Chapter 1

The Basics

If you are familiar with the prerequisites for reading this book, you are probably already familiar with the notion of a *smooth manifold*—a topological manifold equipped with an atlas of coordinate charts whose transition functions are all smooth. (Precise definitions will be found farther down in this chapter.)

You may also have encountered variations on that theme—different classes of manifolds that can be defined by modifying the compatibility condition for charts. For example, a ***C^k manifold*** is one equipped with an atlas whose transition functions are all of class C^k (meaning k times continuously differentiable), and a ***real-analytic manifold*** is one with an atlas whose transition functions are all real-analytic (meaning they are equal to the sum of a convergent power series in a neighborhood of each point).

Another variation on that theme, and the one to which this book is devoted, is a *complex manifold*—this is a topological manifold equipped with an atlas whose transition functions are all holomorphic. While the other classes of manifolds mentioned above are really just slight variations on the theme of smooth manifolds, it turns out that nearly everything changes when we move into the holomorphic category, as you will soon see. That is why the subject of complex manifolds is worth an entire book of its own.

In this chapter we introduce the main definitions, and describe some examples and basic properties of complex manifolds.

Definitions

The most basic type of manifold is a ***topological manifold***: this is a second-countable Hausdorff topological space with the property that every point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$ for some fixed n, called the

dimension of the manifold. (In this book, all manifolds are understood to be manifolds without boundary unless otherwise specified.)

By adding extra structure to a topological manifold, we can obtain other types of manifolds. Differential geometry is concerned primarily with ***smooth manifolds***, which are topological manifolds endowed with *smooth structures*, defined as follows: If M is a topological manifold of dimension n, a ***coordinate chart*** (often called just a ***chart***) for M is a pair (U, φ), where U is an open subset of M and φ is a homeomorphism from U to an open subset of $\mathbb{R}^n$. An ***atlas*** for M is a collection of charts whose domains cover M. Given two charts (U, φ) and (V, ψ) with overlapping domains, their ***transition functions*** are the composite maps $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ and their inverses $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$. Two charts are said to be ***smoothly compatible*** if their domains are disjoint or their transition functions are smooth as maps between open subsets of $\mathbb{R}^n$. (Here and throughout the book, ***smooth*** means infinitely differentiable or of class C^∞.) A ***smooth atlas*** for M is an atlas with the property that any two charts in the atlas are smoothly compatible with each other. Finally, a ***smooth structure*** for M is a smooth atlas that is ***maximal***, meaning that it is not properly contained in any larger smooth atlas; to say that $\mathscr{A}$ is a maximal smooth atlas just means that every chart that is smoothly compatible with every chart in $\mathscr{A}$ is already in $\mathscr{A}$.

The definition of a complex manifold is, at first glance, just a minor modification of the definition of smooth manifolds. The main change is that we require each transition function to be ***holomorphic***, meaning that it is continuous and each of its complex-valued component functions has complex partial derivatives with respect to each of the independent complex variables $z^1, \dots, z^n$. (We will explore properties of holomorphic functions in more depth below; for now, it suffices to know that they are smooth and that compositions of holomorphic functions are holomorphic.) To apply this requirement to the transition functions for topological manifolds, we choose the following standard identification between $\mathbb{R}^{2n}$ and $\mathbb{C}^n$:

$$(x^1, y^1, \dots, x^n, y^n) \leftrightarrow (x^1 + iy^1, \dots, x^n + iy^n).$$

(As in [**LeeSM**] and [**LeeRM**], we index coordinate functions with upper indices to be consistent with the Einstein summation convention, described later in this chapter.) With this identification, it makes sense to ask whether a map between open subsets of $\mathbb{R}^{2n}$ is holomorphic.

Now suppose M is a $2n$-dimensional topological manifold. If (U, φ) and (V, ψ) are two coordinate charts for M, we say they are ***holomorphically compatible*** if $U \cap V = \varnothing$ or both transition functions are holomorphic under our standard identification of $\varphi(U \cap V)$ and $\psi(U \cap V)$ as open subsets of $\mathbb{C}^n$. A ***holomorphic atlas*** for M is an atlas with the property that any two charts in the atlas are holomorphically compatible with each other, and a ***holomorphic structure*** for M is a maximal holomorphic atlas. An ***n-dimensional complex manifold*** (or ***holomorphic manifold***) is a topological manifold of dimension $2n$ endowed with a given holomorphic structure. A complex manifold of dimension 1 is called a ***complex curve***, and one of dimension 2 is called a ***complex surface***. A complex manifold of dimension 3 or higher is sometimes called a ***complex threefold***, ***fourfold***, etc. When it is necessary to distinguish between the dimension of an n-dimensional complex manifold and the dimension of its underlying topological $2n$-manifold, we call n the ***complex dimension*** (denoted by $\dim_{\mathbb{C}} M$) and $2n$ the ***real dimension*** (denoted by $\dim_{\mathbb{R}} M$). Any one of the charts in the maximal holomorphic atlas is called a ***holomorphic coordinate chart***, and the complex-valued coordinate functions $(z^1, \dots, z^n)$ (where $z^j = x^j + iy^j$) are called ***holomorphic coordinates***. We denote the complex conjugate of z^j by $\bar{z}^j = x^j - iy^j$.

Holomorphic structures on manifolds are traditionally called ***complex structures***, but that term risks confusion with complex structures on vector bundles, to be discussed below.

Because all holomorphic functions are smooth (see Thm. 1.21 below), a holomorphic atlas is also a smooth atlas and thus determines a unique smooth structure on M; thus every complex manifold is also a smooth manifold in a canonical way. On the other hand, it is important to note that a given even-dimensional smooth manifold may have many different holomorphic structures that induce the given smooth structure (see Problem 1-4), or it may have none at all. The simplest example of an even-dimensional smooth manifold that carries no holomorphic structure is $\mathbb{S}^4$; see the discussion following Theorem 1.63 for more detail.

Proposition 1.1. *Let M be a topological manifold.*

(a) *Every holomorphic atlas $\mathcal{A}$ for M is contained in a unique maximal holomorphic atlas, called the* ***holomorphic structure determined by $\mathcal{A}$****.*

(b) *Two holomorphic atlases for M determine the same holomorphic structure if and only if their union is a holomorphic atlas.*

Proof. The proof is essentially identical to that of its smooth counterpart [**LeeSM**, Prop. 1.17]. □

To turn a set into a complex manifold using the definitions directly, it would be necessary to go through the separate steps of constructing a topology, verifying that it is a manifold, and then constructing a holomorphic structure for it. But in most cases the following shortcut can be used.

Lemma 1.2 (Complex Manifold Chart Lemma). *Let M be a set, and suppose we are given a collection $\{U_\alpha\}_{\alpha\in A}$ of subsets of M together with maps $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$, such that the following properties are satisfied:*

(i) *For each α, φ_α is a bijection between U_α and an open subset $\varphi_\alpha(U_\alpha) \subseteq \mathbb{C}^n$.*

(ii) *For each α and β, the sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{C}^n$.*

(iii) *When* $U_\alpha \cap U_\beta \neq \varnothing$, *the map* $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ *is holomorphic.*

(iv) *Countably many of the sets* U_α *cover* M.

(v) *Whenever* p, q *are distinct points in* M, *either there exists some* U_α *containing both* p *and* q *or there exist disjoint sets* U_α, U_β *with* $p \in U_\alpha$ *and* $q \in U_\beta$.

Then M *has a unique structure as a complex manifold such that each* $(U_\alpha, \varphi_\alpha)$ *is a holomorphic chart.*

► **Exercise 1.3.** Prove this lemma by verifying that the proof of Lemma 1.35 of [**LeeSM**] goes through in this setting.

Some Examples

Before we go much further, we should have a few examples of complex manifolds to think about. We will introduce many more examples in Chapter 2.

Example 1.4 (Complex *n*-Space). It follows from Proposition 1.1(a) that $\mathbb{C}^n$ has a canonical holomorphic structure determined by the holomorphic atlas consisting of the single coordinate chart $(\mathbb{C}^n, \mathrm{Id}_{\mathbb{C}^n})$. Similarly, the canonical holomorphic structure on every open subset $U \subseteq \mathbb{C}^n$ is defined by the single chart (U, Id_U). When working with $\mathbb{C}$, $\mathbb{C}^n$, or their open subsets, we always use this holomorphic structure, typically without further comment. Here are some specific open subsets that will play important roles in what follows:

- For any $p \in \mathbb{C}^n$ and any $r > 0$, the ***(open) ball of radius r around p*** is the set $B_r(p) = \{z \in \mathbb{C}^n : |z - p| < r\}$, where $|\cdot|$ denotes the norm associated with the ***Euclidean inner product*** on $\mathbb{C}^n \approx \mathbb{R}^{2n}$, which can be written in complex coordinates as $\langle z, w \rangle = z \cdot \overline{w} = \sum_{j=1}^n z^j \overline{w}^j$. The ***unit ball*** of real dimension $2n$, denoted by $\mathbb{B}^{2n}$, is the open ball of radius 1 about the origin in $\mathbb{C}^n$.
- An open ball in $\mathbb{C}$ is called a ***disk***, and the notation is modified accordingly. Thus $D_r(p)$ represents the disk of radius r about $p \in \mathbb{C}$, and the ***unit disk*** is the disk $D_1(0)$, denoted by $\mathbb{D}$.
- A ***polydisk*** is a Cartesian product of open disks, that is, an open subset of the form $D_{r_1}(p^1) \times \cdots \times D_{r_n}(p^n) \subseteq \mathbb{C}^n$ for a point $p = (p^1, \dots, p^n) \in \mathbb{C}^n$ and positive real numbers $r_1, \dots, r_n$. When the radii are all equal, we use the notation $D_r^n(p)$ for the polydisk $D_r(p^1) \times \cdots \times D_r(p^n)$. //

Example 1.5 (Open Submanifolds). Somewhat more generally, if M is a complex n-manifold and U is an open subset of M, we can define a canonical holomorphic structure on U consisting of all holomorphic charts for M whose domains are contained in U. With this holomorphic structure, U is a complex n-manifold, called an ***open submanifold of M***. //

Example 1.6 (Complex Vector Spaces). If V is a finite-dimensional complex vector space, any choice of ordered basis $(b_1, \dots, b_n)$ defines an isomorphism $B : \mathbb{C}^n \to V$ by

$$B(z^1, \dots, z^n) = z^j b_j. \tag{1.1}$$

(Here and throughout the book, we use the ***Einstein summation convention***: each index name that appears twice in the same monomial term, once as an upper index and once as a lower one, is understood to be summed over all possible values of that index, typically from 1 to the dimension of the space. In formula (1.1), since V has dimension n, the implied summation is from 1 to n.) Interpreting B^{-1} as a global chart thus defines a holomorphic structure on V. Since the transition map between any two such charts is an invertible complex-linear transformation and therefore holomorphic along with its inverse, this structure is independent of the choice of basis. We will call this the ***standard holomorphic structure on V***. //

Example 1.7 (0-Manifolds). A topological 0-manifold is just a countable discrete space. Each point has a unique map to $\mathbb{C}^0 = \{0\}$, and the transition functions between these maps are vacuously holomorphic, so every 0-manifold has a canonical holomorphic structure. //

Example 1.8 (Product Manifolds). If $M_1, \dots, M_k$ are complex manifolds, their Cartesian product $M_1 \times \cdots \times M_k$ (with the product topology) is a complex manifold whose dimension is the sum of the dimensions of the factors, with products of holomorphic coordinate maps providing holomorphic coordinates. //

Example 1.9 (Complex Projective Spaces). The next examples are, after $\mathbb{C}^n$ itself, the most important complex manifolds of all. For any nonnegative integer n, we define the ***complex projective space of dimension n***, denoted by $\mathbb{CP}^n$, to be the set of complex 1-dimensional subspaces of $\mathbb{C}^{n+1}$, which we can identify with the quotient of $\mathbb{C}^{n+1} \smallsetminus \{0\}$ by the equivalence relation defined by $w \sim w'$ if and only if $w' = \lambda w$ for some nonzero complex number λ. We endow $\mathbb{CP}^n$ with the quotient topology. By this definition, $\mathbb{CP}^0$ is a single point.

We denote the equivalence class of a point $w = (w^0, w^1, \dots, w^n) \in \mathbb{C}^{n+1} \smallsetminus \{0\}$ by $[w] = [w^0, \dots, w^n]$. The complex numbers $(w^0, \dots, w^n)$ are traditionally called ***homogeneous coordinates*** of the point $[w]$; but be careful about using this terminology, because they are not actually coordinates in the usual sense. The same point $[w]$ is represented by any homogeneous coordinates of the form $(\lambda w^0, \dots, \lambda w^n)$ with $\lambda \neq 0$, so there is not a one-to-one correspondence between points and homogeneous coordinates, even in a small neighborhood of a point.

We can construct honest coordinates for $\mathbb{CP}^n$ as follows. For each $\alpha = 0, \dots, n$, let $U_\alpha \subseteq \mathbb{CP}^n$ be the open subset $U_\alpha = \{[w] \in \mathbb{CP}^n : w^\alpha \neq 0\}$, and define a map

$\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ by

$$\varphi_\alpha([w^0, \dots, w^n]) = \left(\frac{w^0}{w^\alpha}, \dots, \frac{w^{\alpha-1}}{w^\alpha}, \frac{w^{\alpha+1}}{w^\alpha}, \dots, \frac{w^n}{w^\alpha} \right).$$

It is continuous by the characteristic property of the quotient topology [**LeeTM**, Thm. 3.70], and it is a homeomorphism because it has a continuous inverse given by

$$\varphi_\alpha^{-1}(z^1, \dots, z^n) = [z^1, \dots, z^{\alpha-1}, 1, z^\alpha, \dots, z^n].$$

Thus each $(U_\alpha, \varphi_\alpha)$ is a coordinate chart, called ***affine coordinates for*** $\mathbb{CP}^n$. Coupled with the facts that $\mathbb{CP}^n$ is Hausdorff and second-countable (Exercise 1.10), this shows that $\mathbb{CP}^n$ is a topological manifold of real dimension $2n$. It is compact and connected, because it is the image of the surjective continuous map $q : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ given by $q(w^0, \dots, w^n) = [w^0, \dots, w^n]$, where $\mathbb{S}^{2n+1}$ is the set of unit vectors in $\mathbb{C}^{n+1}$.

For $\alpha < \beta$, the transition function between these charts can be computed explicitly as

$$\varphi_\alpha \circ \varphi_\beta^{-1}(z^1, \dots, z^n) = \left(\frac{z^1}{z^\alpha}, \dots, \widehat{\frac{z^\alpha}{z^\alpha}}, \dots, \frac{1}{z^\alpha}, \dots, \frac{z^n}{z^\alpha} \right),$$

where the hat indicates that the term in position α is omitted, and the $1/z^\alpha$ term is in position β; the formula for $\alpha > \beta$ is similar. These transition functions are all holomorphic, so they turn $\mathbb{CP}^n$ into a complex manifold of dimension n. //

► **Exercise 1.10.** Verify that $\mathbb{CP}^n$ is Hausdorff and second-countable.

