrary topological n-manifold M. Each point in M is in the domain of a coordinate map $\varphi \colon U \to \hat{U} \subseteq \mathbb{R}^n$.
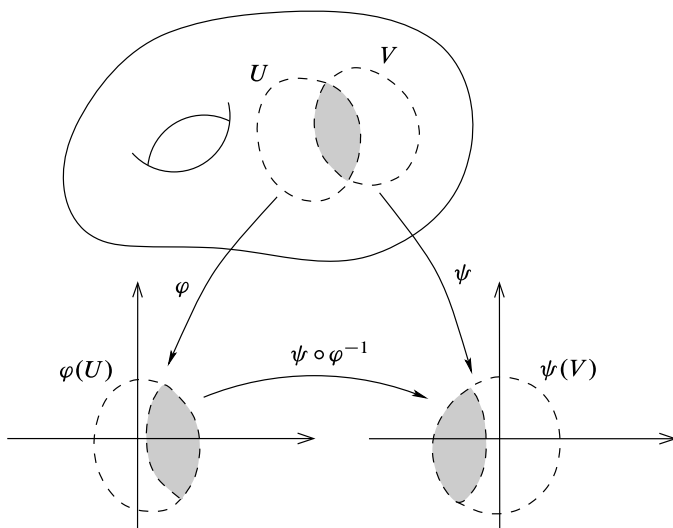
Fig. 1.6   A transition map

A plausible definition of a smooth function on M would be to say that $f : M \to \mathbb{R}$ is smooth if and only if the composite function $f \circ \varphi^{-1} \colon \widehat{U} \to \mathbb{R}$ is smooth in the sense of ordinary calculus. But this will make sense only if this property is independent of the choice of coordinate chart. To guarantee this independence, we will restrict our attention to "smooth charts." Since smoothness is not a homeomorphism-invariant property, the way to do this is to consider the collection of all smooth charts as a new kind of structure on M.

With this motivation in mind, we now describe the details of the construction.

Let M be a topological n-manifold. If (U, φ), (V, ψ) are two charts such that $U \cap V \neq \varnothing$, the composite map $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$ is called the ***transition map from φ to ψ*** (Fig. 1.6). It is a composition of homeomorphisms, and is therefore itself a homeomorphism. Two charts (U, φ) and (V, ψ) are said to be ***smoothly compatible*** if either $U \cap V = \varnothing$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism. Since $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open subsets of $\mathbb{R}^n$, smoothness of this map is to be interpreted in the ordinary sense of having continuous partial derivatives of all orders.

We define an ***atlas for M*** to be a collection of charts whose domains cover M. An atlas $\mathcal{A}$ is called a ***smooth atlas*** if any two charts in $\mathcal{A}$ are smoothly compatible with each other.

To show that an atlas is smooth, we need only verify that each transition map $\psi \circ \varphi^{-1}$ is smooth whenever (U, φ) and (V, ψ) are charts in $\mathcal{A}$; once we have proved this, it follows that $\psi \circ \varphi^{-1}$ is a diffeomorphism because its inverse $\left(\psi \circ \varphi^{-1} \right)^{-1} = \varphi \circ \psi^{-1}$ is one of the transition maps we have already shown to be smooth. Alternatively, given two particular charts (U, φ) and (V, ψ), it is often easiest to show that

they are smoothly compatible by verifying that $\psi \circ \varphi^{-1}$ is smooth and injective with nonsingular Jacobian at each point, and appealing to Corollary C.36.

Our plan is to define a "smooth structure" on M by giving a smooth atlas, and to define a function $f : M \to \mathbb{R}$ to be smooth if and only if $f \circ \varphi^{-1}$ is smooth in the sense of ordinary calculus for each coordinate chart (U, φ) in the atlas. There is one minor technical problem with this approach: in general, there will be many possible atlases that give the "same" smooth structure, in that they all determine the same collection of smooth functions on M. For example, consider the following pair of atlases on $\mathbb{R}^n$:

$$\mathcal{A}_1 = \left\{ \left(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n} \right) \right\},$$

$$\mathcal{A}_2 = \left\{ \left(B_1(x), \mathrm{Id}_{B_1(x)} \right) : x \in \mathbb{R}^n \right\}.$$

Although these are different smooth atlases, clearly a function $f : \mathbb{R}^n \to \mathbb{R}$ is smooth with respect to either atlas if and only if it is smooth in the sense of ordinary calculus.

We could choose to define a smooth structure as an equivalence class of smooth atlases under an appropriate equivalence relation. However, it is more straightforward to make the following definition: a smooth atlas $\mathcal{A}$ on M is **maximal** if it is not properly contained in any larger smooth atlas. This just means that any chart that is smoothly compatible with every chart in $\mathcal{A}$ is already in $\mathcal{A}$. (Such a smooth atlas is also said to be **complete**.)

Now we can define the main concept of this chapter. If M is a topological manifold, a **smooth structure on M** is a maximal smooth atlas. A **smooth manifold** is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M. When the smooth structure is understood, we usually omit mention of it and just say "M is a smooth manifold." Smooth structures are also called **differentiable structures** or **C^∞ structures** by some authors. We also use the term **smooth manifold structure** to mean a manifold topology together with a smooth structure.

We emphasize that a smooth structure is an additional piece of data that must be added to a topological manifold before we are entitled to talk about a "smooth manifold." In fact, a given topological manifold may have many different smooth structures (see Example 1.23 and Problem 1-6). On the other hand, it is not always possible to find a smooth structure on a given topological manifold: there exist topological manifolds that admit no smooth structures at all. (The first example was a compact 10-dimensional manifold found in 1960 by Michel Kervaire [Ker60].)

It is generally not very convenient to define a smooth structure by explicitly describing a maximal smooth atlas, because such an atlas contains very many charts. Fortunately, we need only specify *some* smooth atlas, as the next proposition shows.

Proposition 1.17. *Let M be a topological manifold.*

(a) *Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called the **smooth structure determined by $\mathcal{A}$**.*

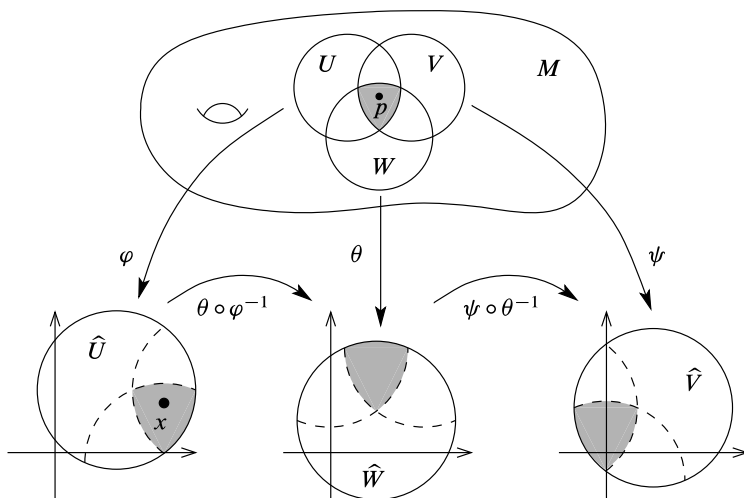(b) *Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.*

Fig. 1.7 Proof of Proposition 1.17(a)

Proof. Let $\mathcal{A}$ be a smooth atlas for M, and let $\overline{\mathcal{A}}$ denote the set of all charts that are smoothly compatible with every chart in $\mathcal{A}$. To show that $\overline{\mathcal{A}}$ is a smooth atlas, we need to show that any two charts of $\overline{\mathcal{A}}$ are smoothly compatible with each other, which is to say that for any (U, φ), $(V, \psi) \in \overline{\mathcal{A}}$, the map $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$ is smooth.

Let $x = \varphi(p) \in \varphi(U \cap V)$ be arbitrary. Because the domains of the charts in $\mathcal{A}$ cover M, there is some chart $(W, \theta) \in \mathcal{A}$ such that $p \in W$ (Fig. 1.7). Since every chart in $\overline{\mathcal{A}}$ is smoothly compatible with (W, θ), both of the maps $\theta \circ \varphi^{-1}$ and $\psi \circ \theta^{-1}$ are smooth where they are defined. Since $p \in U \cap V \cap W$, it follows that $\psi \circ \varphi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ \varphi^{-1})$ is smooth on a neighborhood of x. Thus, $\psi \circ \varphi^{-1}$ is smooth in a neighborhood of each point in $\varphi(U \cap V)$. Therefore, $\overline{\mathcal{A}}$ is a smooth atlas. To check that it is maximal, just note that any chart that is smoothly compatible with every chart in $\overline{\mathcal{A}}$ must in particular be smoothly compatible with every chart in $\mathcal{A}$, so it is already in $\overline{\mathcal{A}}$. This proves the existence of a maximal smooth atlas containing $\mathcal{A}$. If $\mathcal{B}$ is any other maximal smooth atlas containing $\mathcal{A}$, each of its charts is smoothly compatible with each chart in $\mathcal{A}$, so $\mathcal{B} \subseteq \overline{\mathcal{A}}$. By maximality of $\mathcal{B}$, $\mathcal{B} = \overline{\mathcal{A}}$.

The proof of (b) is left as an exercise.   □

▶ **Exercise 1.18.** Prove Proposition 1.17(b).

For example, if a topological manifold M can be covered by a single chart, the smooth compatibility condition is trivially satisfied, so any such chart automatically determines a smooth structure on M.

It is worth mentioning that the notion of smooth structure can be generalized in several different ways by changing the compatibility requirement for charts. For example, if we replace the requirement that charts be smoothly compatible by the weaker requirement that each transition map $\psi \circ \varphi^{-1}$ (and its inverse) be of

class C^k, we obtain the definition of a C^k *structure*. Similarly, if we require that each transition map be real-analytic (i.e., expressible as a convergent power series in a neighborhood of each point), we obtain the definition of a ***real-analytic structure***, also called a C^ω *structure*. If M has even dimension $n = 2m$, we can identify $\mathbb{R}^{2m}$ with $\mathbb{C}^m$ and require that the transition maps be complex-analytic; this determines a ***complex-analytic structure***. A manifold endowed with one of these structures is called a C^k *manifold*, ***real-analytic manifold***, or ***complex manifold***, respectively. (Note that a C^0 manifold is just a topological manifold.) We do not treat any of these other kinds of manifolds in this book, but they play important roles in analysis, so it is useful to know the definitions.

Local Coordinate Representations

If M is a smooth manifold, any chart (U, φ) contained in the given maximal smooth atlas is called a ***smooth chart***, and the corresponding coordinate map φ is called a ***smooth coordinate map***. It is useful also to introduce the terms ***smooth coordinate domain*** or ***smooth coordinate neighborhood*** for the domain of a smooth coordinate chart. A ***smooth coordinate ball*** means a smooth coordinate domain whose image under a smooth coordinate map is a ball in Euclidean space. A ***smooth coordinate cube*** is defined similarly.

It is often useful to restrict attention to coordinate balls whose closures sit nicely inside larger coordinate balls. We say a set $B \subseteq M$ is a ***regular coordinate ball*** if there is a smooth coordinate ball $B' \supseteq \bar{B}$ and a smooth coordinate map $\varphi \colon B' \to \mathbb{R}^n$ such that for some positive real numbers $r < r'$,

$$\varphi(B) = B_r(0), \qquad \varphi(\bar{B}) = \bar{B}_r(0), \quad \text{and} \quad \varphi(B') = B_{r'}(0).$$

Because $\bar{B}$ is homeomorphic to $\bar{B}_r(0)$, it is compact, and thus every regular coordinate ball is precompact in M. The next proposition gives a slight improvement on Lemma 1.10 for smooth manifolds. Its proof is a straightforward adaptation of the proof of that lemma.

Proposition 1.19. *Every smooth manifold has a countable basis of regular coordinate balls.*

▶ **Exercise 1.20.** Prove Proposition 1.19.

Here is how one usually thinks about coordinate charts on a smooth manifold. Once we choose a smooth chart (U, φ) on M, the coordinate map $\varphi \colon U \to \widehat{U} \subseteq \mathbb{R}^n$ can be thought of as giving a temporary *identification* between U and $\widehat{U}$. Using this identification, while we work in this chart, we can think of U simultaneously as an open subset of M and as an open subset of $\mathbb{R}^n$. You can visualize this identification by thinking of a "grid" drawn on U representing the preimages of the coordinate lines under φ (Fig. 1.8). Under this identification, we can represent a point $p \in U$ by its coordinates $(x^1, \ldots, x^n) = \varphi(p)$, and think of this n-tuple as *being* the
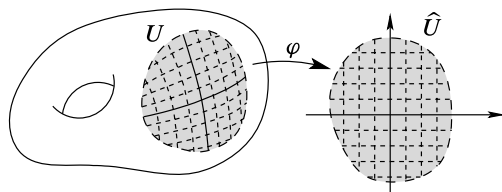
Fig. 1.8 A coordinate grid

point p. We typically express this by saying "$(x^1, \ldots, x^n)$ is the (local) coordinate representation for p" or "$p = (x^1, \ldots, x^n)$" in local coordinates."

Another way to look at it is that by means of our identification $U \leftrightarrow \hat{U}$, we can think of φ as the identity map and suppress it from the notation. This takes a bit of getting used to, but the payoff is a huge simplification of the notation in many situations. You just need to remember that the identification is in general only local, and depends heavily on the choice of coordinate chart.

You are probably already used to such identifications from your study of multivariable calculus. The most common example is ***polar coordinates*** (r, θ) in the plane, defined implicitly by the relation $(x, y) = (r \cos \theta, r \sin \theta)$ (see Example C.37). On an appropriate open subset such as $U = \{(x, y) : x > 0\} \subseteq \mathbb{R}^2$, (r, θ) can be expressed as smooth functions of (x, y), and the map that sends (x, y) to the corresponding (r, θ) is a smooth coordinate map with respect to the standard smooth structure on $\mathbb{R}^2$. Using this map, we can write a given point $p \in U$ either as $p = (x, y)$ in standard coordinates or as $p = (r, \theta)$ in polar coordinates, where the two coordinate representations are related by $(r, \theta) = \left(\sqrt{x^2 + y^2}, \tan^{-1} y/x\right)$ and $(x, y) = (r \cos \theta, r \sin \theta)$. Other polar coordinate charts can be obtained by restricting (r, θ) to other open subsets of $\mathbb{R}^2 \smallsetminus \{0\}$.

The fact that manifolds do not come with any predetermined choice of coordinates is both a blessing and a curse. The flexibility to choose coordinates more or less arbitrarily can be a big advantage in approaching problems in manifold theory, because the coordinates can often be chosen to simplify some aspect of the problem at hand. But we pay for this flexibility by being obliged to ensure that any objects we wish to define globally on a manifold are not dependent on a particular choice of coordinates. There are generally two ways of doing this: either by writing down a coordinate-dependent definition and then proving that the definition gives the same results in any coordinate chart, or by writing down a definition that is manifestly coordinate-independent (often called an *invariant definition*). We will use the coordinate-dependent approach in a few circumstances where it is notably simpler, but for the most part we will give coordinate-free definitions whenever possible. The need for such definitions accounts for much of the abstraction of modern manifold theory. One of the most important skills you will need to acquire in order to use manifold theory effectively is an ability to switch back and forth easily between invariant descriptions and their coordinate counterparts.

Examples of Smooth Manifolds

Before proceeding further with the general theory, let us survey some examples of smooth manifolds.