Example 1.11 (Projectivization of a Vector Space). For some purposes, it is useful to construct projective spaces starting with different complex vector spaces in place of $\mathbb{C}^{n+1}$ itself. Suppose V is an n-dimensional complex vector space with $n > 0$. The ***projectivization of V***, denoted by $\mathbb{P}(V)$, is the set of 1-dimensional complex subspaces of V, endowed with the quotient topology obtained from the equivalence relation on $V \smallsetminus \{0\}$ given by $v_1 \sim v_2$ if $v_2 = \lambda v_1$ for some $\lambda \in \mathbb{C} \smallsetminus \{0\}$. A choice of basis for V yields an isomorphism $V \cong \mathbb{C}^n$ that descends to a bijection $\mathbb{P}(V) \to \mathbb{CP}^{n-1}$, which we can use to give $\mathbb{P}(V)$ the structure of a complex manifold. In the next chapter, we will see that the holomorphic structure obtained in this way is independent of the choice of basis (see Exercise 2.10). //

Example 1.12 (Complex Grassmannians). Suppose V is an n-dimensional complex vector space with $n > 0$, and k is a nonnegative integer less than or equal to n. Let $\mathrm{G}_k(V)$ be the set of k-dimensional complex-linear subspaces of V, called a ***complex Grassmannian***. (The case $k = 1$ is exactly the projective space $\mathbb{P}(V)$.) We can construct complex coordinates on $\mathrm{G}_k(V)$ as follows. Choose a subspace $P \subseteq V$ of dimension k and a complementary $(n-k)$-dimensional subspace Q, and write $V = P \oplus Q$. Then the graph of each complex-linear map $X : P \to Q$ is a

k-dimensional subspace $\Gamma(X) \subseteq V$, and every subspace whose intersection with Q is trivial is the graph of a unique such map. Let $U_Q \subseteq \mathrm{G}_k(V)$ denote the set of such subspaces. By choosing bases for P and Q, we obtain a bijection from U_Q to the vector space $\mathrm{M}((n-k) \times k, \mathbb{C})$ of complex $(n-k) \times k$ matrices, whose matrix entries we can use as coordinates on U_Q. The argument in [**LeeSM**, Example 1.36] (adapted in an obvious way to the complex case) shows that when two such charts overlap, the matrix entries in the new chart are rational functions of the original ones, so any two such charts overlap holomorphically. The arguments of that example also show that hypotheses (iv) and (v) of the chart lemma are satisfied, so $\mathrm{G}_k(V)$ is a complex manifold of dimension $(n-k)k$. Problem 1-5 shows that it is compact. //

Holomorphic Maps

We define holomorphic maps between complex manifolds in the same way as one defines smooth maps between smooth manifolds: if M and N are complex manifolds, a ***holomorphic map*** from M to N is a map $f : M \to N$ with the property that for every $p \in M$ there exist holomorphic coordinate charts (U, φ) for M and (V, ψ) for N whose domains contain p and $f(p)$, respectively, such that $f(U) \subseteq V$ and the composite map $\psi \circ f \circ \varphi^{-1}$ is holomorphic as a map from $\varphi(U)$ to $\psi(V)$. The function $\widehat{f} = \psi \circ f \circ \varphi^{-1}$ is called the ***coordinate representation of f*** with respect to the given holomorphic coordinates. As is the case in smooth manifold theory (see [**LeeSM**, pp. 15–16]), one often uses a coordinate map to temporarily identify an open subset of a manifold with an open subset of $\mathbb{C}^n$, and uses the same notation for a map and its coordinate representation.

When the codomain of a map f is $\mathbb{C}^k$ (or an open subset of $\mathbb{C}^k$) with its canonical holomorphic structure, we can always use the identity map as a holomorphic coordinate chart on $\mathbb{C}^k$, so being holomorphic is equivalent to the requirement that for each $p \in M$, there is a holomorphic chart (U, φ) for M whose domain contains p such that $f \circ \varphi^{-1}$ is holomorphic from $\varphi(U)$ to $\mathbb{C}^k$. It is standard practice to reserve the term ***holomorphic function*** for holomorphic maps whose codomains are open subsets of $\mathbb{C}$ (scalar-valued holomorphic functions) or $\mathbb{C}^k$ (vector valued holomorphic functions); the terms *holomorphic map* and *holomorphic mapping* can refer to maps between arbitrary complex manifolds.

If M is a complex manifold, the notation $\mathcal{O}(M)$ means the set of all holomorphic functions from M to $\mathbb{C}$. This applies, in particular, to any open submanifold of M: if $U \subseteq M$ is open, $\mathcal{O}(U)$ is the set of holomorphic functions from U to $\mathbb{C}$.

A bijective holomorphic map with holomorphic inverse is called a ***biholomorphism***, and a biholomorphism from a complex manifold to itself is called an ***automorphism***. More generally, a map $F : M \to N$ is called a ***local biholomorphism*** if every $p \in M$ has a neighborhood U such that $F|_U$ is a biholomorphism onto an open subset of N.

The following facts about holomorphic maps are proved just like their smooth analogues [**LeeSM**, Props. 2.6 and 2.10 and Example 2.14(b)].

Proposition 1.13.

(a) *The restriction of a holomorphic map to an open subset is holomorphic.*

(b) *If a map f has the property that each point in the domain has a neighborhood U on which the restriction $f|_U$ is holomorphic, then f is holomorphic.*

(c) *Every constant map between complex manifolds is holomorphic.*

(d) *The identity map of every complex manifold is holomorphic.*

(e) *The inclusion map of every open submanifold is holomorphic.*

(f) *Every holomorphic coordinate chart is a biholomorphism onto its image.*

(g) *Every composition of holomorphic maps between complex manifolds is holomorphic.*

Two complex manifolds are said to be ***biholomorphic*** if there is a biholomorphism between them. For example, if V is an n-dimensional complex vector space, any choice of basis determines a complex-linear isomorphism between V and $\mathbb{C}^n$, so all such vector spaces are biholomorphic to $\mathbb{C}^n$. Similarly, a choice of basis yields a biholomorphism between $\mathbb{P}(V)$ and $\mathbb{CP}^{n-1}$, and between $\mathrm{G}_k(V)$ and $\mathrm{G}_k(\mathbb{C}^n)$ for each k. It is easy to check that being biholomorphic is an equivalence relation on the class of all complex manifolds. The main subject matter of this book is properties of complex manifolds that are preserved by biholomorphisms.

Because holomorphic maps are smooth, biholomorphic manifolds are automatically diffeomorphic. However, the converse might not be true: Example 1.31 and Problem 1-4 describe complex manifolds that are diffeomorphic but not biholomorphic.

Covering Manifolds and Quotient Manifolds

In this section, we discuss some ways to produce new complex manifolds from old ones. Recall that a ***covering map*** is a surjective continuous map $\pi : M \to N$ between connected and locally path-connected topological spaces such that every point of N has a neighborhood U that is ***evenly covered***, meaning that $\pi^{-1}(U)$ is a disjoint union of connected open subsets each of which is mapped homeomorphically onto U by π. A covering map $\pi : M \to N$ is said to be ***normal*** if for some $x \in M$, the induced subgroup $\pi_*\big(\pi_1(M,x)\big) \subseteq \pi_1(N,\pi(x))$ is a normal subgroup (meaning it is invariant under conjugation). Equivalently, π is normal if the group of covering automorphisms (homeomorphisms $\varphi : M \to M$ satisfying $\pi \circ \varphi = \pi$) acts transitively on each fiber $\pi^{-1}(y)$. A discussion of the properties of covering maps can be found in [**LeeTM**, Chaps. 11 & 12].

Suppose $\pi : M \to N$ is a covering map. If M and N are smooth manifolds and π is a local diffeomorphism, then it is called a ***smooth covering map***. Properties of smooth covering maps are discussed in [**LeeSM**, pp. 91–95]. Similarly, if M and N are complex manifolds and π is a local biholomorphism, it is called a ***holomorphic covering map***.

► **Exercise 1.14.** Suppose $\pi : M \to N$ is a holomorphic covering map. Show that every point of M is in the image of a ***holomorphic local section*** of π, that is, a holomorphic map $\sigma : U \to M$ defined on an open subset $U \subseteq N$ such that $\pi \circ \sigma = \mathrm{Id}_U$.

The next proposition shows that every covering space of a connected complex manifold is a complex manifold in a natural way.

Proposition 1.15 (Coverings of Complex Manifolds are Complex Manifolds). *Suppose M is a connected complex manifold and $\pi : E \to M$ is a (topological) covering map. Then E is a topological manifold and has a unique holomorphic structure such that π is a holomorphic covering map.*

Proof. Proposition 4.40 in [**LeeSM**] shows that E is a topological manifold and has a unique smooth structure such that π is a smooth covering map. We can define holomorphic charts on E as follows: Given a point $p \in E$, let U be an evenly covered neighborhood of $\pi(p)$. After shrinking U if necessary, we can find a holomorphic coordinate map $\varphi : U \to \mathbb{C}^n$. Let $\widetilde{U}$ be the connected component of $\pi^{-1}(U)$ containing p, and define $\widetilde{\varphi} = \varphi \circ \pi : \widetilde{U} \to \mathbb{C}^n$. The argument in the proof of [**LeeSM**, Prop. 4.40] shows that when two such charts $\left(\widetilde{U}, \widetilde{\varphi}\right)$ and $\left(\widetilde{V}, \widetilde{\psi}\right)$ overlap, in a neighborhood of each point the transition function can be expressed as $\widetilde{\psi}^{-1} \circ \widetilde{\varphi}^{-1} = \psi^{-1} \circ \varphi^{-1}$, which in this case is holomorphic. Then π is a local biholomorphism because its coordinate representation is the identity with respect to the holomorphic coordinates $\left(\widetilde{U}, \widetilde{\varphi}\right)$ on E and (U, φ) on M.

If $\widetilde{E}$ is the same topological space E with another holomorphic structure such that $\pi : \widetilde{E} \to M$ is a holomorphic covering map, then because π is a local biholomorphism, each of the charts constructed above must be a holomorphic chart for $\widetilde{E}$, so the holomorphic structure of $\widetilde{E}$ is the same as the one constructed above. □

Under certain circumstances, we can also put holomorphic structures on manifolds *covered by* complex manifolds. Suppose Γ is a ***discrete Lie group*** (i.e., a countable group with the discrete topology). Recall that an action of Γ on a manifold M is ***free*** if $g \cdot x = x$ for some $g \in \Gamma$ and $x \in M$ implies g is the identity; and it is ***proper*** if the map $\Gamma \times M \to M \times M$ given by $(g, x) \mapsto (g \cdot x, x)$ is a proper map, meaning that the preimage of every compact set is compact. (See [**LeeSM**, pp. 543–544].) If M is a complex manifold, the action is ***holomorphic*** if the map $x \mapsto g \cdot x$ is holomorphic for each $g \in \Gamma$.

Theorem 1.16 (Holomorphic Quotient Manifold Theorem). *Suppose Γ is a discrete Lie group acting holomorphically, freely, and properly on a complex manifold M. Then the quotient space M/Γ has a unique complex manifold structure such that the quotient map $q: M \to M/\Gamma$ is a holomorphic normal covering map.*

Proof. Smooth manifold theory shows that M/Γ has a unique smooth manifold structure such that q is a smooth normal covering map [**LeeSM**, Thm. 21.13]. To define a complex manifold structure on M/Γ, let $U \subseteq M/\Gamma$ be any evenly covered open set, and choose a smooth local section $\sigma: U \to M$. Because M is a complex manifold, $\sigma(U)$ has a covering by holomorphic charts $(U_\alpha, \varphi_\alpha)$, and for each such chart we can define $(\sigma^{-1}(U_\alpha), \varphi_\alpha \circ \sigma))$ as a chart for M/Γ. For a fixed local section σ, all of these charts are holomorphically compatible with each other. If $\widetilde{\sigma}: U \to M$ is any other local section, there is an element $g \in \Gamma$ such that $\widetilde{\sigma}(x) = g \cdot \sigma(x)$ for all $x \in U$; and the fact that $x \mapsto g \cdot x$ is a biholomorphism of M with inverse $x \mapsto g^{-1} \cdot x$ guarantees that the charts obtained from $\widetilde{\sigma}$ will be holomorphically compatible with those obtained from σ. □

A ***complex Lie group*** is a complex manifold G endowed with a group structure such that the multiplication map $m: G \times G \to G$ and the inversion map $i: G \to G$ are holomorphic. Here are some simple examples; we will see more in the next chapter (see Example 2.26).

- Every countable discrete group is a 0-dimensional complex Lie group.
- Every finite-dimensional complex vector space is a complex Lie group under addition.
- The group $\mathrm{GL}(n, \mathbb{C})$ of invertible $n \times n$ complex matrices is a complex Lie group of dimension n^2, with the matrix entries as global holomorphic coordinates. The component functions of the multiplication map are holomorphic polynomials in the matrix entries, and those of the inversion map are holomorphic rational functions.
- Given any n-dimensional complex vector space V, the group $\mathrm{GL}(V)$ of complex linear automorphisms of V becomes a Lie group isomorphic to $\mathrm{GL}(n, \mathbb{C})$ once we choose a basis for V, and the resulting holomorphic structure is independent of the choice of basis.

Corollary 1.17. *Suppose G is a connected complex Lie group and $\Gamma \subseteq G$ is a discrete subgroup. The left coset space G/Γ is a complex manifold, and the quotient map $\pi: G \to G/\Gamma$ is a holomorphic normal covering map. If Γ is also a normal subgroup, then G/Γ is a complex Lie group and π is a group homomorphism.*

Proof. The left coset space G/Γ is the quotient of G by the action of Γ by right translation. This action is holomorphic by the definition of a complex Lie group, and the proof of Theorem 21.17 in [**LeeSM**] shows that it is free and proper. Thus

Theorem 1.16 above shows that G/Γ has the structure of a complex manifold and π is a holomorphic normal covering map.