Example 1.21 (0-Dimensional Manifolds). A topological manifold M of dimension 0 is just a countable discrete space. For each point $p \in M$, the only neighborhood of p that is homeomorphic to an open subset of $\mathbb{R}^0$ is $\{p\}$ itself, and there is exactly one coordinate map $\varphi \colon \{p\} \to \mathbb{R}^0$. Thus, the set of all charts on M trivially satisfies the smooth compatibility condition, and each 0-dimensional manifold has a unique smooth structure.   //

Example 1.22 (Euclidean Spaces). For each nonnegative integer n, the Euclidean space $\mathbb{R}^n$ is a smooth n-manifold with the smooth structure determined by the atlas consisting of the single chart $(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})$. We call this the ***standard smooth structure on $\mathbb{R}^n$*** and the resulting coordinate map ***standard coordinates***. Unless we explicitly specify otherwise, we always use this smooth structure on $\mathbb{R}^n$. With respect to this smooth structure, the smooth coordinate charts for $\mathbb{R}^n$ are exactly those charts (U, φ) such that φ is a diffeomorphism (in the sense of ordinary calculus) from U to another open subset $\hat{U} \subseteq \mathbb{R}^n$.   //

Example 1.23 (Another Smooth Structure on $\mathbb{R}$). Consider the homeomorphism $\psi \colon \mathbb{R} \to \mathbb{R}$ given by

$$\psi(x) = x^3. \tag{1.1}$$

The atlas consisting of the single chart $(\mathbb{R}, \psi)$ defines a smooth structure on $\mathbb{R}$. This chart is not smoothly compatible with the standard smooth structure, because the transition map $\mathrm{Id}_{\mathbb{R}} \circ \psi^{-1}(y) = y^{1/3}$ is not smooth at the origin. Therefore, the smooth structure defined on $\mathbb{R}$ by ψ is not the same as the standard one. Using similar ideas, it is not hard to construct many distinct smooth structures on any given positive-dimensional topological manifold, as long as it has one smooth structure to begin with (see Problem 1-6).   //

Example 1.24 (Finite-Dimensional Vector Spaces). Let V be a finite-dimensional real vector space. Any norm on V determines a topology, which is independent of the choice of norm (Exercise B.49). With this topology, V is a topological n-manifold, and has a natural smooth structure defined as follows. Each (ordered) basis $(E_1, \ldots, E_n)$ for V defines a basis isomorphism $E \colon \mathbb{R}^n \to V$ by

$$E(x) = \sum_{i=1}^{n} x^i E_i.$$

This map is a homeomorphism, so (V, E^{-1}) is a chart. If $(\widetilde{E}_1, \ldots, \widetilde{E}_n)$ is any other basis and $\widetilde{E}(x) = \sum_j x^j \widetilde{E}_j$ is the corresponding isomorphism, then there is some invertible matrix (A_i^j) such that $E_i = \sum_j A_i^j \widetilde{E}_j$ for each i. The transition map between the two charts is then given by $\widetilde{E}^{-1} \circ E(x) = \widetilde{x}$, where $\widetilde{x} = (\widetilde{x}^1, \ldots, \widetilde{x}^n)$

is determined by

$$\sum_{j=1}^{n} \tilde{x}^j \, \tilde{E}_j = \sum_{i=1}^{n} x^i \, E_i = \sum_{i,j=1}^{n} x^i \, A_i^j \, \tilde{E}_j .$$

It follows that $\tilde{x}^j = \sum_i A_i^j \, x^i$. Thus, the map sending x to $\tilde{x}$ is an invertible linear map and hence a diffeomorphism, so any two such charts are smoothly compatible. The collection of all such charts thus defines a smooth structure, called the **standard smooth structure on** V.   //

The Einstein Summation Convention

This is a good place to pause and introduce an important notational convention that is commonly used in the study of smooth manifolds. Because of the proliferation of summations such as $\sum_i x^i \, E_i$ in this subject, we often abbreviate such a sum by omitting the summation sign, as in

$$E(x) = x^i \, E_i, \quad \text{an abbreviation for } E(x) = \sum_{i=1}^{n} x^i \, E_i .$$

We interpret any such expression according to the following rule, called the **Einstein summation convention**: if the same index name (such as i in the expression above) appears exactly twice in any monomial term, once as an upper index and once as a lower index, that term is understood to be summed over all possible values of that index, generally from 1 to the dimension of the space in question. This simple idea was introduced by Einstein to reduce the complexity of expressions arising in the study of smooth manifolds by eliminating the necessity of explicitly writing summation signs. We use the summation convention systematically throughout the book (except in the appendices, which many readers will look at before the rest of the book).

Another important aspect of the summation convention is the positions of the indices. We always write basis vectors (such as E_i) with lower indices, and components of a vector with respect to a basis (such as x^i) with upper indices. These index conventions help to ensure that, in summations that make mathematical sense, each index to be summed over typically appears twice in any given term, once as a lower index and once as an upper index. Any index that is implicitly summed over is a "dummy index," meaning that the value of such an expression is unchanged if a different name is substituted for each dummy index. For example, $x^i \, E_i$ and $x^j \, E_j$ mean exactly the same thing.

Since the coordinates of a point $(x^1, \ldots, x^n) \in \mathbb{R}^n$ are also its components with respect to the standard basis, in order to be consistent with our convention of writing components of vectors with upper indices, we need to use upper indices for these coordinates, and we do so throughout this book. Although this may seem awkward at first, in combination with the summation convention it offers enormous advantages

when we work with complicated indexed sums, not the least of which is that expressions that are not mathematically meaningful often betray themselves quickly by violating the index convention. (The main exceptions are expressions involving the Euclidean dot product $x \cdot y = \sum_i x^i y^i$, in which the same index appears twice in the upper position, and the standard symplectic form on $\mathbb{R}^{2n}$, which we will define in Chapter 22. We always explicitly write summation signs in such expressions.)

More Examples

Now we continue with our examples of smooth manifolds.

Example 1.25 (Spaces of Matrices). Let $\mathrm{M}(m \times n, \mathbb{R})$ denote the set of $m \times n$ matrices with real entries. Because it is a real vector space of dimension mn under matrix addition and scalar multiplication, $\mathrm{M}(m \times n, \mathbb{R})$ is a smooth mn-dimensional manifold. (In fact, it is often useful to *identify* $\mathrm{M}(m \times n, \mathbb{R})$ with $\mathbb{R}^{mn}$, just by stringing all the matrix entries out in a single row.) Similarly, the space $\mathrm{M}(m \times n, \mathbb{C})$ of $m \times n$ complex matrices is a vector space of dimension $2mn$ over $\mathbb{R}$, and thus a smooth manifold of dimension $2mn$. In the special case in which $m = n$ (square matrices), we abbreviate $\mathrm{M}(n \times n, \mathbb{R})$ and $\mathrm{M}(n \times n, \mathbb{C})$ by $\mathrm{M}(n, \mathbb{R})$ and $\mathrm{M}(n, \mathbb{C})$, respectively.   //

Example 1.26 (Open Submanifolds). Let U be any open subset of $\mathbb{R}^n$. Then U is a topological n-manifold, and the single chart (U, Id_U) defines a smooth structure on U.

More generally, let M be a smooth n-manifold and let $U \subseteq M$ be any open subset. Define an atlas on U by

$$\mathcal{A}_U = \{\text{smooth charts } (V, \varphi) \text{ for } M \text{ such that } V \subseteq U\}.$$

Every point $p \in U$ is contained in the domain of some chart (W, φ) for M; if we set $V = W \cap U$, then $(V, \varphi|_V)$ is a chart in $\mathcal{A}_U$ whose domain contains p. Therefore, U is covered by the domains of charts in $\mathcal{A}_U$, and it is easy to verify that this is a smooth atlas for U. Thus any open subset of M is itself a smooth n-manifold in a natural way. Endowed with this smooth structure, we call any open subset an *open submanifold of M*. (We will define a more general class of submanifolds in Chapter 5.)   //

Example 1.27 (The General Linear Group). The *general linear group* $\mathrm{GL}(n, \mathbb{R})$ is the set of invertible $n \times n$ matrices with real entries. It is a smooth n^2-dimensional manifold because it is an open subset of the n^2-dimensional vector space $\mathrm{M}(n, \mathbb{R})$, namely the set where the (continuous) determinant function is nonzero.   //

Example 1.28 (Matrices of Full Rank). The previous example has a natural generalization to rectangular matrices of full rank. Suppose $m < n$, and let $\mathrm{M}_m(m \times n, \mathbb{R})$ denote the subset of $\mathrm{M}(m \times n, \mathbb{R})$ consisting of matrices of rank m. If A is an arbitrary such matrix, the fact that $\mathrm{rank}\, A = m$ means that A has some nonsingular $m \times m$ submatrix. By continuity of the determinant function, this same submatrix

has nonzero determinant on a neighborhood of A in $M(m \times n, \mathbb{R})$, which implies that A has a neighborhood contained in $M_m(m \times n, \mathbb{R})$. Thus, $M_m(m \times n, \mathbb{R})$ is an open subset of $M(m \times n, \mathbb{R})$, and therefore is itself a smooth mn-dimensional manifold. A similar argument shows that $M_n(m \times n, \mathbb{R})$ is a smooth mn-manifold when $n < m$.   //