If Γ is a normal subgroup, then elementary group theory shows that G/Γ is a group and π is a homomorphism. To see that the group operations in G/Γ are holomorphic, just note that given any pair of points $p, q \in G/\Gamma$, we can choose neighborhoods U of p and V of q on which there exist holomorphic local sections $\sigma : U \to G$ and $\tau : V \to G$. Then the multiplication map $\widetilde{m} : G/\Gamma \times G/\Gamma \to G/\Gamma$ can be written in a neighborhood of (p,q) as $\pi \circ m \circ (\sigma \times \tau)$:

$$\begin{tikzcd}
G \times G \arrow[r, "m"] & G \arrow[d, "\pi"] \\
G/\Gamma \times G/\Gamma \arrow[u, "\sigma \times \tau"] \arrow[r, "\widetilde{m}"'] & G/\Gamma.
\end{tikzcd}$$

This is a composition of holomorphic maps and thus holomorphic. A similar argument applies to inversion. □

Example 1.18 (Complex Tori). Suppose V is an n-dimensional complex vector space, considered as an abelian complex Lie group. A ***lattice*** in V is a discrete additive subgroup $\Lambda \subseteq V$ generated by $2n$ vectors $v_1, \dots, v_{2n}$ that are linearly independent over $\mathbb{R}$. Corollary 1.17 shows that V/Λ is an n-dimensional complex Lie group, called a ***complex torus***. When $n = 0$, it is just a single point. When $n > 0$, the real-linear isomorphism $A : \mathbb{R}^{2n} \to V$ given by $A\left(x^1, \dots, x^{2n}\right) = x^j v_j$ descends to a diffeomorphism from $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ to V/Λ; since $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ is diffeomorphic to the $2n$-torus $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, so is V/Λ. Thus the complex tori defined by different lattices are all diffeomorphic to each other. They are typically not biholomorphic, however; see Problem 1-4 for an example. //

Example 1.19 (Hopf Manifolds). Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be an ordered n-tuple of real numbers with $0 < \lambda_j < 1$, and define an action of $\mathbb{Z}$ on $\mathbb{C}^n \smallsetminus \{0\}$ by $k \cdot z = ((\lambda_1)^k z^1, \dots, (\lambda_n)^k z^n)$. This action is holomorphic, free, and proper, so the quotient $H_\lambda = (\mathbb{C}^n \smallsetminus \{0\})/\mathbb{Z}$ is an n-dimensional complex manifold called a ***Hopf manifold***. Regarding $\mathbb{S}^{2n-1}$ as the set of unit vectors in $\mathbb{C}^n$, we define a smooth map $A : \mathbb{S}^{2n-1} \times \mathbb{R} \to \mathbb{C}^n \smallsetminus \{0\}$ by $A(z,t) = ((\lambda_1)^t z^1, \dots, (\lambda_n)^t z^n)$; if $\pi : \mathbb{C}^n \smallsetminus \{0\} \to H_\lambda$ is the quotient map, one can check that $\pi \circ A$ makes the same identifications as the quotient map from $\mathbb{S}^{2n-1} \times \mathbb{R}$ to $\mathbb{S}^{2n-1} \times (\mathbb{R}/\mathbb{Z}) \approx \mathbb{S}^{2n-1} \times \mathbb{S}^1$, so all Hopf manifolds are diffeomorphic to $\mathbb{S}^{2n-1} \times \mathbb{S}^1$. //

Example 1.20 (Iwasawa Manifolds). Consider the subgroup $G \subseteq \mathrm{GL}(3, \mathbb{C})$ consisting of matrices of the form

$$\begin{pmatrix} 1 & z^1 & z^3 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{pmatrix}$$

for $z^1, z^2, z^3 \in \mathbb{C}$. It is a complex Lie group, biholomorphic to $\mathbb{C}^3$, with multiplication given by

$$(z^1, z^2, z^3) \cdot (w^1, w^2, w^3) = (z^1 + w^1, z^2 + w^2, z^3 + w^3 + z^1 w^2).$$

For a discrete subgroup $\Gamma \subseteq G$, the left coset space G/Γ is a complex 3-manifold by Corollary 1.17. An ***Iwasawa manifold*** is a left coset space of the form G/Γ for a discrete subgroup Γ that is ***cocompact***, meaning that G/Γ is compact. (Some authors use quotients by left Γ-actions in their definitions, corresponding to right coset spaces; group inversion in G induces a biholomorphism between the left and right coset spaces, so there is no real difference.) The simplest example is the ***standard Iwasawa manifold***, obtained by taking Γ to be the subgroup consisting of matrices in which z^1, z^2, z^3 are ***Gaussian integers***, that is, complex numbers of the form $m + ni$ for $m, n \in \mathbb{Z}$. It is cocompact by the result of Problem 1-1. //

Some Complex Analysis

In this book, we assume you are familiar with basic undergraduate-level complex analysis in one variable; if your complex analysis is rusty, this would be a good time to review. (Some suggested texts are listed in the Preface.)

Recall the definition of a holomorphic function of one complex variable: if $W \subseteq \mathbb{C}$ is an open subset and $f : W \to \mathbb{C}$ is a function, then f is said to be ***holomorphic*** if it has a complex derivative at each point $p \in W$, defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Holomorphic functions are sometimes called ***complex-analytic***, or just ***analytic*** if there can be no confusion with real-analytic functions.

For convenience, let us recall some basic facts from the one-variable theory. In these statements, W represents an arbitrary open subset of $\mathbb{C}$ and $f : W \to \mathbb{C}$ is an arbitrary holomorphic function. We write the standard coordinate on $\mathbb{C}$ as $z = x + iy$.

- CAUCHY INTEGRAL FORMULA: If $a \in W$ and $r > 0$ is chosen so that the closed disk $\overline{D}_r(a)$ is contained in W, then the following formula holds for all z in the open disk $D_r(a)$:

$$h(z) = \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{h(\zeta)}{\zeta - z}\, d\zeta.$$

- CAUCHY–RIEMANN EQUATIONS: The real and imaginary parts u and v of f satisfy the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

- Power Series Expansion: For each $a \in W$, if the disk $D_r(a)$ is contained in W, then $f|_D$ is equal to a convergent series in powers of $(z-a)$. It has complex derivatives of all orders, which may be computed by differentiating the series term-by-term.
- Zeros Are Isolated and Have Finite Order: If $f(a) = 0$ for some $a \in W$ and f is not identically zero, then there is a disk $D_r(a) \subseteq W$ such that $f(z) \neq 0$ for $z \in D_r(a) \smallsetminus \{a\}$; and there is a positive integer m (called the ***order*** or ***multiplicity*** of the zero), such that $f(z) = (z-a)^m h(z)$ for some holomorphic function h that does not vanish at a. The order of a zero is equal to the smallest integer m such that $f^{(m)}(a) \neq 0$. A zero of order 1 is called a ***simple zero***.
- Maximum Principle: If W is connected and $|f(z)|$ attains a maximum at a point $z \in W$, then f is constant.
- Liouville's Theorem: If $W = \mathbb{C}$ and f is bounded, then it is constant.
- Riemann's Removable Singularity Theorem: If $W = \widetilde{W} \smallsetminus \{a\}$ for some open set $\widetilde{W}$ and some point $a \in \widetilde{W}$, and f is bounded, then f extends to a holomorphic function on all of $\widetilde{W}$.

For our study of complex manifolds, we need to extend some of the results of the one-variable theory to functions of several complex variables. Many of these results will look familiar, but some properties of holomorphic functions are decidedly different in higher dimensions.

We begin with the official definition of holomorphic functions of several variables. Suppose $U \subseteq \mathbb{C}^n$ is an open subset and $f : U \to \mathbb{C}$. For $p = (p^1, \dots, p^n) \in U$ and $j \in \{1, \dots, n\}$, we say f has a ***complex partial derivative at p with respect to z^j*** if the following limit exists:

$$\frac{\partial f}{\partial z^j}(p) = \lim_{h \to 0} \frac{f(p^1, \dots, p^j + h, \dots, p^n) - f(p^1, \dots, p^n)}{h}, \tag{1.2}$$

where the limit is taken over all h in some punctured disk centered at the origin in $\mathbb{C}$. Such a function is said to be ***holomorphic*** if it is continuous and has a complex partial derivative with respect to each variable $z^1, \dots, z^n$ at each point of U. More generally, a vector-valued function $F : U \to \mathbb{C}^k$ is said to be holomorphic if each of its component functions is holomorphic.

Our definition of holomorphic functions is essentially the same as the one-variable definition, except in that case the assumption of continuity is not needed because a simple argument shows that continuity follows from the existence of a complex derivative. It is worth noting, in fact, that the continuity assumption is actually not needed in higher dimensions either: the German mathematician Friedrich

Hartogs proved in 1906 [**Har06**] that a function that has complex partial derivatives at every point of an open subset of $\mathbb{C}^n$ is automatically continuous. That proof (which can be found in [**Kra01**, Section 2.4]) is difficult, though, so it is much more convenient simply to assume continuity as part of our definition.

In one complex variable, there are several equivalent ways to characterize holomorphic functions: having a complex derivative everywhere, or having continuous partial derivatives that satisfy the Cauchy–Riemann equations, or being the sum of a convergent power series in a neighborhood of each point. There are similar equivalent characterizations for holomorphic functions of several variables.

Theorem 1.21. *Let $U \subseteq \mathbb{C}^n$ be open and $f : U \to \mathbb{C}$. The following are equivalent.*

(a) *f is holomorphic (i.e., it is continuous and has a complex partial derivative with respect to each variable at each point of U).*

(b) *f is smooth and satisfies the following Cauchy–Riemann equations:*

$$\frac{\partial u}{\partial x^j} = \frac{\partial v}{\partial y^j}, \qquad \frac{\partial u}{\partial y^j} = -\frac{\partial v}{\partial x^j}, \qquad j = 1, \dots, n,$$

where $z^j = x^j + iy^j$ and $f(z) = u(z) + iv(z)$.

(c) *For each $p = (p^1, \dots, p^n) \in U$, there exists a neighborhood of p in U on which f is equal to the sum of an absolutely convergent power series of the form*

$$f(z) = \sum_{k_1,\dots,k_n=0}^{\infty} a_{k_1\dots k_n}(z^1 - p^1)^{k_1} \cdots (z^n - p^n)^{k_n}. \tag{1.3}$$

Remarks.

- In the decomposition $f(z) = u(z) + iv(z)$ in part (b), it is understood that $u(z)$ and $v(z)$ are real. The same applies to $z^j = x^j + iy^j$ and everywhere in the book when we write such a decomposition, unless otherwise specified.
- In (c), the reason we insist on absolute convergence is that a sum over multiple indices can be ordered in various ways, and absolute convergence ensures that the ordering of terms does not matter.

Proof. We will prove (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c).

Suppose f satisfies (a). Because f is holomorphic in each variable separately, the one-variable theory shows that it satisfies the Cauchy–Riemann equations with respect to each variable. To show that it is smooth, given $p \in U$, choose $r > 0$ such that the closed polydisk $\overline{D}^n_r(p)$ is contained in U. Because f is holomorphic in each variable separately, we can apply the single-variable version of the Cauchy integral formula repeatedly to obtain the following for all $z \in D_r^n(p)$:

$$
\begin{aligned}
&f(z^1, \dots, z^n)\\
&= \frac{1}{2\pi i} \int\limits_{|\zeta^n - p^n| = r} \frac{f(z^1, \dots, z^{n-1}, \zeta^n)}{\zeta^n - z^n}\, d\zeta^n\\
&= \frac{1}{(2\pi i)^2} \int\limits_{|\zeta^n - p^n| = r} \int\limits_{|\zeta^{n-1} - p^{n-1}| = r} \frac{f(z^1, \dots, \zeta^{n-1}, \zeta^n)}{(\zeta^n - z^n)(\zeta^{n-1} - z^{n-1})}\, d\zeta^{n-1} d\zeta^n \qquad (1.4)\\
&\ \ \vdots\\
&= \frac{1}{(2\pi i)^n} \int\limits_{|\zeta^n - p^n| = r} \cdots \int\limits_{|\zeta^1 - p^1| = r} \frac{f(\zeta^1, \dots, \zeta^n)}{(\zeta^n - z^n) \cdots (\zeta^1 - z^1)}\, d\zeta^1 \cdots d\zeta^n.
\end{aligned}
$$

Since the domain of integration is compact and the integrand is continuous in all variables and smooth as a function of (the real and imaginary parts of) $z^1, \dots, z^n$, we can differentiate under the integral sign as often as we like with respect to x^j and y^j to conclude that f is smooth. This proves (b).

To prove that f also satisfies (c), note that

$$
\frac{1}{\zeta^j - z^j} = \frac{1}{(\zeta^j - p^j) - (z^j - p^j)} = \frac{1}{\zeta^j - p^j} \, \frac{1}{1 - \left(\dfrac{z^j - p^j}{\zeta^j - p^j} \right)},
$$

and since $|z^j - p^j|/|\zeta^j - p^j| < 1$ on the domain of integration in (1.4), we can expand the last fraction on the right in a power series to obtain

$$
\frac{1}{\zeta^j - z^j} = \frac{1}{\zeta^j - p^j} \sum_{k=0}^{\infty} \left(\frac{z^j - p^j}{\zeta^j - p^j} \right)^k,
$$

which converges uniformly and absolutely for z^j in any closed disk $\overline{D}_{r'}(p^j)$ with $0 < r' < r$ by comparison with the geometric series $\sum_k (r'/r)^k$. Inserting this formula for each variable into (1.4), we conclude that f satisfies (1.3) with coefficients

$$
a_{k_1 \dots k_n} = \int\limits_{|\zeta^n - p^n| = r} \cdots \int\limits_{|\zeta^1 - p^1| = r} \frac{f(\zeta^1, \dots, \zeta^n)}{(\zeta^n - p^n)^{k_n + 1} \cdots (\zeta^1 - p^1)^{k_1 + 1}}\, d\zeta^1 \cdots d\zeta^n.
$$

This completes the proof that (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c).

Conversely, if f satisfies (b), then it is certainly continuous, and the one-variable theory implies that it has a complex derivative with respect to each variable, so it also satisfies (a).

Finally, assume f satisfies (c), and let $p \in U$ be arbitrary. There is some closed polydisk $\overline{D}{}_r^n(p)$ contained in U and centered at p on which the series converges absolutely. Because the series converges at $z_0 = (p^1 + r, \dots, p^n + r)$, the terms in the series for $f(z_0)$ are all uniformly bounded, which means there is a constant C such that

$$|a_{k_1\dots k_n}|r^{k_1}\cdots r^{k_n} \le C.$$

On a polydisk $D^n_{r'}(p)$ for any $0 < r' < r$, the terms of the series satisfy the following bound:

$$\sum_{k_1,\dots,k_n=0}^{\infty} \left| a_{k_1\dots k_n}(z^1-p^1)^{k_1}\cdots(z^n-p^n)^{k_n}\right| \le \sum_{k_1,\dots,k_n=0}^{\infty} C\left(\frac{r'}{r}\right)^{k_1}\cdots\left(\frac{r'}{r}\right)^{k_n}, \tag{1.5}$$

and the series on the right is an iterated convergent geometric series. Therefore, the series for f converges uniformly and absolutely on $D^n_{r'}(p)$ by the Weierstrass M-test, so f is continuous there, and in particular at p. If we fix all the variables but z^j, we obtain a convergent power series in z^j, which is therefore holomorphic in z^j by the one-variable theory, thus proving (a). □

Next we enumerate the basic properties of holomorphic functions that we will use throughout the book.