Example 1.29 (Spaces of Linear Maps). Suppose V and W are finite-dimensional real vector spaces, and let $L(V; W)$ denote the set of linear maps from V to W. Then because $L(V; W)$ is itself a finite-dimensional vector space (whose dimension is the product of the dimensions of V and W), it has a natural smooth manifold structure as in Example 1.24. One way to put global coordinates on it is to choose bases for V and W, and represent each $T \in L(V; W)$ by its matrix, which yields an isomorphism of $L(V; W)$ with $M(m \times n, \mathbb{R})$ for $m = \dim W$ and $n = \dim V$.   //

Example 1.30 (Graphs of Smooth Functions). If $U \subseteq \mathbb{R}^n$ is an open subset and $f: U \to \mathbb{R}^k$ is a smooth function, we have already observed above (Example 1.3) that the graph of f is a topological n-manifold in the subspace topology. Since $\Gamma(f)$ is covered by the single graph coordinate chart $\varphi: \Gamma(f) \to U$ (the restriction of π_1), we can put a canonical smooth structure on $\Gamma(f)$ by declaring the graph coordinate chart $(\Gamma(f), \varphi)$ to be a smooth chart.   //

Example 1.31 (Spheres). We showed in Example 1.4 that the n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is a topological n-manifold. We put a smooth structure on $\mathbb{S}^n$ as follows. For each $i = 1, \ldots, n+1$, let $\left(U_i^\pm, \varphi_i^\pm \right)$ denote the graph coordinate charts we constructed in Example 1.4. For any distinct indices i and j, the transition map $\varphi_i^\pm \circ \left(\varphi_j^\pm \right)^{-1}$ is easily computed. In the case $i < j$, we get

$$\varphi_i^\pm \circ \left(\varphi_j^\pm \right)^{-1} \left(u^1, \ldots, u^n \right) = \left(u^1, \ldots, \widehat{u^i}, \ldots, \pm \sqrt{1 - |u|^2}, \ldots, u^n \right)$$

(with the square root in the jth position), and a similar formula holds when $i > j$. When $i = j$, an even simpler computation gives $\varphi_i^+ \circ \left(\varphi_i^- \right)^{-1} = \varphi_i^- \circ \left(\varphi_i^+ \right)^{-1} = \mathrm{Id}_{\mathbb{B}^n}$. Thus, the collection of charts $\left\{ \left(U_i^\pm, \varphi_i^\pm \right) \right\}$ is a smooth atlas, and so defines a smooth structure on $\mathbb{S}^n$. We call this its ***standard smooth structure***.   //

Example 1.32 (Level Sets). The preceding example can be generalized as follows. Suppose $U \subseteq \mathbb{R}^n$ is an open subset and $\Phi: U \to \mathbb{R}$ is a smooth function. For any $c \in \mathbb{R}$, the set $\Phi^{-1}(c)$ is called a ***level set of*** Φ. Choose some $c \in \mathbb{R}$, let $M = \Phi^{-1}(c)$, and suppose it happens that the total derivative $D\Phi(a)$ is nonzero for each $a \in \Phi^{-1}(c)$. Because $D\Phi(a)$ is a row matrix whose entries are the partial derivatives $(\partial \Phi / \partial x^1(a), \ldots, \partial \Phi / \partial x^n(a))$, for each $a \in M$ there is some i such that $\partial \Phi / \partial x^i(a) \neq 0$. It follows from the implicit function theorem (Theorem C.40, with x^i playing the role of y) that there is a neighborhood U_0 of a such that $M \cap U_0$ can be expressed as a graph of an equation of the form

$$x^i = f \left(x^1, \ldots, \widehat{x^i}, \ldots, x^n \right),$$

for some smooth real-valued function f defined on an open subset of $\mathbb{R}^{n-1}$. Therefore, arguing just as in the case of the n-sphere, we see that M is a topological

manifold of dimension $(n - 1)$, and has a smooth structure such that each of the graph coordinate charts associated with a choice of f as above is a smooth chart. In Chapter 5, we will develop the theory of smooth submanifolds, which is a far-reaching generalization of this construction.   //

Example 1.33 (Projective Spaces). The n-dimensional real projective space $\mathbb{R}\mathbb{P}^n$ is a topological n-manifold by Example 1.5. Let us check that the coordinate charts (U_i, φ_i) constructed in that example are all smoothly compatible. Assuming for convenience that $i > j$, it is straightforward to compute that

$$\varphi_j \circ \varphi_i^{-1}(u^1, \ldots, u^n) = \left(\frac{u^1}{u^j}, \ldots, \frac{u^{j-1}}{u^j}, \frac{u^{j+1}}{u^j}, \ldots, \frac{u^{i-1}}{u^j}, \frac{1}{u^j}, \frac{u^i}{u^j}, \ldots, \frac{u^n}{u^j} \right),$$

which is a diffeomorphism from $\varphi_i(U_i \cap U_j)$ to $\varphi_j(U_i \cap U_j)$.   //

Example 1.34 (Smooth Product Manifolds). If $M_1, \ldots, M_k$ are smooth manifolds of dimensions $n_1, \ldots, n_k$, respectively, we showed in Example 1.8 that the product space $M_1 \times \cdots \times M_k$ is a topological manifold of dimension $n_1 + \cdots + n_k$, with charts of the form $(U_1 \times \cdots \times U_k, \varphi_1 \times \cdots \times \varphi_k)$. Any two such charts are smoothly compatible because, as is easily verified,

$$(\psi_1 \times \cdots \times \psi_k) \circ (\varphi_1 \times \cdots \times \varphi_k)^{-1} = \left(\psi_1 \circ \varphi_1^{-1} \right) \times \cdots \times \left(\psi_k \circ \varphi_k^{-1} \right),$$

which is a smooth map. This defines a natural smooth manifold structure on the product, called the ***product smooth manifold structure***. For example, this yields a smooth manifold structure on the n-torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$.   //

In each of the examples we have seen so far, we constructed a smooth manifold structure in two stages: we started with a topological space and checked that it was a topological manifold, and then we specified a smooth structure. It is often more convenient to combine these two steps into a single construction, especially if we start with a set that is not already equipped with a topology. The following lemma provides a shortcut—it shows how, given a set with suitable "charts" that overlap smoothly, we can use the charts to define both a topology and a smooth structure on the set.

Lemma 1.35 (Smooth Manifold Chart Lemma). *Let M be a set, and suppose we are given a collection $\{U_\alpha\}$ of subsets of M together with maps $\varphi_\alpha \colon U_\alpha \to \mathbb{R}^n$, such that the following properties are satisfied:*

  (i) *For each α, φ_α is a bijection between U_α and an open subset $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$.*

  (ii) *For each α and β, the sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$.*

  (iii) *Whenever $U_\alpha \cap U_\beta \neq \varnothing$, the map $\varphi_\beta \circ \varphi_\alpha^{-1} \colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth.*

  (iv) *Countably many of the sets U_α cover M.*

  (v) *Whenever p, q are distinct points in M, either there exists some U_α containing both p and q or there exist disjoint sets U_α, U_β with $p \in U_\alpha$ and $q \in U_\beta$.*

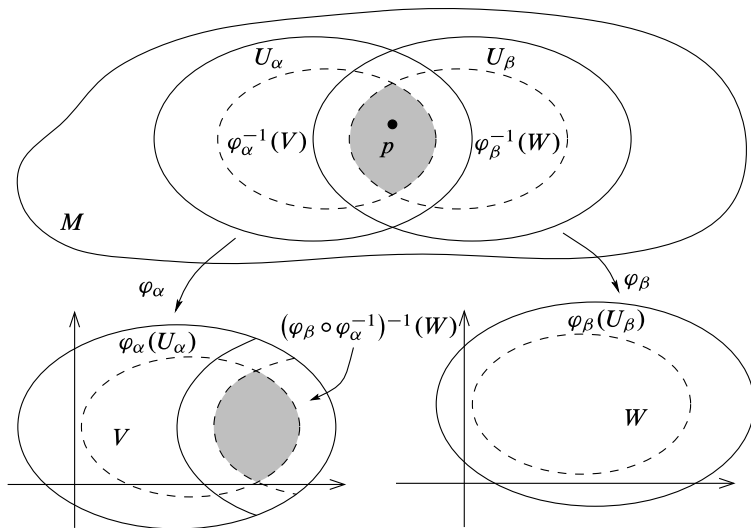*Then M has a unique smooth manifold structure such that each $(U_\alpha, \varphi_\alpha)$ is a smooth chart.*