Proposition 1.22 (Compositions of Holomorphic Functions are Holomorphic). *Suppose $Z \subseteq \mathbb{C}^m$ and $W \subseteq \mathbb{C}^n$ are open subsets and $f : Z \to W$ and $g : W \to \mathbb{C}^k$ are holomorphic functions. Then $g \circ f : Z \to \mathbb{C}^k$ is holomorphic.*

Proof. Certainly $g \circ f$ is smooth, so we just need to check that it satisfies the Cauchy–Riemann equations. Let us write the variables in Z as $z^j = x^j + iy^j$, those in W as $w^j = u^j + iv^j$, and the component functions of f and g as $f^k(z) = U^k(z) + iV^k(z)$, $g^l(w) = A^l(w) + iB^l(w)$. Applying the real-variable chain rule (and using the summation convention), we find

$$\frac{\partial(A^l\circ f)}{\partial x^j} - \frac{\partial(B^l\circ f)}{\partial y^j} = \frac{\partial A^l}{\partial u^k}\frac{\partial U^k}{\partial x^j} + \frac{\partial A^l}{\partial v^k}\frac{\partial V^k}{\partial x^j} - \frac{\partial B^l}{\partial u^k}\frac{\partial U^k}{\partial y^j} - \frac{\partial B^l}{\partial v^k}\frac{\partial V^k}{\partial y^j}.$$

Using the Cauchy–Riemann equations for g to replace $\partial A^l/\partial u^k$ by $\partial B^l/\partial v^k$ and $\partial A^l/\partial v^k$ by $-\partial B^l/\partial u^k$ and then applying the Cauchy–Riemann equations for f, we see that this expression is identically zero. A similar computation shows that the composition also satisfies the other set of Cauchy–Riemann equations. □

Proposition 1.23. *Suppose $f, g : U \to \mathbb{C}$ are holomorphic functions on an open subset $U \subseteq \mathbb{C}^n$. Then $f + g$, $f - g$, and fg are holomorphic on U, and f/g is holomorphic on $U \smallsetminus g^{-1}(0)$.*

► **Exercise 1.24.** Prove this proposition.

It follows easily from the two preceding propositions, for example, that all polynomial functions of $z^1, \dots, z^n$ are holomorphic on $\mathbb{C}^n$, and all rational functions (quotients of polynomials) are holomorphic wherever their denominators are nonzero.

Our next proposition relates partial derivatives with respect to complex variables to those with respect to real variables. If $f = u + iv$ is a complex-valued function, the notation $\partial f/\partial x^j$ denotes the complex-valued function $\partial u/\partial x^j + i\partial v/\partial x^j$, and similarly with y^j derivatives.

Proposition 1.25. *Suppose U is an open subset of $\mathbb{C}^n$ and $f : U \to \mathbb{C}$ is holomorphic. Writing $z^j = x^j + iy^j$ for $j = 1, \dots, n$, we have*

$$\frac{\partial f}{\partial z^j} = \frac{\partial f}{\partial x^j} = \frac{1}{i}\frac{\partial f}{\partial y^j}. \tag{1.6}$$

Proof. Note that the existence of the limit in (1.2) as h approaches zero through all complex values implies that we obtain the same limit if we restrict h to approach zero through real values only or imaginary values only. Thus for any $p \in U$,

$$\frac{\partial f}{\partial z^j}(p) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(p^1, \dots, p^j + h, \dots, p^n) - f(p^1, \dots, p^n)}{h} = \frac{\partial f}{\partial x^j}(p),$$

$$\frac{\partial f}{\partial z^j}(p) = \lim_{\substack{k \to 0 \\ k \in \mathbb{R}}} \frac{f(p^1, \dots, p^j + ik, \dots, p^n) - f(p^1, \dots, p^n)}{ik} = \frac{1}{i}\frac{\partial f}{\partial y^j}(p). \qquad \square$$

Next we establish some important properties of multivariable power series.

Proposition 1.26. *Suppose f is a holomorphic function given by an absolutely convergent power series of the form* (1.3) *on a polydisk $D^n_r(p) \subseteq \mathbb{C}^n$. The complex partial derivatives of f of all orders exist and are given by absolutely convergent power series on the same polydisk, which can be computed by differentiating the series term by term.*

Proof. For any $0 < r' < r_1 < r$, the series converges absolutely on $\overline{D}^n_{r_1}(p)$, and thus the proof of Theorem 1.21 shows that it converges uniformly and absolutely on $D^n_{r'}(p)$. Note that the complex derivative $\partial f/\partial z^j$ is equal to the real partial derivative $\partial f/\partial x^j$ by Proposition 1.25. A standard result in real analysis [**Rud76**, Thm. 7.17] shows that we can differentiate the power series term by term with respect to x^j on $D^n_{r'}(p)$ provided the differentiated series converges uniformly there.

With notation as in (1.5), the differentiated series satisfies

$$\begin{aligned}
&\sum_{k_1,\dots,k_n=0}^{\infty} \left|\frac{\partial}{\partial x^j}\left(a_{k_1\dots k_n}(z^1-p^1)^{k_1}\cdots(z^n-p^n)^{k_n}\right)\right| \\
&\quad = \sum_{k_1,\dots,k_n=0}^{\infty} \left|a_{k_1\dots k_n}(z^1-p^1)^{k_1}\cdots k_j(z^j-p^j)^{k_j-1}\cdots(z^n-p^n)^{k_n}\right| \\
&\quad \le \sum_{k_1,\dots,k_n=0}^{\infty} C\left(\frac{r'}{r_1}\right)^{k_1}\cdots k_j\left(\frac{r'}{r_1}\right)^{k_j-1}\cdots\left(\frac{r'}{r_1}\right)^{k_n}.
\end{aligned}$$

The last expression is an iterated sum in which $n-1$ of the sums are convergent geometric series, while the jth one is the series $\sum_k kx^{k-1}$, which converges absolutely for $|x|<1$ by the ratio test. Thus we may apply the Weierstrass M-test again to conclude that the differentiated series converges uniformly and absolutely on $D^n_{r'}(p)$, and therefore is equal to the derivative of f there. Since every point in $D^n_r(p)$ lies in $D^n_{r'}(p)$ for some $0<r'<r_1<r$, it follows that $\partial f/\partial z^j$ is equal to the sum of the differentiated series on all of $D^n_r(p)$. It then follows by induction that the same is true of all higher complex derivatives. □

Corollary 1.27. *If f is a holomorphic function given by an absolutely convergent power series of the form* (1.3) *on a polydisk $D^n_r(p)\subseteq\mathbb{C}^n$, then the power series is given explicitly by the following formula, called the* ***Taylor series of f centered at p****:*

$$f(z)=\sum_{k_1,\dots,k_n=0}^{\infty}\frac{1}{k_1!\cdots k_n!}\,\frac{\partial^{k_1+\dots+k_n}f(p)}{(\partial z^1)^{k_1}\cdots(\partial z^n)^{k_n}}(z^1-p^1)^{k_1}\cdots(z^n-p^n)^{k_n}.$$

Proof. Just differentiate (1.3) repeatedly term-by-term and evaluate at $z=p$ to determine the coefficients $a_{k_1\dots k_n}$. □

Proposition 1.28 (Identity Theorem). *Suppose $W\subseteq\mathbb{C}^n$ is a connected open subset, and $f,g\colon W\to\mathbb{C}$ are holomorphic functions that agree on a nonempty open subset of W. Then $f\equiv g$ on W.*

Proof. Set $h=f-g$, so $h\equiv 0$ on a nonempty open subset $U_0\subseteq W$. Let

$$U=\{p\in W : h \text{ and its complex partial derivatives of all orders vanish at } p\}.$$

Then U is nonempty because $U_0\subseteq U$. We will show that it is open and closed in W, which implies by connectivity that it is all of W.

Suppose $p\in U$. Then h is equal to a convergent power series in a neighborhood of p, and Corollary 1.27 shows that every term in the series is zero. Thus U is open in W.

Now suppose $p \in W$ is a limit point of U. There is a sequence of points $p_j \in U$ converging to p, and the hypothesis implies that all partial derivatives of h vanish at each p_j. Thus by continuity, they also vanish at p, showing that $p \in U$. Thus U is closed in W. $\square$

Corollary 1.29 (Identity Theorem for Manifolds). *Suppose M and N are complex manifolds with M connected, and $f, g : M \to N$ are holomorphic maps that agree on a nonempty open subset of M. Then $f \equiv g$ on M.*

Proof. Proposition 1.28 applied to local coordinate representations of f and g shows that the set of points where f and g agree along with their partial derivatives of all orders is both open and closed in M, hence all of M. $\square$

Proposition 1.30 (Liouville's Theorem). *Every holomorphic function that is defined on all of $\mathbb{C}^n$ and bounded is constant.*

Proof. Suppose $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic and bounded. Given any point $z \in \mathbb{C}^n$, the function $g(\zeta) = f(\zeta z)$ is a bounded holomorphic function defined on all of $\mathbb{C}$, so it is constant by the one-variable version of Liouville's theorem. In particular, this means $f(z) = f(0)$. Since z is arbitrary, this shows f is constant. $\square$

Liouville's theorem allows us to give our first example of two complex manifolds that are diffeomorphic but not biholomorphic.

Example 1.31 (The Unit Ball is Not Biholomorphic to $\mathbb{C}^n$). We know that $\mathbb{B}^{2n}$ and $\mathbb{C}^n$ are diffeomorphic (see [**LeeSM**, Example 2.14]). But if $F : \mathbb{C}^n \to \mathbb{B}^{2n}$ is any holomorphic map, each of its coefficient functions is a bounded holomorphic function on $\mathbb{C}^n$ and therefore constant. Thus there is no biholomorphism between $\mathbb{B}^{2n}$ and $\mathbb{C}^n$. //

Proposition 1.32 (The Maximum Principle). *Suppose $f : U \to \mathbb{C}$ is a holomorphic function on a connected open set $U \subseteq \mathbb{C}^n$. If $|f(z)|$ attains a maximum value at some point in U, then f is constant.*

Proof. Suppose $|f(z)|$ attains a maximum value at $z_0 \in U$. Let $c = f(z_0)$, and set $W = \{z \in U : f(z) = c\}$. Then W is nonempty because $z_0 \in W$, and it is closed in U by continuity. Given $z_1 \in W$, choose $\varepsilon > 0$ such that the ball $B_\varepsilon(z_1)$ is contained in U. For each $w \in \mathbb{C}^n$ with $|w| = 1$, the function $g(\zeta) = f(z_1 + \zeta w)$ is holomorphic on the disk $D_\varepsilon(0) \subseteq \mathbb{C}$ and achieves its maximum modulus at $\zeta = 0$. By the one-variable maximum principle, therefore, g is constant. Since w is arbitrary, this shows f is constant on $B_\varepsilon(z_1)$. Thus W is open, and by connectivity it is all of U. $\square$

This result too has an immediate, and somewhat surprising, application to complex manifolds.

Corollary 1.33. *Let M be a connected compact complex manifold. Then every globally defined holomorphic function from M to $\mathbb{C}$ is constant.*

Proof. Suppose $f \in \mathcal{O}(M)$. By compactness, the continuous function $|f|$ attains a maximum value at a point $z_0 \in M$. In a holomorphic coordinate ball centered at z_0, the coordinate representation of f is a holomorphic function on an open ball in $\mathbb{C}^n$ that attains its maximum modulus at the origin, so it is constant on the entire coordinate domain. Thus by the identity theorem, it is constant on all of M. □

One of the most striking features of holomorphic functions is described in the next proposition, which shows in particular that uniform limits of holomorphic functions are holomorphic. It is worth noting that the analogous result for smooth functions, or even real-analytic functions, is not true.

Proposition 1.34. *Suppose $U \subseteq \mathbb{C}^n$ is open and $f_k : U \to \mathbb{C}$ is a sequence of holomorphic functions that converge uniformly on compact subsets of U to a function $f : U \to \mathbb{C}$. Then f is holomorphic.*

Proof. Given $p \in U$, choose $r > 0$ such that $\overline{D}^n_r(p) \subseteq U$. For all $z \in D^n_r(p)$, we can apply the Cauchy integral formula to f_k, and uniform convergence guarantees that

$$\begin{aligned} f(z) &= \lim_{k\to\infty} \frac{1}{(2\pi i)^n} \int\limits_{|\zeta^n - p^n| = r} \cdots \int\limits_{|\zeta^1 - p^1| = r} \frac{f_k(\zeta^1, \dots, \zeta^n)}{(\zeta^n - z^n)\cdots(\zeta^1 - z^1)}\, d\zeta^1 \cdots d\zeta^n \\ &= \frac{1}{(2\pi i)^n} \int\limits_{|\zeta^n - p^n| = r} \cdots \int\limits_{|\zeta^1 - p^1| = r} \frac{f(\zeta^1, \dots, \zeta^n)}{(\zeta^n - z^n)\cdots(\zeta^1 - z^1)}\, d\zeta^1 \cdots d\zeta^n. \end{aligned}$$

The integrand in the last expression is continuous in all variables and smooth in $z^1, \dots, z^n$, so we can differentiate under the integral sign with respect to x^j and y^j as many times as we like to conclude that f is smooth. In particular, since the integrand is holomorphic in $z^1, \dots, z^n$, we see that f satisfies the Cauchy–Riemann equations on $D^n_r(p)$. □

Our next result is a little less elementary, so its one-variable analogue is not always covered in undergraduate complex analysis texts. We will use it only once, when we study sections of holomorphic vector bundles (Thm. 3.13).

Proposition 1.35 (Montel's Theorem). *Suppose $U \subseteq \mathbb{C}^n$ is open and $f_k : U \to \mathbb{C}$ is a sequence of holomorphic functions that are uniformly bounded, meaning there is some $C > 0$ such that $|f_k(z)| < C$ for all $k \geq 1$ and all $z \in U$. Then there is a subsequence $\{f_{k_j}\}_{j=1}^\infty$ that converges uniformly on compact subsets of U to a holomorphic function defined on all of U.*

Proof. For any closed polydisk $\overline{D}^n_r(p) \subseteq U$, we can use Cauchy's formula to write each f_k on $D^n_r(p)$ in the form

$$f_k(z) = \frac{1}{(2\pi i)^n} \int\limits_{|\zeta^n - p^n| = r} \cdots \int\limits_{|\zeta^1 - p^1| = r} \frac{f_k(\zeta^1, \dots, \zeta^n)}{(\zeta^n - z^n) \cdots (\zeta^1 - z^1)}\, d\zeta^1 \cdots d\zeta^n.$$

Differentiating under the integral sign, we obtain

$$\frac{\partial f_k(z)}{\partial x^j} = \frac{1}{(2\pi i)^n} \int \cdots \int \frac{f_k(\zeta^1, \dots, \zeta^n)}{(\zeta^n - z^n) \cdots (\zeta^j - z^j)^2 \cdots (\zeta^1 - z^1)}\, d\zeta^1 \cdots d\zeta^n.$$

A simple computation shows that a contour integral over a circle c of radius r satisfies $\left|\int_c h(\zeta) d\zeta\right| \leq 2\pi r \sup_c |h|$. Applying this in turn to each contour integral in the above formula gives

$$\left|\frac{\partial f_k(z)}{\partial x^j}\right| \leq \frac{C}{r}.$$

This shows that the partial derivatives of f_k with respect to $x^1, \dots, x^n$ are uniformly bounded on $D^n_r(p)$, and then the Cauchy–Riemann equations show the same is true of the derivatives with respect to $y^1, \dots, y^n$. Therefore, each f_k satisfies a Lipschitz estimate of the form $|f_k(z_1) - f_k(z_2)| \leq (C'/r)|z_1 - z_2|$ there. By continuity, the same bound holds on the closed polydisk $\overline{D}^n_r(p)$. Thus the functions f_k are uniformly bounded and uniformly equicontinuous on $\overline{D}^n_r(p)$, so the Arzelà–Ascoli theorem [**Rud76**, Thm. 7.25] guarantees that a subsequence $\{f_{k_j}\}_{j=1}^\infty$ converges uniformly there.