Fig. 1.9 The smooth manifold chart lemma

Proof. We define the topology by taking all sets of the form $\varphi_\alpha^{-1}(V)$, with V an open subset of $\mathbb{R}^n$, as a basis. To prove that this is a basis for a topology, we need to show that for any point p in the intersection of two basis sets $\varphi_\alpha^{-1}(V)$ and $\varphi_\beta^{-1}(W)$, there is a third basis set containing p and contained in the intersection. It suffices to show that $\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W)$ is itself a basis set (Fig. 1.9). To see this, observe that (iii) implies that $\left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)^{-1}(W)$ is an open subset of $\varphi_\alpha(U_\alpha \cap U_\beta)$, and (ii) implies that this set is also open in $\mathbb{R}^n$. It follows that

$$\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W) = \varphi_\alpha^{-1}\left(V \cap \left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)^{-1}(W)\right)$$

is also a basis set, as claimed.

Each map φ_α is then a homeomorphism onto its image (essentially by definition), so M is locally Euclidean of dimension n. The Hausdorff property follows easily from (v), and second-countability follows from (iv) and the result of Exercise A.22, because each U_α is second-countable. Finally, (iii) guarantees that the collection $\{(U_\alpha, \varphi_\alpha)\}$ is a smooth atlas. It is clear that this topology and smooth structure are the unique ones satisfying the conclusions of the lemma. $\square$

Example 1.36 (Grassmann Manifolds). Let V be an n-dimensional real vector space. For any integer $0 \le k \le n$, we let $\mathrm{G}_k(V)$ denote the set of all k-dimensional linear subspaces of V. We will show that $\mathrm{G}_k(V)$ can be naturally given the structure of a smooth manifold of dimension $k(n-k)$. With this structure, it is called a *Grassmann manifold*, or simply a *Grassmannian*. In the special case $V = \mathbb{R}^n$, the Grassmannian $\mathrm{G}_k\left(\mathbb{R}^n\right)$ is often denoted by some simpler notation such as $\mathrm{G}_{k,n}$ or $\mathrm{G}(k,n)$. Note that $\mathrm{G}_1\left(\mathbb{R}^{n+1}\right)$ is exactly the n-dimensional projective space $\mathbb{RP}^n$.

The construction of a smooth structure on $G_k(V)$ is somewhat more involved than the ones we have done so far, but the basic idea is just to use linear algebra to construct charts for $G_k(V)$, and then apply the smooth manifold chart lemma. We will give a shorter proof that $G_k(V)$ is a smooth manifold in Chapter 21 (see Example 21.21).

Let P and Q be any complementary subspaces of V of dimensions k and $n - k$, respectively, so that V decomposes as a direct sum: $V = P \oplus Q$. The graph of any linear map $X \colon P \to Q$ can be identified with a k-dimensional subspace $\Gamma(X) \subseteq V$, defined by

$$\Gamma(X) = \{v + Xv : v \in P\}.$$

Any such subspace has the property that its intersection with Q is the zero subspace. Conversely, any subspace $S \subseteq V$ that intersects Q trivially is the graph of a unique linear map $X \colon P \to Q$, which can be constructed as follows: let $\pi_P \colon V \to P$ and $\pi_Q \colon V \to Q$ be the projections determined by the direct sum decomposition; then the hypothesis implies that $\pi_P|_S$ is an isomorphism from S to P. Therefore, $X = (\pi_Q|_S) \circ (\pi_P|_S)^{-1}$ is a well-defined linear map from P to Q, and it is straightforward to check that S is its graph.

Let $L(P; Q)$ denote the vector space of linear maps from P to Q, and let U_Q denote the subset of $G_k(V)$ consisting of k-dimensional subspaces whose intersections with Q are trivial. The assignment $X \mapsto \Gamma(X)$ defines a map $\Gamma \colon L(P; Q) \to U_Q$, and the discussion above shows that Γ is a bijection. Let $\varphi = \Gamma^{-1} \colon U_Q \to L(P; Q)$. By choosing bases for P and Q, we can identify $L(P; Q)$ with $M((n - k) \times k, \mathbb{R})$ and hence with $\mathbb{R}^{k(n-k)}$, and thus we can think of (U_Q, φ) as a coordinate chart. Since the image of each such chart is all of $L(P; Q)$, condition (i) of Lemma 1.35 is clearly satisfied.

Now let (P', Q') be any other such pair of subspaces, and let $\pi_{P'}, \pi_{Q'}$ be the corresponding projections and $\varphi' \colon U_{Q'} \to L(P'; Q')$ the corresponding map. The set $\varphi(U_Q \cap U_{Q'}) \subseteq L(P; Q)$ consists of all linear maps $X \colon P \to Q$ whose graphs intersect Q' trivially. To see that this set is open in $L(P; Q)$, for each $X \in L(P; Q)$, let $I_X \colon P \to V$ be the map $I_X(v) = v + Xv$, which is a bijection from P to the graph of X. Because $\Gamma(X) = \operatorname{Im} I_X$ and $Q' = \operatorname{Ker} \pi_{P'}$, it follows from Exercise B.22(d) that the graph of X intersects Q' trivially if and only if $\pi_{P'} \circ I_X$ has full rank. Because the matrix entries of $\pi_{P'} \circ I_X$ (with respect to any bases) depend continuously on X, the result of Example 1.28 shows that the set of all such X is open in $L(P; Q)$. Thus property (ii) in the smooth manifold chart lemma holds.

We need to show that the transition map $\varphi' \circ \varphi^{-1}$ is smooth on $\varphi(U_Q \cap U_{Q'})$. Suppose $X \in \varphi(U_Q \cap U_{Q'}) \subseteq L(P; Q)$ is arbitrary, and let S denote the subspace $\Gamma(X) \subseteq V$. If we put $X' = \varphi' \circ \varphi^{-1}(X)$, then as above, $X' = (\pi_{Q'}|_S) \circ (\pi_{P'}|_S)^{-1}$ (see Fig. 1.10). To relate this map to X, note that $I_X \colon P \to S$ is an isomorphism, so we can write

$$X' = (\pi_{Q'}|_S) \circ I_X \circ (I_X)^{-1} \circ (\pi_{P'}|_S)^{-1} = (\pi_{Q'} \circ I_X) \circ (\pi_{P'} \circ I_X)^{-1}.$$
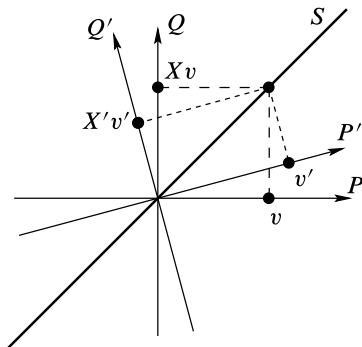
Fig. 1.10 Smooth compatibility of coordinates on $G_k(V)$

To show that this depends smoothly on X, define linear maps $A\colon P \to P'$, $B\colon P \to Q'$, $C\colon Q \to P'$, and $D\colon Q \to Q'$ as follows:

$$A = \pi_{P'}\big|_P, \qquad B = \pi_{Q'}\big|_P, \qquad C = \pi_{P'}\big|_Q, \qquad D = \pi_{Q'}\big|_Q.$$

Then for $v \in P$, we have

$$(\pi_{P'} \circ I_X)v = (A + CX)v, \qquad (\pi_{Q'} \circ I_X)v = (B + DX)v,$$

from which it follows that $X' = (B + DX)(A + CX)^{-1}$. Once we choose bases for P, Q, P', and Q', all of these linear maps are represented by matrices. Because the matrix entries of $(A + CX)^{-1}$ are rational functions of those of $A + CX$ by Cramer's rule, it follows that the matrix entries of X' depend smoothly on those of X. This proves that $\varphi' \circ \varphi^{-1}$ is a smooth map, so the charts we have constructed satisfy condition (iii) of Lemma 1.35.

To check condition (iv), we just note that $G_k(V)$ can in fact be covered by *finitely* many of the sets U_Q: for example, if $(E_1, \ldots, E_n)$ is any fixed basis for V, any partition of the basis elements into two subsets containing k and $n - k$ elements determines appropriate subspaces P and Q, and any subspace S must have trivial intersection with Q for at least one of these partitions (see Exercise B.9). Thus, $G_k(V)$ is covered by the finitely many charts determined by all possible partitions of a fixed basis.

Finally, the Hausdorff condition (v) is easily verified by noting that for any two k-dimensional subspaces $P, P' \subseteq V$, it is possible to find a subspace Q of dimension $n - k$ whose intersections with both P and P' are trivial, and then P and P' are both contained in the domain of the chart determined by, say, (P, Q).   //

Manifolds with Boundary

In many important applications of manifolds, most notably those involving integration, we will encounter spaces that would be smooth manifolds except that they
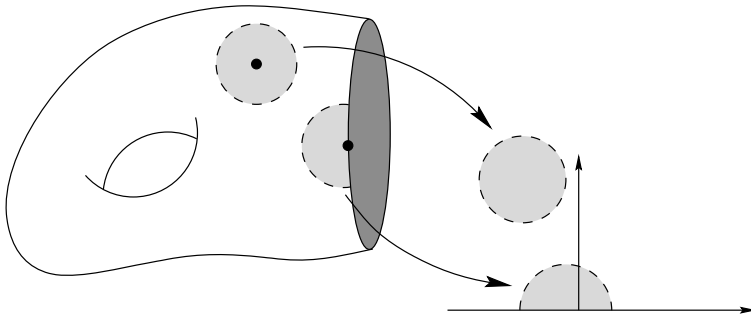
Fig. 1.11 A manifold with boundary

have a "boundary" of some sort. Simple examples of such spaces include closed intervals in $\mathbb{R}$, closed balls in $\mathbb{R}^n$, and closed hemispheres in $\mathbb{S}^n$. To accommodate such spaces, we need to extend our definition of manifolds.

Points in these spaces will have neighborhoods modeled either on open subsets of $\mathbb{R}^n$ or on open subsets of the ***closed n-dimensional upper half-space*** $\mathbb{H}^n \subseteq \mathbb{R}^n$, defined as

$$\mathbb{H}^n = \left\{ \left(x^1, \ldots, x^n \right) \in \mathbb{R}^n : x^n \geq 0 \right\}.$$

We will use the notations $\operatorname{Int} \mathbb{H}^n$ and $\partial \mathbb{H}^n$ to denote the interior and boundary of $\mathbb{H}^n$, respectively, as a subset of $\mathbb{R}^n$. When $n > 0$, this means

$$\operatorname{Int} \mathbb{H}^n = \left\{ \left(x^1, \ldots, x^n \right) \in \mathbb{R}^n : x^n > 0 \right\},$$

$$\partial \mathbb{H}^n = \left\{ \left(x^1, \ldots, x^n \right) \in \mathbb{R}^n : x^n = 0 \right\}.$$

In the $n = 0$ case, $\mathbb{H}^0 = \mathbb{R}^0 = \{0\}$, so $\operatorname{Int} \mathbb{H}^0 = \mathbb{R}^0$ and $\partial \mathbb{H}^0 = \varnothing$.

An ***n-dimensional topological manifold with boundary*** is a second-countable Hausdorff space M in which every point has a neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$ or to a (relatively) open subset of $\mathbb{H}^n$ (Fig. 1.11). An open subset $U \subseteq M$ together with a map $\varphi \colon U \to \mathbb{R}^n$ that is a homeomorphism onto an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ will be called a ***chart for M***, just as in the case of manifolds. When it is necessary to make the distinction, we will call (U, φ) an ***interior chart*** if $\varphi(U)$ is an open subset of $\mathbb{R}^n$ (which includes the case of an open subset of $\mathbb{H}^n$ that does not intersect $\partial \mathbb{H}^n$), and a ***boundary chart*** if $\varphi(U)$ is an open subset of $\mathbb{H}^n$ such that $\varphi(U) \cap \partial \mathbb{H}^n \neq \varnothing$. A boundary chart whose image is a set of the form $B_r(x) \cap \mathbb{H}^n$ for some $x \in \partial \mathbb{H}^n$ and $r > 0$ is called a ***coordinate half-ball***.

A point $p \in M$ is called an ***interior point of M*** if it is in the domain of some interior chart. It is a ***boundary point of M*** if it is in the domain of a boundary chart that sends p to $\partial \mathbb{H}^n$. The ***boundary of M*** (the set of all its boundary points) is denoted by ∂M; similarly, its ***interior***, the set of all its interior points, is denoted by $\operatorname{Int} M$.

It follows from the definition that each point $p \in M$ is either an interior point or a boundary point: if p is not a boundary point, then either it is in the domain of an interior chart or it is in the domain of a boundary chart (U, φ) such that $\varphi(p) \notin \partial \mathbb{H}^n$,

in which case the restriction of φ to $U \cap \varphi^{-1}(\operatorname{Int} \mathbb{H}^n)$ is an interior chart whose domain contains p. However, it is not obvious that a given point cannot be simultaneously an interior point with respect to one chart and a boundary point with respect to another. In fact, this cannot happen, but the proof requires more machinery than we have available at this point. For convenience, we state the theorem here.

Theorem 1.37 (Topological Invariance of the Boundary). *If M is a topological manifold with boundary, then each point of M is either a boundary point or an interior point, but not both. Thus ∂M and $\operatorname{Int} M$ are disjoint sets whose union is M.*

For the proof, see Problem 17-9. Later in this chapter, we will prove a weaker version of this result for smooth manifolds with boundary (Theorem 1.46), which will be sufficient for most of our purposes.

Be careful to observe the distinction between these new definitions of the terms *boundary* and *interior* and their usage to refer to the boundary and interior of a subset of a topological space. A manifold with boundary may have nonempty boundary in this new sense, irrespective of whether it has a boundary as a subset of some other topological space. If we need to emphasize the difference between the two notions of boundary, we will use the terms **topological boundary** and **manifold boundary** as appropriate. For example, the closed unit ball $\overline{\mathbb{B}}^n$ is a manifold with boundary (see Problem 1-11), whose manifold boundary is $\mathbb{S}^{n-1}$. Its topological boundary as a subset of $\mathbb{R}^n$ happens to be the sphere as well. However, if we think of $\overline{\mathbb{B}}^n$ as a topological space in its own right, then as a subset of itself, it has empty topological boundary. And if we think of it as a subset of $\mathbb{R}^{n+1}$ (considering $\mathbb{R}^n$ as a subset of $\mathbb{R}^{n+1}$ in the obvious way), its topological boundary is all of $\overline{\mathbb{B}}^n$. Note that $\mathbb{H}^n$ is itself a manifold with boundary, and its manifold boundary is the same as its topological boundary as a subset of $\mathbb{R}^n$. Every interval in $\mathbb{R}$ is a 1-manifold with boundary, whose manifold boundary consists of its endpoints (if any).

The nomenclature for manifolds with boundary is traditional and well established, but it must be used with care. Despite their name, manifolds with boundary are *not* in general manifolds, because boundary points do not have locally Euclidean neighborhoods. (This is a consequence of the theorem on invariance of the boundary.) Moreover, a manifold with boundary might have empty boundary—there is nothing in the definition that requires the boundary to be a nonempty set. On the other hand, a manifold is also a manifold with boundary, whose boundary is empty. Thus, every manifold is a manifold with boundary, but a manifold with boundary is a manifold if and only if its boundary is empty (see Proposition 1.38 below).