Every $p \in U$ is contained in some polydisk $D^n_r(p)$ such that $\overline{D}^n_r(p) \subseteq U$. The set of all such polydisks is an open cover of U, and thus U is covered by countably many such polydisks. Let $\{V_m\}_{m=1}^\infty$ be such a countable cover. By the above argument, we may choose a subsequence $\{f_{1,j}\}_{j=1}^\infty$ of the original sequence that converges uniformly on $\overline{V}_1$. From that subsequence, we may choose a further subsequence $\{f_{2,j}\}_{j=1}^\infty$ that also converges uniformly on $\overline{V}_2$. Continuing by induction, for each m we get a subsequence $\{f_{m,j}\}_{j=1}^\infty$ converging uniformly on $\overline{V}_1 \cup \cdots \cup \overline{V}_m$, such that the mth sequence is a subsequence of the $(m-1)$st one. Finally, let $\{f_{k_j}\}_{j=1}^\infty$ be the diagonal subsequence $f_{k_j} = f_{j,j}$. If $K \subseteq U$ is any compact set, there is some m such that $K \subseteq V_1 \cup \cdots \cup V_m$. Since $\{f_{k_j}\}$ is a subsequence of $\{f_{i,j}\}$ for each i, it converges uniformly on K. By Proposition 1.34, the limit function is holomorphic. □

So far, all these facts about holomorphic functions of several variables have been straightforward generalizations of standard facts about holomorphic functions of one variable. The next result, however, is radically different from anything in the one-variable theory. It was proved by Friedrich Hartogs in 1906 [**Har06**].

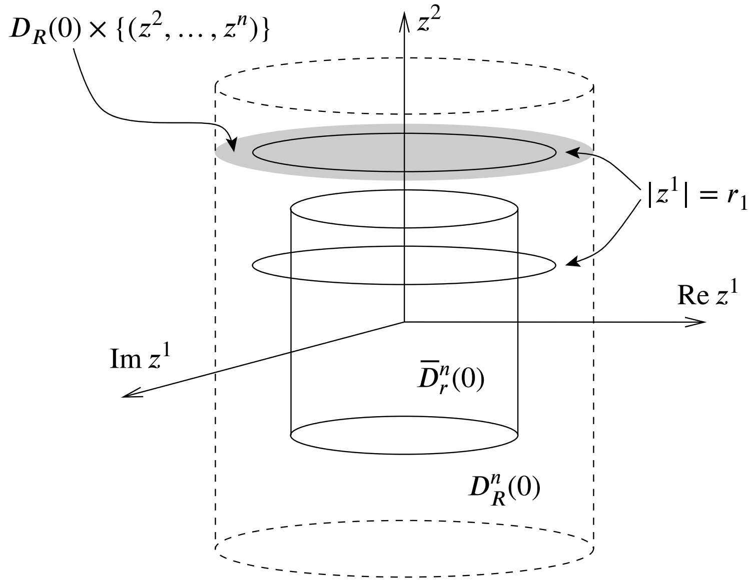

Figure 1.1. Proof of Hartogs's extension theorem

Theorem 1.36 (Hartogs's Extension Theorem). *Let $n \geq 2$, and let $\Omega \subseteq \mathbb{C}^n$ be an open set of the form $D^n_R(p) \smallsetminus \overline{D}^n_r(p)$ for some $p \in \mathbb{C}^n$ and $0 < r < R$. Every holomorphic function $f : \Omega \to \mathbb{C}$ has a unique extension to a holomorphic function on all of $D^n_R(p)$.*

Proof. After a translation, we may assume that $p = 0$. Choose any r_1 such that $r < r_1 < R$. As long as $r < |z^2| < R$, the function $z^1 \mapsto f(z^1, \ldots, z^n)$ is holomorphic on the entire disk $D_R(0) \subseteq \mathbb{C}$ (see Fig. 1.1), so Cauchy's formula shows that

$$f(z^1, \ldots, z^n) = \frac{1}{2\pi i} \int_{|\zeta| = r_1} \frac{f(\zeta, z^2, \ldots, z^n)}{\zeta - z^1}\, d\zeta.$$

But this formula actually makes sense for all $(z^1, \ldots, z^n) \in D^n_{r_1}(0)$ because the integration contour is contained in Ω in that case, and it defines a holomorphic function f_1 there by differentiation under the integral sign. Because f_1 agrees with f on the open subset of $D^n_{r_1}(0)$ where $r < |z^2| < r_1$, the identity theorem shows that it agrees on the entire connected set $D^n_{r_1}(0) \smallsetminus \overline{D}^n_r(0)$. Thus we can define a holomorphic function on all of $D^n_R(0)$ by letting it be equal to f on Ω and to f_1 on $D^n_{r_1}(0)$. Uniqueness follows immediately from the identity theorem. □

This theorem is false in the case $n = 1$, because there are many holomorphic functions with isolated singularities, such as $1/z$ or $e^{1/z}$, which are holomorphic on annuli centered at a singular point but have no holomorphic extensions across that point. Hartogs's theorem implies that singularities of holomorphic functions in two or more variables are never isolated. Moreover, it says something important about zeros of holomorphic functions as well. In one complex variable, zeros of holomorphic functions of one variable are always isolated. But if a holomorphic function f had an isolated zero at $p \in \mathbb{C}^n$ with $n \geq 2$, then $1/f$ would have an isolated singularity, which is impossible. Thus zeros of holomorphic functions of more than one variable are never isolated either.

The Complexified Tangent and Cotangent Bundles

Now we introduce some extensions to the theory of smooth manifolds that we will need for working with complex-valued functions. Writing such a function as $f = u + iv$, we would like to express its differential as $df = du + i\,dv$. But this is not an ordinary 1-form in the sense that the term is used in smooth manifold theory: sections of a real vector bundle like the cotangent bundle can be multiplied by real numbers, but not by complex ones.

To make sense of this, we make the following definition. If V is a real vector space, we define the ***complexification of V***, denoted by $V_{\mathbb{C}}$, to be the vector space $V \oplus V$ with multiplication by complex numbers defined as follows:

$$(a + ib)(u, v) = (au - bv, av + bu) \quad \text{for } a + ib \in \mathbb{C}.$$

Together with the usual addition in $V \oplus V$, it turns $V_{\mathbb{C}}$ into a vector space over $\mathbb{C}$. The map $u \mapsto (u, 0)$ is a real-linear isomorphism from V onto the (real) subspace $V \oplus \{0\} \subseteq V_{\mathbb{C}}$, and we typically identify V with its image under this map, thus considering V itself to be a real-linear subspace of $V_{\mathbb{C}}$. With this identification, we can write $(u, v) = u + iv$, and we can think of $V_{\mathbb{C}}$ as consisting of the set of all linear combinations of elements of V with complex coefficients.

If $(b_1, \dots, b_n)$ is any basis for V (over $\mathbb{R}$), then $\big((b_1, 0), \dots, (b_n, 0)\big)$ is a basis for $V_{\mathbb{C}}$ over $\mathbb{C}$, which under our identification we can just write as $(b_1, \dots, b_n)$. It follows that the complex dimension of $V_{\mathbb{C}}$ is the same as the real dimension of V.

For example, the complexification of $\mathbb{R}^n$ can be naturally identified with $\mathbb{C}^n$.

If $L \colon V \to W$ is a linear map between real vector spaces, it extends canonically to a complex-linear map $L_{\mathbb{C}} \colon V_{\mathbb{C}} \to W_{\mathbb{C}}$, called the ***complexification of L***, satisfying $L_{\mathbb{C}}(u + iv) = L(u) + iL(v)$. In cases where it will not cause confusion, we will often denote the complexification of a linear map L by the same symbol L.

► **Exercise 1.37.** Show that the assignment $V \mapsto V_{\mathbb{C}}$, $L \mapsto L_{\mathbb{C}}$ defines a covariant functor from the category of real vector spaces to the category of complex ones.

The next exercise describes an alternative definition of the complexification.

► **Exercise 1.38.** Let V be a real vector space. Give the space $V \otimes_{\mathbb{R}} \mathbb{C}$ (the abstract tensor product of V and $\mathbb{C}$, considered as real vector spaces), the structure of a complex vector space with the usual addition and with scalar multiplication defined by

$$\alpha\Big(\sum_{j=1}^{k} v_j \otimes \beta_j\Big) = \sum_{j=1}^{k} v_j \otimes (\alpha\beta_j),$$

for $v_j \in V$ and $\alpha, \beta_j \in \mathbb{C}$. Show that this turns $V \otimes_{\mathbb{R}} \mathbb{C}$ into a complex vector space, which is canonically isomorphic to $V_{\mathbb{C}}$ via the map $(u, v) \mapsto u \otimes 1 + v \otimes i$.

► **Exercise 1.39.** Suppose V is a real vector space.

(a) Given $w = (u, v) \in V_{\mathbb{C}}$, define the ***conjugate of w*** by $\overline{w} = (u, -v)$. Show that the map $w \mapsto \overline{w}$ is a bijective conjugate-linear map from $V_{\mathbb{C}}$ to itself satisfying $\overline{\overline{w}} = w$ for all $w \in V_{\mathbb{C}}$. (A map $F : V \to W$ between complex vector spaces is said to be ***conjugate-linear*** if it is linear over $\mathbb{R}$ and satisfies $F(\alpha v) = \bar{\alpha} F(v)$ for all $\alpha \in \mathbb{C}$ and $v \in V$.)

(b) An element $w \in V_{\mathbb{C}}$ is said to be ***real*** if $\overline{w} = w$. Show that w is real if and only if it lies in the real subspace $V \subseteq V_{\mathbb{C}}$ defined above.

(c) For $w \in V_{\mathbb{C}}$, define $\operatorname{Re} w = \frac{1}{2}(w + \overline{w})$ and $\operatorname{Im} w = \frac{1}{2i}(w - \overline{w})$. Show that $\operatorname{Re} w$ and $\operatorname{Im} w$ are real, and $w = \operatorname{Re} w + i \operatorname{Im} w$.

The complexification functor can be adapted easily to vector bundles. First we establish some definitions.

Suppose M is a topological space. A ***complex vector bundle of rank k over M*** is defined analogously to a real vector bundle (e.g., as in [**LeeSM**, Chap. 10]): it is a topological space E together with a continuous surjective map $\pi : E \to M$ such that each fiber $E_p = \pi^{-1}(p)$ is given the structure of a k-dimensional complex vector space, and each $p \in M$ has a neighborhood U over which there exists a ***local trivialization***, which is a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ that restricts to a complex-linear isomorphism from E_q to $\{q\} \times \mathbb{C}^k$ for each $q \in U$. This means, in particular, that the following diagram commutes, where $\pi_1 : U \times \mathbb{C}^k \to U$ is the projection on the first factor:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi|_{\pi^{-1}(U)}"'] & & U \times \mathbb{C}^k \arrow[dl, "\pi_1"] \\
& U. &
\end{tikzcd}$$

If M and E are smooth manifolds, π is a smooth map, and the local trivializations can be chosen to be diffeomorphisms, it is a ***smooth complex vector bundle***; and if M and E are complex manifolds, π is holomorphic, and the local trivializations can be chosen to be biholomorphisms, it is a ***holomorphic vector bundle***. Any open cover of M such that E admits a trivialization over each of the open sets of the cover is called a ***trivializing cover for*** $\boldsymbol{E}$. If there is a global trivialization (that is, a local trivialization over all of M), the bundle is said to be a ***trivial bundle***. A ***line bundle*** is a (real or complex) vector bundle of rank 1.

If $\pi : E \to M$ and $\pi' : E' \to M$ are complex vector bundles over M, a map $F : E \to E'$ is called a ***bundle homomorphism*** if $\pi' \circ F = \pi$ and for each $p \in M$, the map $F|_{E_p} : E_p \to E'_p$ is a complex-linear map. A bundle homomorphism that is also a homeomorphism between E and E' is called a ***bundle isomorphism***, and the bundles E and E' are said to be ***isomorphic***, denoted by $E \cong E'$, if there is a bundle isomorphism between them. If the bundles are smooth and F is a diffeomorphism, it is called a ***smooth isomorphism***, and if the bundles are holomorphic and F is a biholomorphism, it is a ***holomorphic isomorphism***. In each of these cases, it is easy to check that the inverse map is also a bundle isomorphism. (For some purposes, it is useful to introduce a more general notion of vector bundle homomorphisms between bundles over different manifolds, and the kind we have defined here is identified as a ***bundle homomorphism over*** $\boldsymbol{M}$; see [**LeeSM**, Chap. 10] for details. Since we will not have any need for that extra generality, we always understand bundle homomorphisms to be the type we have defined here.)

Most of the standard constructions used for real vector bundles, such as Whitney sums [**LeeSM**, Example 10.7] and smooth subbundles [**LeeSM**, pp. 264–266], carry over in obvious ways to smooth complex bundles.

We will have much more to say about holomorphic vector bundles in Chapter 3; for now we focus attention on smooth bundles.

If $\pi : E \to M$ is a smooth (real or complex) vector bundle, a ***(global) section of E*** is a continuous map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{Id}_M$. For any open subset $U \subseteq M$, a ***local section of E over U*** is a continuous map $\sigma : U \to E$ satisfying $\sigma \circ \pi = \mathrm{Id}_U$. Every smooth vector bundle has a smooth ***zero section*** ζ, for which $\zeta(p)$ is the zero element of E_p for each $p \in M$. Any section that is not equal to the zero section will be called a ***nontrivial section***. A ***rough (local or global) section of E*** is a map $\sigma : U \to E$ ***rough (local or global) section of E*** is a map $\sigma : U \to E$ satisfying $\sigma \circ \pi = \mathrm{Id}_U$, but not assumed to be smooth or even continuous. We denote the space of smooth global sections of E by $\Gamma(E)$. A ***local frame for E*** is an ordered k-tuple of local sections $(\sigma_1, \dots, \sigma_k)$ over an open set $U \subseteq M$ whose values at each $p \in U$ form a basis for the fiber E_p.

If $\pi : E \to M$ is a smooth rank-k real vector bundle over a smooth manifold M, we define the ***complexification of E*** to be the set $E_{\mathbb{C}} = \bigcup_{p \in M} (E_p)_{\mathbb{C}}$ together with the obvious projection $\pi_{\mathbb{C}} : E_{\mathbb{C}} \to M$. For each smooth local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, we define a local trivialization $\Phi_{\mathbb{C}} : \pi_{\mathbb{C}}^{-1}(U) \to U \times \mathbb{C}^k$ by

$$\Phi_{\mathbb{C}}(\xi) = \big(\pi_{\mathbb{C}}(\xi),\ \big(\Phi|_{E_{\pi_{\mathbb{C}}(\xi)}}\big)_{\mathbb{C}}(\xi)\big).$$

Wherever two such trivializations (U, Φ) and (V, Ψ) overlap, [**LeeSM**, Lemma 10.15] shows that we can write $\Psi \circ \Phi^{-1}(p, v) = (p, \tau(p)v)$ for some smooth transition function $\tau : U \cap V \to \mathrm{GL}(k, \mathbb{R})$, and it is straightforward to check that the transition function from $\Phi_{\mathbb{C}}$ to $\Psi_{\mathbb{C}}$ is the same: $\Psi_{\mathbb{C}} \circ \Phi_{\mathbb{C}}^{-1}(p, v) = (p, \tau(p)v)$, where now we are considering τ as a map into $\mathrm{GL}(k, \mathbb{C})$. It follows from the vector bundle chart lemma [**LeeSM**, Lemma 10.6] (adapted in the obvious way for complex vector bundles) that $\pi_{\mathbb{C}} : E_{\mathbb{C}} \to M$ has a unique structure as a smooth rank-k complex vector bundle, with the maps constructed above as smooth local trivializations.