Even though the term *manifold with boundary* encompasses manifolds as well, we will often use redundant phrases such as **manifold without boundary** if we wish to emphasize that we are talking about a manifold in the original sense, and **manifold with or without boundary** to refer to a manifold with boundary if we wish emphasize that the boundary might be empty. (The latter phrase will often appear when our primary interest is in manifolds, but the results being discussed are just as easy to state and prove in the more general case of manifolds with boundary.) Note that the word "manifold" without further qualification always means a manifold

without boundary. In the literature, you will also encounter the terms ***closed manifold*** to mean a compact manifold without boundary, and ***open manifold*** to mean a noncompact manifold without boundary.

Proposition 1.38. *Let M be a topological n-manifold with boundary.*

(a) *Int M is an open subset of M and a topological n-manifold without boundary.*
(b) *∂M is a closed subset of M and a topological $(n - 1)$-manifold without boundary.*
(c) *M is a topological manifold if and only if $\partial M = \varnothing$.*
(d) *If $n = 0$, then $\partial M = \varnothing$ and M is a 0-manifold.*

▶ **Exercise 1.39.** Prove the preceding proposition. For this proof, you may use the theorem on topological invariance of the boundary when necessary. Which parts require it?

The topological properties of manifolds that we proved earlier in the chapter have natural extensions to manifolds with boundary, with essentially the same proofs as in the manifold case. For the record, we state them here.

Proposition 1.40. *Let M be a topological manifold with boundary.*

(a) *M has a countable basis of precompact coordinate balls and half-balls.*
(b) *M is locally compact.*
(c) *M is paracompact.*
(d) *M is locally path-connected.*
(e) *M has countably many components, each of which is an open subset of M and a connected topological manifold with boundary.*
(f) *The fundamental group of M is countable.*

▶ **Exercise 1.41.** Prove the preceding proposition.

Smooth Structures on Manifolds with Boundary

To see how to define a smooth structure on a manifold with boundary, recall that a map from an arbitrary subset $A \subseteq \mathbb{R}^n$ to $\mathbb{R}^k$ is said to be smooth if in a neighborhood of each point of A it admits an extension to a smooth map defined on an open subset of $\mathbb{R}^n$ (see Appendix C, p. 645). Thus, if U is an open subset of $\mathbb{H}^n$, a map $F \colon U \to \mathbb{R}^k$ is smooth if for each $x \in U$, there exists an open subset $\widetilde{U} \subseteq \mathbb{R}^n$ containing x and a smooth map $\widetilde{F} \colon \widetilde{U} \to \mathbb{R}^k$ that agrees with F on $\widetilde{U} \cap \mathbb{H}^n$ (Fig. 1.12). If F is such a map, the restriction of F to $U \cap \operatorname{Int} \mathbb{H}^n$ is smooth in the usual sense. By continuity, all partial derivatives of F at points of $U \cap \partial \mathbb{H}^n$ are determined by their values in $\operatorname{Int} \mathbb{H}^n$, and therefore in particular are independent of the choice of extension. It is a fact (which we will neither prove nor use) that $F \colon U \to \mathbb{R}^k$ is smooth in this sense if and only if F is continuous, $F|_{U \cap \operatorname{Int} \mathbb{H}^n}$ is smooth, and the partial derivatives of $F|_{U \cap \operatorname{Int} \mathbb{H}^n}$ of all orders have continuous extensions to all of U. (One direction is obvious; the other direction depends on a lemma of Émile Borel, which shows that there is a smooth function defined in the lower half-space whose derivatives all match those of F on $U \cap \partial \mathbb{H}^n$. See, e.g., [Hör90, Thm. 1.2.6].)
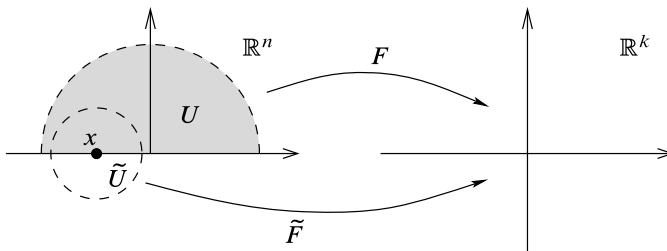
Fig. 1.12 Smoothness of maps on open subsets of $\mathbb{H}^n$

For example, let $\mathbb{B}^2 \subseteq \mathbb{R}^2$ be the open unit disk, let $U = \mathbb{B}^2 \cap \mathbb{H}^2$, and define $f: U \to \mathbb{R}$ by $f(x, y) = \sqrt{1 - x^2 - y^2}$. Because f extends smoothly to all of $\mathbb{B}^2$ (by the same formula), f is a smooth function on U. On the other hand, although $g(x, y) = \sqrt{y}$ is continuous on U and smooth in $U \cap \operatorname{Int} \mathbb{H}^2$, it has no smooth extension to any neighborhood of the origin in $\mathbb{R}^2$ because $\partial g / \partial y \to \infty$ as $y \to 0$. Thus g is not smooth on U.

Now let M be a topological manifold with boundary. As in the manifold case, a ***smooth structure for M*** is defined to be a maximal smooth atlas—a collection of charts whose domains cover M and whose transition maps (and their inverses) are smooth in the sense just described. With such a structure, M is called a ***smooth manifold with boundary***. Every smooth manifold is automatically a smooth manifold with boundary (whose boundary is empty).

Just as for smooth manifolds, if M is a smooth manifold with boundary, any chart in the given smooth atlas is called a ***smooth chart for M***. ***Smooth coordinate balls***, ***smooth coordinate half-balls***, and ***regular coordinate balls in M*** are defined in the obvious ways. In addition, a subset $B \subseteq M$ is called a ***regular coordinate half-ball*** if there is a smooth coordinate half-ball $B' \supseteq \bar{B}$ and a smooth coordinate map $\varphi: B' \to \mathbb{H}^n$ such that for some $r' > r > 0$ we have

$$\varphi(B) = B_r(0) \cap \mathbb{H}^n, \qquad \varphi\big(\bar{B}\big) = \bar{B}_r(0) \cap \mathbb{H}^n, \quad \text{and} \quad \varphi\big(B'\big) = B_{r'}(0) \cap \mathbb{H}^n.$$

▶ **Exercise 1.42.** Show that every smooth manifold with boundary has a countable basis consisting of regular coordinate balls and half-balls.

▶ **Exercise 1.43.** Show that the smooth manifold chart lemma (Lemma 1.35) holds with "$\mathbb{R}^n$" replaced by "$\mathbb{R}^n$ or $\mathbb{H}^n$" and "smooth manifold" replaced by "smooth manifold with boundary."

▶ **Exercise 1.44.** Suppose M is a smooth n-manifold with boundary and U is an open subset of M. Prove the following statements:

(a) U is a topological n-manifold with boundary, and the atlas consisting of all smooth charts (V, φ) for M such that $V \subseteq U$ defines a smooth structure on U. With this topology and smooth structure, U is called an ***open submanifold with boundary***.
(b) If $U \subseteq \operatorname{Int} M$, then U is actually a smooth manifold (without boundary); in this case we call it an ***open submanifold of M***.
(c) $\operatorname{Int} M$ is an open submanifold of M (without boundary).

One important result about smooth manifolds that does *not* extend directly to smooth manifolds with boundary is the construction of smooth structures on finite products (see Example 1.8). Because a product of half-spaces $\mathbb{H}^n \times \mathbb{H}^m$ is not itself a half-space, a finite product of smooth manifolds with boundary cannot generally be considered as a smooth manifold with boundary. (Instead, it is an example of a *smooth manifold with corners*, which we will study in Chapter 16.) However, we do have the following result.

Proposition 1.45. *Suppose $M_1, \ldots, M_k$ are smooth manifolds and N is a smooth manifold with boundary. Then $M_1 \times \cdots \times M_k \times N$ is a smooth manifold with boundary, and $\partial(M_1 \times \cdots \times M_k \times N) = M_1 \times \cdots \times M_k \times \partial N$.*

Proof. Problem 1-12. □

For smooth manifolds with boundary, the following result is often an adequate substitute for the theorem on invariance of the boundary.