What this really amounts to in practice is that, given any smooth local frame $(b_1, \dots, b_k)$ for E, we can write a section of $E_{\mathbb{C}}$ locally as a sum $f^j b_j$, where now the coefficient functions f^j are allowed to be complex-valued.

► **Exercise 1.40.** Let $E \to M$ be a smooth real vector bundle. Show that every smooth (local or global) section of $E_{\mathbb{C}}$ can be written uniquely as a sum $\alpha + i\beta$, where α and β are smooth local or global sections of E.

The result of Exercise 1.39 shows that for any real vector bundle $E \to M$, conjugation defines a smooth conjugate-linear bundle homomorphism from $E_{\mathbb{C}}$ to itself, and the set of real elements (those satisfying $\overline{w} = w$) forms a real-linear subbundle canonically isomorphic to the original bundle E. It is important to note that the existence of such a conjugation operator is a special feature of complexifications: in fact, as Problem 1-6 shows, a complex vector bundle admits such a conjugation operator if and only if it is isomorphic to the complexification of a real bundle.

When we apply this construction to the tangent and cotangent bundles of a smooth manifold M, we obtain the ***complexified tangent bundle*** $T_{\mathbb{C}}M$ and the ***complexified cotangent bundle*** $T^*_{\mathbb{C}}M$, respectively. A section of $T_{\mathbb{C}}M$, called a ***complex vector field***, can be written locally as a linear combination of coordinate vector fields with complex-valued coefficient functions, or as a sum of a real vector field plus i times another real vector field. A complex vector field $Z = X + iY$ acts on a smooth real-valued function f by $Zf = Xf + iYf$, and on a complex-valued function $f = u + iv$ by the same formula, where we interpret Xf to mean $Xu + iXv$ and similarly for Y. The Lie bracket operation can be extended to pairs of smooth complex vector fields by complex bilinearity: $[X_1 + iY_1, X_2 + iY_2] = \big([X_1, X_2] - [Y_1, Y_2]\big) + i\big([X_1, Y_2] + [Y_2, X_1]\big)$. It is straightforward to check that the formula $[fV, gW] = fg[V, W] + f(Vg)W - g(Wf)V$ holds equally well when the vector fields V, W and the functions f, g are allowed to be complex.

Similarly, a section of $T^*_{\mathbb{C}}M$ is called a ***complex* 1*-form*** or a ***complex covector field***, and can be written locally as a linear combination of coordinate 1-forms with complex coefficients, or as a sum of a real 1-form plus i times another real 1-form. With this construction, we are now justified in writing $df = du + i\,dv$ whenever $f = u + iv$ is a complex-valued smooth function.

► **Exercise 1.41.** Prove that there is a canonical smooth bundle isomorphism between $T^*_{\mathbb{C}}M$ and the bundle $\operatorname{Hom}_{\mathbb{C}}(T_{\mathbb{C}}M, \mathbb{C})$ whose fiber at a point $p \in M$ is the space of complex-linear maps from $(T_pM)_{\mathbb{C}}$ to $\mathbb{C}$.

Let us specialize to the case of $\mathbb{C}^n$, with its standard holomorphic coordinates $z^j = x^j + iy^j$. Considering $\mathbb{C}^n$ as a smooth manifold of (real) dimension $2n$, we can use (x^j, y^j) as smooth global coordinates. We have a smooth global coframe $\{dx^j, dy^j\}$ for $T^*\mathbb{C}^n$, which is therefore also a coframe for $T^*_{\mathbb{C}}\mathbb{C}^n$. Consider the $2n$ complex 1-forms $dz^j = dx^j + i\,dy^j$ and $d\bar{z}^j = dx^j - i\,dy^j$. Because we can solve for $dx^j = \frac{1}{2}(dz^j + d\bar{z}^j)$ and $dy^j = \frac{1}{2i}(dz^j - d\bar{z}^j)$, it follows that $\{dz^j, d\bar{z}^j\}$ is also a smooth coframe for $T^*_{\mathbb{C}}\mathbb{C}^n$, and arbitrary complex 1-forms can also be expressed in terms of this coframe. In particular, if $f : U \to \mathbb{C}$ is a smooth function on an open subset $U \subseteq \mathbb{C}^n$, we can write

$$df = \frac{\partial f}{\partial x^j}\,dx^j + \frac{\partial f}{\partial y^j}\,dy^j = A_j\,dz^j + B_j\,d\bar{z}^j$$

for some coefficient functions A_j and B_j. (When using the summation convention, the understanding is that an upper index "in the denominator" is to be treated as a lower index.) To see what these coefficients are, just substitute the formulas for dx^j and dy^j in terms of dz^j, $d\bar{z}^j$ and collect terms:

$$\begin{aligned} df &= \frac{\partial f}{\partial x^j}\left(\frac{dz^j + d\bar{z}^j}{2}\right) + \frac{\partial f}{\partial x^j}\left(\frac{dz^j - d\bar{z}^j}{2i}\right) \\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x^j} - i\frac{\partial f}{\partial y^j}\right)dz^j + \frac{1}{2}\left(\frac{\partial f}{\partial x^j} + i\frac{\partial f}{\partial y^j}\right)d\bar{z}^j. \end{aligned} \tag{1.7}$$

Motivated by this calculation, we define $2n$ smooth complex vector fields $\partial/\partial z^j$ and $\partial/\partial \bar{z}^j$ on $\mathbb{C}^n$ by

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right), \qquad \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right). \tag{1.8}$$

(Be sure to notice that the negative sign appears in the formula for $\partial/\partial z^j$, not $\partial/\partial \bar{z}^j$; this is not a typo!) A simple computation shows that $\{\partial/\partial z^j, \partial/\partial \bar{z}^j\}$ is the smooth global frame for $T_{\mathbb{C}}\mathbb{C}^n$ dual to $\{dz^j, d\bar{z}^j\}$. For a smooth complex-valued function f defined on an open subset $U \subseteq \mathbb{C}^n$, formula (1.7) can be rewritten in terms of this frame as

$$df = \frac{\partial f}{\partial z^j}\,dz^j + \frac{\partial f}{\partial \bar{z}^j}\,d\bar{z}^j. \tag{1.9}$$

In the special case in which f is a holomorphic function on an open subset of $\mathbb{C}^n$, you will notice that we had already defined the expression $\partial f/\partial z^j$ by equation (1.2); now we seem to have introduced a different meaning for the same expression. The next proposition ensures that the two definitions are equivalent for holomorphic functions.

Proposition 1.42. *Suppose $U \subseteq \mathbb{C}^n$ is open. Let $f : U \to \mathbb{C}$ be any smooth function, and let $\partial/\partial z^j$ and $\partial/\partial \bar{z}^j$ be the complex vector fields on U defined by* (1.8).

(a) *f is holomorphic if and only if $\partial f/\partial \bar{z}^j = 0$ for $j = 1, \dots, n$.*

(b) *If f is holomorphic, then for each j, the expression $\partial f/\partial z^j$ obtained by applying the complex vector field $\partial/\partial z^j$ to f is equal to the complex partial derivative defined by* (1.2).

Proof. After we substitute $f = u + iv$ into the equation $\partial f/\partial \bar{z}^j = 0$ and separate its real and imaginary parts, it becomes the jth pair of Cauchy–Riemann equations for f, thus proving (a). Then (b) follows from Proposition 1.25. □

One must be careful not to read too much into the expressions $\partial f/\partial z^j$ and $\partial f/\partial \bar{z}^j$ when f is merely smooth: despite the notation, they are not partial derivatives in the ordinary sense, because, for example, it does not make sense to take a derivative of a function with respect to z^1 while holding $z^2, \dots, z^n, \bar{z}^1, \dots, \bar{z}^n$ fixed. If you fix $\bar{z}^1$, then z^1 remains fixed as well. However, there is a sense in which these operators behave like partial derivatives, which we now explain.

Suppose p is any (not necessarily holomorphic) complex-valued polynomial function of the real variables $\{x^j, y^j\}$:

$$p(x, y) = \sum_{\substack{l_1,\dots,l_n\\ m_1,\dots,m_n}} a_{l_1,\dots,l_n,m_1,\dots,m_n} (x^1)^{l_1} \cdots (x^n)^{l_n} (y^1)^{m_1} \cdots (y^n)^{m_n}.$$

Substituting $x^j = \frac{1}{2}(z^j + \bar{z}^j)$ and $y^j = \frac{1}{2i}(z^j - \bar{z}^j)$ and collecting like terms, we can express p as a polynomial expression in $z^j, \bar{z}^j$, which we denote by $\tilde{p}$:

$$\begin{aligned}\tilde{p}(z) &= p\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right)\\ &= \sum_{\substack{l_1,\dots,l_n\\ m_1,\dots,m_n}} \tilde{a}_{l_1,\dots,l_n,m_1,\dots,m_n} (z^1)^{l_1} \cdots (z^n)^{l_n} (\bar{z}^1)^{m_1} \cdots (\bar{z}^n)^{m_n}.\end{aligned}$$

To separate the dependence on z^j and $\bar{z}^j$, we can introduce new independent variables w^j in place of $\bar{z}^j$. Let $q : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be the polynomial function

$$q(z, w) = p\left(\frac{z + w}{2}, \frac{z - w}{2i}\right),$$

so that $\tilde{p}(z) = q(z, \bar{z})$. Now it makes sense to ask whether q is independent of $w^1, \dots, w^n$.

► **Exercise 1.43.** Prove that the original polynomial p defines a holomorphic function if and only if $\partial q/\partial w^j = 0$ for each j.

So for a polynomial function p, in this sense we can say p is holomorphic if and only if it depends only on $z^1, \dots, z^n$ with no occurrences of $\bar{z}^1, \dots, \bar{z}^n$. Exactly the same argument can be made when p is a real-analytic function, except then the finite sums above become absolutely convergent infinite series; the absolute convergence ensures that the convergence is not affected by rearranging the terms. In that case as well, a real-analytic function f is holomorphic if and only if it can be written as a power series in $z^1, \dots, z^n$, with no occurrences of $\bar{z}^1, \dots, \bar{z}^n$.

For a function that is merely smooth, these computations do not make sense, because you cannot plug complex numbers into a function that is defined only for real values of $(x^1, \dots, x^n, y^1, \dots, y^n)$. But motivated by the computations above, it is sometimes helpful to think about a holomorphic function intuitively as a "smooth function that is independent of $\bar{z}^1, \dots, \bar{z}^n$."

Complex Coordinate Frames

Now suppose M is a complex manifold and $(z^1, \dots, z^n)$ are local holomorphic coordinates on an open subset $U \subseteq M$. The coordinate map $\varphi : U \to \mathbb{C}^n$ can also be thought of as a smooth coordinate map from U to $\mathbb{R}^{2n}$, with smooth coordinate functions $(x^1, y^1, \dots, x^n, y^n)$ where $z^j = x^j + i y^j$. These coordinates yield smooth coordinate vector fields $(\partial/\partial x^1, \partial/\partial y^1, \dots, \partial/\partial x^n, \partial/\partial y^n)$, which act on a smooth function $f : U \to \mathbb{C}$ by

$$(1.10)\qquad \left.\frac{\partial}{\partial x^j}\right|_p f = \left.\frac{\partial}{\partial x^j}\right|_{\varphi(p)} (f \circ \varphi^{-1}), \qquad \left.\frac{\partial}{\partial y^j}\right|_p f = \left.\frac{\partial}{\partial y^j}\right|_{\varphi(p)} (f \circ \varphi^{-1}),$$

where the expressions on the right-hand sides are ordinary partial derivatives on $\mathbb{R}^{2n}$ (see [**LeeSM**, p. 60]). We define a smooth local complex frame $\{\partial/\partial z^j, \partial/\partial \bar{z}^j\}$ for $T_{\mathbb{C}}M$ by (1.8), where now $\partial/\partial x^j$ and $\partial/\partial y^j$ are interpreted as smooth vector fields on $U \subseteq M$. These vector fields are called ***complex coordinate vector fields***, and the corresponding local frame is called a ***complex coordinate frame***.

Lemma 1.44. *Suppose M is a complex manifold and $f : M \to \mathbb{C}$ is a smooth function. If $(z^1, \dots, z^n)$ are holomorphic coordinates on a subset $U \subseteq M$ and $\{\partial/\partial z^j, \partial/\partial \bar{z}^j\}$ are the corresponding complex coordinate vector fields, then f is holomorphic on U if and only if $\partial f/\partial \bar{z}^j \equiv 0$ on U for $j = 1, \dots, n$.*

Proof. Let $\varphi : U \to \mathbb{C}^n$ be the holomorphic coordinate map, and let $\widehat{U} = \varphi(U) \subseteq \mathbb{C}^n$. It follows from (1.10) together with the definition of $\partial/\partial \bar{z}^j$ that for all $p \in U$,

$$\frac{\partial f}{\partial \bar{z}^j}(p) = \frac{\partial (f \circ \varphi^{-1})}{\partial \bar{z}^j}(\varphi(p)).$$

The lemma then follows from the fact that $f : U \to \mathbb{C}$ is holomorphic by definition if and only if $f \circ \varphi^{-1} : \widehat{U} \to \mathbb{C}$ is holomorphic. □

When M and N are complex manifolds, the ***total derivative*** or ***differential*** of a smooth map $F : M \to N$ at a point $p \in M$ is a real-linear map from T_pM to $T_{F(p)}N$, and its complexification is a complex-linear map from $(T_pM)_{\mathbb{C}}$

to $(T_{F(p)}N)_{\mathbb{C}}$. For smooth manifolds, the differential is often denoted by dF_p, but for reasons that will be explained shortly, in this book we will denote the differential at p (or its complexification) by $DF(p)$, and the associated bundle homomorphism, called the ***global differential of F***, by $DF : T_{\mathbb{C}}M \to T_{\mathbb{C}}N$. The next proposition shows how to compute it in terms of holomorphic coordinates.

Proposition 1.45 (The Total Derivative in Holomorphic Coordinates). *Let M and N be complex manifolds and $F : M \to N$ be a smooth map. Given $p \in M$, let $z^j = x^j + iy^j$ be local holomorphic coordinates for M in a neighborhood of p, and $w^j = u^j + iv^j$ for N in a neighborhood of $F(p)$. In terms of the complex local frames $\{\partial/\partial z^j, \partial/\partial \bar{z}^j\}$ for M and $\{\partial/\partial w^j, \partial/\partial \overline{w}^j\}$ for N, the total derivative of F at p has the following coordinate representation:*

$$DF(p)\Big(\frac{\partial}{\partial z^j}\Big|_p\Big) = \frac{\partial F^k}{\partial z^j}(p)\,\frac{\partial}{\partial w^k}\Big|_{F(p)} + \frac{\partial \overline{F}^k}{\partial z^j}(p)\,\frac{\partial}{\partial \overline{w}^k}\Big|_{F(p)}, \tag{1.11}$$

$$DF(p)\Big(\frac{\partial}{\partial \bar{z}^j}\Big|_p\Big) = \frac{\partial F^k}{\partial \bar{z}^j}(p)\,\frac{\partial}{\partial w^k}\Big|_{F(p)} + \frac{\partial \overline{F}^k}{\partial \bar{z}^j}(p)\,\frac{\partial}{\partial \overline{w}^k}\Big|_{F(p)}. \tag{1.12}$$

Proof. Write the real and imaginary parts of (the coordinate representation of) F as $F = U + iV$. Considering M and N as smooth manifolds, we have the usual coordinate formula for $DF(p)$:

$$\begin{aligned} DF(p)\Big(\frac{\partial}{\partial x^j}\Big|_p\Big) &= \frac{\partial U^k}{\partial x^j}(p)\,\frac{\partial}{\partial u^k}\Big|_{F(p)} + \frac{\partial V^k}{\partial x^j}(p)\,\frac{\partial}{\partial v^k}\Big|_{F(p)},\\ DF(p)\Big(\frac{\partial}{\partial y^j}\Big|_p\Big) &= \frac{\partial U^k}{\partial y^j}(p)\,\frac{\partial}{\partial u^k}\Big|_{F(p)} + \frac{\partial V^k}{\partial y^j}(p)\,\frac{\partial}{\partial v^k}\Big|_{F(p)}. \end{aligned} \tag{1.13}$$

To transform this to holomorphic coordinates, begin with the definitions of $\partial/\partial z^j$ and $\partial/\partial \bar{z}^j$, and use (1.13) together with the complex linearity of $DF(p)$ to obtain

$$\begin{aligned} DF(p)\Big(\frac{\partial}{\partial z^j}\Big|_p\Big) &= \frac{\partial U^k}{\partial z^j}(p)\,\frac{\partial}{\partial u^k}\Big|_{F(p)} + \frac{\partial V^k}{\partial z^j}(p)\,\frac{\partial}{\partial v^k}\Big|_{F(p)},\\ DF(p)\Big(\frac{\partial}{\partial \bar{z}^j}\Big|_p\Big) &= \frac{\partial U^k}{\partial \bar{z}^j}(p)\,\frac{\partial}{\partial u^k}\Big|_{F(p)} + \frac{\partial V^k}{\partial \bar{z}^j}(p)\,\frac{\partial}{\partial v^k}\Big|_{F(p)}. \end{aligned}$$

Now substitute $\partial/\partial u^k = \partial/\partial w^k + \partial/\partial \overline{w}^k$ and $\partial/\partial v^k = i(\partial/\partial w^k - \partial/\partial \overline{w}^k)$ and collect terms:

$$\begin{aligned} DF(p)\Big(\frac{\partial}{\partial z^j}\Big|_p\Big) = \Big(\frac{\partial U^k}{\partial z^j}(p) + i\frac{\partial V^k}{\partial z^j}(p)\Big)\,&\frac{\partial}{\partial w^k}\Big|_{F(p)}\\ &+ \Big(\frac{\partial U^k}{\partial z^j}(p) - i\frac{\partial V^k}{\partial z^j}(p)\Big)\,\frac{\partial}{\partial \overline{w}^k}\Big|_{F(p)}. \end{aligned}$$

This is (1.11), and a similar computation proves (1.12). □

Corollary 1.46. *In addition to the hypotheses of* 1.45*, suppose F is holomorphic. Then in terms of the local frames $\{\partial/\partial z^j, \partial/\partial \bar z^j\}$ and $\{\partial/\partial w^j, \partial/\partial \overline{w}^j\}$, $DF(p)$ is represented by the block-diagonal matrix*

$$\begin{pmatrix} D'F(p) & 0 \\ 0 & \overline{D'F(p)} \end{pmatrix}, \tag{1.14}$$

*where $D'F$ denotes the $n \times n$ complex matrix-valued function $\left(\partial F^k/\partial z^j\right)$, called the **holomorphic Jacobian of F**. Thus the linear map $DF(p)$ is invertible if and only if the holomorphic Jacobian of F is invertible at p.*

Proof. The fact that F is holomorphic means that each component function of its coordinate representation is holomorphic. Thus $\partial F^k/\partial \bar z^j$ vanishes identically, and by conjugation so does $\partial \overline{F}^k/\partial z^j$. Therefore, $DF(p)$ has the given matrix representation by Proposition 1.45. The last statement then follows from the fact that $\det DF(p) = |\det D'F(p)|^2$. □

Proposition 1.47 (Chain Rule for Smooth Functions). *Suppose M and N are complex manifolds, $F: M \to N$ is a smooth map, and $h: N \to \mathbb{C}$ is a smooth function. In terms of local holomorphic coordinates (z^j) for M and (ζ^k) for N,*

$$\frac{\partial(h \circ F)}{\partial z^j} = \frac{\partial h}{\partial \zeta^k}\frac{\partial F^k}{\partial z^j} + \frac{\partial h}{\partial \bar\zeta^k}\frac{\partial \overline{F}^k}{\partial z^j},$$
$$\frac{\partial(h \circ F)}{\partial \bar z^j} = \frac{\partial h}{\partial \zeta^k}\frac{\partial F^k}{\partial \bar z^j} + \frac{\partial h}{\partial \bar\zeta^k}\frac{\partial \overline{F}^k}{\partial \bar z^j}.$$

Proof. Proposition 1.45 shows that the value of $\partial(h \circ F)/\partial z^j$ at $p \in M$ is equal to the $\partial/\partial w$ component of $D(h \circ F)(p)\left(\partial/\partial z^j|_p\right)$ (where w denotes the standard holomorphic coordinate of $\mathbb{C}$). By smooth manifold theory, $D(h \circ F)(p) = Dh(F(p)) \circ DF(p)$, which can be computed by applying the formula of Proposition 1.45 to h and to F and composing the two linear maps. A similar argument applies to the $\bar z^j$ derivative. □

Corollary 1.48 (Chain Rule for Holomorphic Functions). *Under the hypotheses of Proposition* 1.47*, suppose in addition that F and h are holomorphic. Then*

$$d(h \circ F) = \frac{\partial h}{\partial w^k}\frac{\partial F^k}{\partial z^j} dz^j. \qquad \square$$

In the theory of smooth (real) manifolds, the differential of a smooth real-valued function f at a point $p \in M$ can be considered either as a linear map from T_pM to $\mathbb{R}$ (a covector) or as a linear map from T_pM to $T_{f(p)}\mathbb{R}$; in view of the canonical identification between $T_{f(p)}\mathbb{R}$ and $\mathbb{R}$, these are the same map, so it makes sense to use the same notation df_p to denote both of them. But in complex manifold theory, something different happens. Suppose $f = u + iv: M \to \mathbb{C}$ is a complex-valued smooth function on a complex manifold M. On the one hand, df_p denotes the value at p of the complex-valued 1-form $df = du + i\,dv$, an element of $(T_p^*M)_{\mathbb{C}}$, which can also be viewed as a complex-linear map from $(T_pM)_{\mathbb{C}}$ to $\mathbb{C}$ (by Exercise 1.41). Using the coordinate formula (1.9), we find, for example, that

$$df_p\left(\frac{\partial}{\partial \bar{z}^j}\bigg|_p\right) = \frac{\partial f}{\partial \bar{z}^j}(p) \in \mathbb{C}.$$

On the other hand, $Df(p)$ is a complex-linear map from $(T_pM)_{\mathbb{C}}$ to $(T_{f(p)}\mathbb{C})_{\mathbb{C}}$, and Proposition 1.45 shows that

$$Df(p)\left(\frac{\partial}{\partial \bar{z}^j}\bigg|_p\right) = \frac{\partial f}{\partial \bar{z}^j}(p)\,\frac{\partial}{\partial w}\bigg|_{f(p)} + \frac{\partial \bar{f}}{\partial \bar{z}^j}(p)\,\frac{\partial}{\partial \bar{w}}\bigg|_{f(p)} \in (T_{f(p)}\mathbb{C})_{\mathbb{C}}.$$

These are distinctly different objects—for example, if f is holomorphic, then $df(\partial/\partial \bar{z}^j)$ vanishes identically, but $Df(\partial/\partial \bar{z}^j)$ does not. This is why we use different notations for the two kinds of derivatives, and prefer the term "total derivative" for $Df(p)$.

Orientations

The computations we just did lead to another important property of complex manifolds: they all have canonical orientations. (Just to be clear: when we speak of an orientation of a complex manifold, it means an orientation of its underlying smooth real manifold.)

Proposition 1.49. *Every complex manifold has a canonical orientation, uniquely determined by the following two properties:*

(i) *The canonical orientation of $\mathbb{C}^n$ is the one determined by the $2n$-form*

$$\omega_n = dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n. \tag{1.15}$$

(ii) *Every local biholomorphism is orientation-preserving.*

Proof. Let us begin by expressing the real $2n$-form ω_n in terms of the complex coordinates $(z^1, \dots, z^n)$. Observe that for each j, we have $dz^j \wedge d\bar{z}^j = (dx^j + i\,dy^j) \wedge (dx^j - i\,dy^j) = -2i dx^j \wedge dy^j$. Therefore

$$\omega_n = \left(\frac{i}{2}\right)^n dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n. \tag{1.16}$$

Let $U \subseteq \mathbb{C}^n$ be an open subset and $F : U \to \mathbb{C}^n$ be a local biholomorphism. The Jacobian matrix of F has the form (1.14) when expressed in terms of the ordered frame $\left(\partial/\partial z^1, \dots, \partial/\partial z^n, \partial/\partial \bar{z}^1, \dots, \partial/\partial \bar{z}^n\right)$. Note that this order is not the same as the one used in formula (1.16)—they differ by a permutation whose sign is $(-1)^{(n-1)n/2}$, as you can check, and therefore

$$\omega_n = (-1)^{(n-1)n/2}\left(\frac{i}{2}\right)^n dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n.$$

The formula for the pullback of a top-degree form (see [**LeeSM**, Prop. 14.9], which works equally well for complex-valued forms) gives

$$\begin{aligned}
&F^*\left(dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n\right) \\
&\qquad = \det \begin{pmatrix} D'F & 0 \\ 0 & \overline{D'F} \end{pmatrix} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n \\
&\qquad = |\det D'F|^2 dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n,
\end{aligned}$$

and multiplying both sides by $(-1)^{(n-1)n/2}(i/2)^n$ implies

$$F^*\omega_n = |\det D'F|^2 \omega_n.$$

This shows that every biholomorphism between open subsets of $\mathbb{C}^n$ is orientation-preserving.

Now let M be an n-dimensional complex manifold. Because every holomorphic coordinate chart is a local biholomorphism, if there is to be an orientation of M satisfying (i) and (ii), it must be determined in the domain of each holomorphic chart by the pullback of ω_n under the coordinate map, and it is uniquely determined by this property. We just need to verify that the orientations determined by different holomorphic charts agree.

Suppose two holomorphic charts (U, φ) and (V, ψ) overlap. The transition function $\psi \circ \varphi^{-1}$ is a biholomorphism between open subsets of $\mathbb{C}^n$, so the above computation shows that $(\psi \circ \varphi^{-1})^*\omega_n = u\,\omega_n$, where u is the positive smooth function $|\det D'(\psi \circ \varphi^{-1})|^2$. Thus on $U \cap V$ we have

$$\begin{aligned}
\psi^*\omega_n &= \varphi^*(\varphi^{-1})^*\psi^*\omega_n \\
&= \varphi^*\left((\psi \circ \varphi^{-1})^*\omega_n\right) \\
&= \varphi^*\left(u\,\omega_n\right) \\
&= (u \circ \varphi)\varphi^*\omega_n.
\end{aligned}$$

Thus the n-forms determined by φ and ψ are positive multiples of each other, so they determine the same orientation on $U \cap V$.

Finally, we need to show that every local biholomorphism between complex manifolds is orientation-preserving. Suppose $F\colon M \to N$ is a local biholomorphism. Let $p \in M$, and choose holomorphic charts (U, φ) for M and (V, ψ) for N such that $p \in U$, $F(U) \subseteq V$, and $F|_U$ is a biholomorphism onto its image. Then on U,

$$F = \left(\psi^{-1}\right) \circ \left(\psi \circ F \circ \varphi^{-1}\right) \circ \left(\varphi\right).$$

The three maps in parentheses above are all orientation-preserving: the first and third by the way we have defined the orientations on N and M, and the second because it is a biholomorphism between open subsets of $\mathbb{C}^n$. □

Almost Complex Structures

To delve further into the interaction between a holomorphic structure and its underlying smooth structure, we introduce the following linear-algebraic construction. Let V be an n-dimensional complex vector space, and let $V_{\mathbb{R}}$ be its underlying real vector space—the same set as V, but considered only as a vector space over $\mathbb{R}$. Then $V_{\mathbb{R}}$ is a $2n$-dimensional real vector space. The fact that V is a complex vector space is encoded in the rule for multiplying vectors by i, which is the map $J : V \to V$ sending each vector v to iv. By ignoring the complex vector space structure, we can also think of J as a real-linear map $J : V_{\mathbb{R}} \to V_{\mathbb{R}}$ satisfying $J \circ J = -\operatorname{Id}$.

Now suppose V is any vector space over $\mathbb{R}$. A ***complex structure on V*** is a real-linear endomorphism $J : V \to V$ satisfying $J \circ J = -\operatorname{Id}$.

Lemma 1.50. *Suppose V is a real vector space and J is a complex structure on V. Then the multiplication by complex scalars defined by $(a + bi)v = av + bJv$, together with the given vector addition operation, turns the set V into a complex vector space.*

► **Exercise 1.51.** Prove this lemma by showing that complex multiplication is associative and distributive.

To understand a complex structure J on a vector space V more deeply, we need to look at its eigenvalues. The fact that $J \circ J = -\operatorname{Id}$ means that every eigenvalue λ must satisfy $\lambda^2 = -1$. Thus J has no real eigenvalues, and the only possible complex eigenvalues are $\pm i$. To find eigenspaces, therefore, we must complexify V and J. Let $V_{\mathbb{C}}$ be the complexification of V, and denote the complexification of J by $J : V_{\mathbb{C}} \to V_{\mathbb{C}}$. It still satisfies $J \circ J = -\operatorname{Id}$.

Proposition 1.52. *If J is a complex structure on the real vector space V, then $V_{\mathbb{C}}$ has a complete eigenspace decomposition of the form*

$$V_{\mathbb{C}} = V' \oplus V'',$$

where $V' \subseteq V_{\mathbb{C}}$ is the i-eigenspace of J and V'' is the $(-i)$-eigenspace. The eigenspace decomposition of $w \in V_{\mathbb{C}}$ is given by $w = w' + w''$, where

$$w' = \tfrac{1}{2}(w - iJw), \qquad w'' = \tfrac{1}{2}(w + iJw). \tag{1.17}$$

If V is finite-dimensional, then V' and V'' have the same complex dimension.