Theorem 1.46 (Smooth Invariance of the Boundary). *Suppose M is a smooth manifold with boundary and $p \in M$. If there is some smooth chart (U, φ) for M such that $\varphi(U) \subseteq \mathbb{H}^n$ and $\varphi(p) \in \partial\mathbb{H}^n$, then the same is true for every smooth chart whose domain contains p.*

Proof. Suppose on the contrary that p is in the domain of a smooth interior chart (U, ψ) and also in the domain of a smooth boundary chart (V, φ) such that $\varphi(p) \in \partial\mathbb{H}^n$. Let $\tau = \varphi \circ \psi^{-1}$ denote the transition map; it is a homeomorphism from $\psi(U \cap V)$ to $\varphi(U \cap V)$. The smooth compatibility of the charts ensures that both τ and τ^{-1} are smooth, in the sense that locally they can be extended, if necessary, to smooth maps defined on open subsets of $\mathbb{R}^n$.

Write $x_0 = \psi(p)$ and $y_0 = \varphi(p) = \tau(x_0)$. There is some neighborhood W of y_0 in $\mathbb{R}^n$ and a smooth function $\eta \colon W \to \mathbb{R}^n$ that agrees with τ^{-1} on $W \cap \varphi(U \cap V)$. On the other hand, because we are assuming that ψ is an interior chart, there is an open Euclidean ball B that is centered at x_0 and contained in $\varphi(U \cap V)$, so τ itself is smooth on B in the ordinary sense. After shrinking B if necessary, we may assume that $B \subseteq \tau^{-1}(W)$. Then $\eta \circ \tau|_B = \tau^{-1} \circ \tau|_B = \mathrm{Id}_B$, so it follows from the chain rule that $D\eta(\tau(x)) \circ D\tau(x)$ is the identity map for each $x \in B$. Since $D\tau(x)$ is a square matrix, this implies that it is nonsingular. It follows from Corollary C.36 that τ (considered as a map from B to $\mathbb{R}^n$) is an open map, so $\tau(B)$ is an open subset of $\mathbb{R}^n$ that contains $y_0 = \varphi(p)$ and is contained in $\varphi(V)$. This contradicts the assumption that $\varphi(V) \subseteq \mathbb{H}^n$ and $\varphi(p) \in \partial\mathbb{H}^n$. □

Problems

1-1. Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff. (This space is called the ***line with two origins***.)

1-2. Show that a disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.

1-3. A topological space is said to be ***σ-compact*** if it can be expressed as a union of countably many compact subspaces. Show that a locally Euclidean Hausdorff space is a topological manifold if and only if it is σ-compact.

1-4. Let M be a topological manifold, and let $\mathcal{U}$ be an open cover of M.
   (a) Assuming that each set in $\mathcal{U}$ intersects only finitely many others, show that $\mathcal{U}$ is locally finite.
   (b) Give an example to show that the converse to (a) may be false.
   (c) Now assume that the sets in $\mathcal{U}$ are precompact in M, and prove the converse: if $\mathcal{U}$ is locally finite, then each set in $\mathcal{U}$ intersects only finitely many others.

1-5. Suppose M is a locally Euclidean Hausdorff space. Show that M is second-countable if and only if it is paracompact and has countably many connected components. [Hint: assuming M is paracompact, show that each component of M has a locally finite cover by precompact coordinate domains, and extract from this a countable subcover.]

1-6. Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, show that it has uncountably many distinct ones. [Hint: first show that for any $s > 0$, $F_s(x) = |x|^{s-1}x$ defines a homeomorphism from $\mathbb{B}^n$ to itself, which is a diffeomorphism if and only if $s = 1$.]

1-7. Let N denote the ***north pole*** $(0,\dots,0,1) \in \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$, and let S denote the ***south pole*** $(0,\dots,0,-1)$. Define the ***stereographic projection*** $\sigma \colon \mathbb{S}^n \smallsetminus \{N\} \to \mathbb{R}^n$ by

$$\sigma\left(x^1,\dots,x^{n+1}\right) = \frac{(x^1,\dots,x^n)}{1-x^{n+1}}.$$

Let $\widetilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \smallsetminus \{S\}$.
   (a) For any $x \in \mathbb{S}^n \smallsetminus \{N\}$, show that $\sigma(x) = u$, where $(u,0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$ (Fig. 1.13). Similarly, show that $\widetilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace. (For this reason, $\widetilde{\sigma}$ is called ***stereographic projection from the south pole***.)
   (b) Show that σ is bijective, and

$$\sigma^{-1}\left(u^1,\dots,u^n\right) = \frac{(2u^1,\dots,2u^n,|u|^2-1)}{|u|^2+1}.$$

   (c) Compute the transition map $\widetilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(\mathbb{S}^n \smallsetminus \{N\},\sigma)$ and $(\mathbb{S}^n \smallsetminus \{S\},\widetilde{\sigma})$ defines a smooth structure on $\mathbb{S}^n$. (The coordinates defined by σ or $\widetilde{\sigma}$ are called ***stereographic coordinates***.)
   (d) Show that this smooth structure is the same as the one defined in Example 1.31.
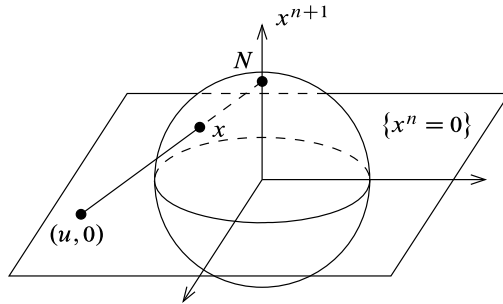*(Used on pp. 201, 269, 301, 345, 347, 450.)*

Fig. 1.13 Stereographic projection

1-8.   By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle $\mathbb{S}^1$ as a subset of the complex plane. An ***angle function*** on a subset $U \subseteq \mathbb{S}^1$ is a continuous function $\theta \colon U \to \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$. Show that there exists an angle function θ on an open subset $U \subseteq \mathbb{S}^1$ if and only if $U \neq \mathbb{S}^1$. For any such angle function, show that (U, θ) is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure. (*Used on pp. 37, 152, 176.*)

1-9.   ***Complex projective n-space***, denoted by $\mathbb{CP}^n$, is the set of all 1-dimensional complex-linear subspaces of $\mathbb{C}^{n+1}$, with the quotient topology inherited from the natural projection $\pi \colon \mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{CP}^n$. Show that $\mathbb{CP}^n$ is a compact $2n$-dimensional topological manifold, and show how to give it a smooth structure analogous to the one we constructed for $\mathbb{RP}^n$. (We use the correspondence

$$\left(x^1 + iy^1, \ldots, x^{n+1} + iy^{n+1}\right) \leftrightarrow \left(x^1, y^1, \ldots, x^{n+1}, y^{n+1}\right)$$

to identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$.) (*Used on pp. 48, 96, 172, 560, 561.*)

1-10. Let k and n be integers satisfying $0 < k < n$, and let $P, Q \subseteq \mathbb{R}^n$ be the linear subspaces spanned by $(e_1, \ldots, e_k)$ and $(e_{k+1}, \ldots, e_n)$, respectively, where e_i is the ith standard basis vector for $\mathbb{R}^n$. For any k-dimensional subspace $S \subseteq \mathbb{R}^n$ that has trivial intersection with Q, show that the coordinate representation $\varphi(S)$ constructed in Example 1.36 is the unique $(n-k) \times k$ matrix B such that S is spanned by the columns of the matrix $\binom{I_k}{B}$, where I_k denotes the $k \times k$ identity matrix.

1-11. Let $M = \overline{\mathbb{B}}^n$, the closed unit ball in $\mathbb{R}^n$. Show that M is a topological manifold with boundary in which each point in $\mathbb{S}^{n-1}$ is a boundary point and each point in $\mathbb{B}^n$ is an interior point. Show how to give it a smooth structure such that every smooth interior chart is a smooth chart for the standard smooth structure on $\mathbb{B}^n$. [Hint: consider the map $\pi \circ \sigma^{-1} \colon \mathbb{R}^n \to \mathbb{R}^n$, where $\sigma \colon \mathbb{S}^n \to \mathbb{R}^n$ is the stereographic projection (Problem 1-7) and π is a projection from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ that omits some coordinate other than the last.]

1-12. Prove Proposition 1.45 (a product of smooth manifolds together with one smooth manifold with boundary is a smooth manifold with boundary).