Proof. Given $w \in V_{\mathbb{C}}$, define $w', w'' \in V_{\mathbb{C}}$ by (1.17). Simple computations show that $Jw' = iw'$ and $Jw'' = -iw''$. Because $w = w' + w''$, this shows that $V_{\mathbb{C}} = V' + V''$. On the other hand, a nonzero vector cannot be an eigenvector with two different eigenvalues, so $V' \cap V'' = \{0\}$, which shows that the sum is direct.

To see that the eigenspaces have the same dimension, note that conjugation (Exercise 1.39) is a bijective real-linear map from $V_{\mathbb{C}}$ to itself, and it interchanges

V' and V''. Thus the underlying real spaces of V' and V'' have the same real dimension, and because the complex dimension is half the real dimension, V' and V'' have the same complex dimension. □

Corollary 1.53. *If a finite-dimensional real vector space admits a complex structure, then it is even-dimensional.*

Proof. If V admits a complex structure, the preceding proposition shows that $V_{\mathbb{C}}$ is even-dimensional. The result follows from the fact that the complex dimension of $V_{\mathbb{C}}$ is equal to the real dimension of V. □

Let us apply this construction to $\mathbb{C}^n$ with its standard complex structure. Let $(X_1, \dots, X_n)$ denote the standard basis for $\mathbb{C}^n$ as a complex vector space, where $X_j = (0, \dots, 1, \dots, 0)$ with a 1 in the jth place. Let $Y_j = JX_j = (0, \dots, i, \dots, 0)$. Then $(X_1, Y_1, \dots, X_n, Y_n)$ is a basis over $\mathbb{R}$ for the underlying real vector space $(\mathbb{C}^n)_{\mathbb{R}}$, and J satisfies $JX_j = Y_j$, $JY_j = -X_j$. From Proposition 1.52, we see that the i-eigenspace $(\mathbb{C}^n)'$ is spanned by $(Z_1, \dots, Z_n)$, where $Z_j = \frac{1}{2}(X_j - iY_j)$, and $(\mathbb{C}^n)''$ is spanned by $(\overline{Z}_1, \dots, \overline{Z}_n)$.

All of these constructions can be applied to vector bundles. If $E \to M$ is a smooth real vector bundle, a ***complex structure on E*** is a smooth bundle endomorphism $J : E \to E$ satisfying $J \circ J = -\mathrm{Id}$.

Consider the case of $\mathbb{C}^n$ as a smooth manifold. For each point $p \in \mathbb{C}^n$, using the standard identification of $T_p\mathbb{C}^n$ with $(\mathbb{C}^n)_{\mathbb{R}}$, we have the following correspondences:

$$\left.\frac{\partial}{\partial x^j}\right|_p \leftrightarrow X_j, \qquad \left.\frac{\partial}{\partial y^j}\right|_p \leftrightarrow Y_j, \qquad \left.\frac{\partial}{\partial z^j}\right|_p \leftrightarrow Z_j.$$

Thus the bundle $T\mathbb{C}^n$ has a canonical complex structure $J_{\mathbb{C}^n}$, which satisfies

$$J_{\mathbb{C}^n}\frac{\partial}{\partial x^j} = \frac{\partial}{\partial y^j}, \qquad J_{\mathbb{C}^n}\frac{\partial}{\partial y^j} = -\frac{\partial}{\partial x^j}.$$

The complexified tangent bundle $T_{\mathbb{C}}\mathbb{C}^n$ splits as $T_{\mathbb{C}}\mathbb{C}^n = T'\mathbb{C}^n \oplus T''\mathbb{C}^n$, with $T'\mathbb{C}^n$ spanned by the complex vector fields $\partial/\partial z^1, \dots, \partial/\partial z^n$, and $T''\mathbb{C}^n$ spanned by $\partial/\partial \bar{z}^1, \dots, \partial/\partial \bar{z}^n$.

Lemma 1.54. *For an open subset $U \subseteq \mathbb{C}^n$, a smooth function $F : U \to \mathbb{C}^m$ is holomorphic if and only if the following relation holds for all $p \in U$:*

$$DF(p) \circ J_{\mathbb{C}^n} = J_{\mathbb{C}^m} \circ DF(p). \tag{1.18}$$

Proof. First suppose that (1.18) holds for all $p \in U$. After both sides are extended by complex linearity to act on complex vectors, the two expressions yield the same result when applied to the elements of the complex coordinate frame

$\{\partial/\partial z^j,\ \partial/\partial \bar z^j\}$. Using (1.12), we obtain

$$
\begin{aligned}
0 &= DF\Big(J_{\mathbb{C}^n}\frac{\partial}{\partial \bar z^j}\Big) - J_{\mathbb{C}^m}\Big(DF\frac{\partial}{\partial \bar z^j}\Big)\\
&= DF\Big(-i\frac{\partial}{\partial \bar z^j}\Big) - J_{\mathbb{C}^m}\Big(DF\frac{\partial}{\partial \bar z^j}\Big)\\
&= -i\frac{\partial F^k}{\partial \bar z^j}\frac{\partial}{\partial w^k} - i\frac{\partial \overline{F}^k}{\partial \bar z^j}\frac{\partial}{\partial \overline{w}^k} - J_{\mathbb{C}^m}\frac{\partial F^k}{\partial \bar z^j}\frac{\partial}{\partial w^k} - J_{\mathbb{C}^m}\frac{\partial \overline{F}^k}{\partial \bar z^j}\frac{\partial}{\partial \overline{w}^k}\\
&= -2i\frac{\partial F^k}{\partial \bar z^j}\frac{\partial}{\partial w^k}.
\end{aligned}
$$

This shows $\partial F^k/\partial \bar z^j \equiv 0$ for all j, k, so F is holomorphic.

Conversely, if F is holomorphic, the computation above shows that both sides of (1.18) yield the same result when applied to $\partial/\partial \bar z^j$, and conjugation shows that the same is true when applied to $\partial/\partial z^j$, using the fact that $\partial \overline{F}^k/\partial z^j = \overline{\partial F^k/\partial \bar z^j} = 0$. Since both sides are linear over $C^\infty(M;\mathbb{C})$, this shows the equation holds when applied to arbitrary vector fields. □

Lemma 1.54 enables us to define a canonical complex structure on the tangent bundle of every complex manifold.

Proposition 1.55. *For every complex manifold M, there is a canonical complex structure on TM, denoted by $J_M : TM \to TM$. If N is another complex manifold and $F : M \to N$ is a smooth map, then F is holomorphic if and only if*

$$DF \circ J_M = J_N \circ DF. \tag{1.19}$$

Proof. Let n be the complex dimension of M. We define J_M as follows: given $p \in M$, choose a holomorphic coordinate chart (U, φ) on a neighborhood of p, and define $J_M : TM|_U \to TM|_U$ by

$$J_M = D\varphi^{-1} \circ J_{\mathbb{C}^n} \circ D\varphi. \tag{1.20}$$

Wherever two holomorphic charts (U, φ) and (V, ψ) overlap, the transition map $\psi \circ \varphi^{-1}$ is a holomorphic map between open subsets of $\mathbb{C}^n$, so its differential commutes with $J_{\mathbb{C}^n}$ by Lemma 1.54. Therefore,

$$
\begin{aligned}
D\psi^{-1} \circ J_{\mathbb{C}^n} \circ D\psi &= D\psi^{-1} \circ J_{\mathbb{C}^n} \circ \big(D\psi \circ D\varphi^{-1}\big) \circ D\varphi\\
&= D\psi^{-1} \circ \big(D\psi \circ D\varphi^{-1}\big) \circ J_{\mathbb{C}^n} \circ D\varphi\\
&= D\varphi^{-1} \circ J_{\mathbb{C}^n} \circ D\varphi,
\end{aligned}
$$

so J_M is well defined. The fact that it satisfies $J_M \circ J_M = -\,\mathrm{Id}$ follows from the corresponding fact for $J_{\mathbb{C}^n}$.

Now let N be a complex m-manifold and $F : M \to N$ be a smooth map. Because (1.19) is a local statement, it suffices to choose arbitrary local holomorphic charts (U, φ) for M and (V, ψ) for N such that $F(U) \subseteq V$, and prove that the restriction of F to U is holomorphic if and only if it satisfies (1.19) there. By definition, F is holomorphic on U if and only if its coordinate representation $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ is holomorphic, which in turn is true if and only if $D\widehat{F} \circ J_{\mathbb{C}^n} = J_{\mathbb{C}^m} \circ D\widehat{F}$ by Lemma 1.54. Using (1.20) for both M and N, we compute

$$\begin{aligned} D\widehat{F} \circ J_{\mathbb{C}^n} - J_{\mathbb{C}^m} \circ D\widehat{F} &= D\psi \circ DF \circ D\varphi^{-1} \circ J_{\mathbb{C}^n} - J_{\mathbb{C}^m} \circ D\psi \circ DF \circ D\varphi^{-1} \\ &= D\psi \circ DF \circ J_M \circ D\varphi^{-1} - D\psi \circ J_N \circ DF \circ D\varphi^{-1} \\ &= D\psi \circ (DF \circ J_M - J_N \circ DF) \circ D\varphi^{-1}. \end{aligned}$$

Since $D\psi$ and $D\varphi^{-1}$ are bundle isomorphisms, this last expression is zero if and only if (1.18) holds, thus completing the proof. □

Proposition 1.56. *Let M be a complex manifold and let $J_M : TM \to TM$ be the associated complex structure on TM. There are smooth subbundles $T'M$, $T''M \subseteq T_{\mathbb{C}}M$ whose fibers at each point are the i-eigenspace and $(-i)$-eigenspace of (the complexification of) J_M, respectively. The complexified tangent bundle decomposes as a Whitney sum: $T_{\mathbb{C}}M = T'M \oplus T''M$. In terms of any local holomorphic coordinates $z^j = x^j + iy^j$, the complex vector fields $\partial/\partial z^j$ defined by* (1.8) *form a local frame for $T'M$; and the vector fields $\partial/\partial \bar{z}^j$ form a local frame for $T''M$.*

Proof. For each $p \in M$, the space $(T_pM)_{\mathbb{C}}$ has such a decomposition by Proposition 1.52. Suppose $z^j = x^j + iy^j$ are holomorphic local coordinates on M. Because the endomorphism J_M is defined by using the coordinate map to transport $J_{\mathbb{C}^n}$ to the manifold, it follows that the vector fields $\partial/\partial z^j$ provide a local frame for $T'M$, as do $\partial/\partial \bar{z}^j$ for $T''M$. Because both subbundles are spanned locally by smooth vector fields, they are smooth. □

We call the bundles $T'M$ and $T''M$ the ***holomorphic tangent bundle*** and ***antiholomorphic tangent bundle of M***, respectively. The fibers $T_p'M$ and $T_p''M$ at a point $p \in M$ are called the ***holomorphic tangent space*** and ***antiholomorphic tangent space at p***, respectively.

The decomposition of $T_{\mathbb{C}}M$ into holomorphic and antiholomorphic tangent bundles allows us to give a coordinate-free interpretation to the holomorphic Jacobian of a holomorphic map. It follows from Proposition 1.55 that if $F : M \to N$ is holomorphic, then $DF(T'M) \subseteq T'N$. In local holomorphic coordinates (z^j) for M and (w^k) for N, Corollary 1.46 shows that the restriction of $DF(p)$ to $T_p'M$ is represented by the holomorphic Jacobian matrix $(\partial F^k(p)/\partial z^j)$. Henceforth, we will use the notation $D'F(p)$ and the term ***holomorphic Jacobian*** to refer either to this complex-linear map from $T'_p M$ to $T'_{F(p)} N$ or to its matrix representation in local holomorphic coordinates.

For a finite-dimensional real vector space with its natural smooth structure, the tangent space at each point is canonically identified with the vector space itself [**LeeSM**, Prop. 3.13]. The following proposition shows that there is a corresponding identification for complex vector spaces.

Proposition 1.57 (Holomorphic Tangent Space to a Complex Vector Space). *Suppose V is a finite-dimensional complex vector space with its standard holomorphic structure. For each $a \in V$, there is a canonical (basis-independent) complex-linear isomorphism $\Phi_a : V \cong T'_a V$. It is natural in the following sense: if $L : V \to W$ is a complex-linear map between finite-dimensional complex vector spaces, then the following diagram commutes for each $a \in V$:*

$$\begin{array}{ccc} V & \xrightarrow{\Phi_a} & T'_a V \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle D'L(a)} \\ W & \xrightarrow[\Phi_{L(a)}]{} & T'_{L(a)} W. \end{array} \tag{1.21}$$

Proof. Given $a, w \in V$, let $\lambda_{a,w} : \mathbb{C} \to V$ be the holomorphic map $\lambda_{a,w}(\tau) = a + \tau w$. We define $\Phi_a : V \to T'_a V$ by

$$\Phi_a(w) = D'(\lambda_{a,w})(0)\left(\left.\frac{\partial}{\partial \tau}\right|_0\right).$$

The definition shows that this is independent of any choice of basis for V. To see that it satisfies the required conditions, choose any basis for V and let $(z^1, \dots, z^n)$ be the corresponding linear coordinates. Then a simple computation based on Corollary 1.46 shows that Φ_a has the coordinate representation

$$\Phi_a(w^1, \dots, w^n) = w^j \left.\frac{\partial}{\partial z^j}\right|_a,$$

which shows that it is a complex-linear isomorphism. If W is another finite-dimensional complex vector space and $L : V \to W$ is a complex-linear map, then in terms of any linear coordinates $(\zeta^1, \dots, \zeta^m)$ for W, we see that

$$D'L(a)(\Phi_a(w^1, \dots, w^n)) = L^j_k w^k \left.\frac{\partial}{\partial \zeta^j}\right|_{L(a)} = \Phi_{L(a)}(L(w^1, \dots, w^n)),$$

which proves (1.21). □

For a complex manifold M, we have now introduced several different varieties of tangent bundles: TM, $T_{\mathbb{C}}M$, $T'M$, and $T''M$. In case you are not confused enough already, we now define one more: $T_J M$ is the complex vector bundle with the same total space as the ordinary tangent bundle TM, but endowed with the complex vector space structure on fibers determined by J_M as in Lemma 1.50.

Proposition 1.58. *Let M be a complex n-manifold. Then $T_J M$ is a smooth rank-n complex vector bundle over M. The complex vector bundles $T_J M$ and $T'M$ are isomorphic via the map $\xi : T_J M \to T'M$ given by $\xi(v) = v - iJv$.*

Proof. Problem 1-7. □

For easy reference, here is a summary of all of these bundles. Suppose M is a complex n-manifold.

- $\boldsymbol{TM}$: The ordinary tangent bundle of the smooth manifold M. It is a real vector bundle of rank $2n$.
- $\boldsymbol{T_{\mathbb{C}}M}$: The complexified tangent bundle, a complex vector bundle of rank $2n$.
- $\boldsymbol{T'M}$: The holomorphic tangent bundle, a complex rank-n vector subbundle of $T_{\mathbb{C}}M$. Its fiber at each point is the i-eigenspace of J_M.
- $\boldsymbol{T''M}$: The antiholomorphic tangent bundle, another complex rank-n vector subbundle of $T_{\mathbb{C}}M$, whose fibers are $(-i)$-eigenspaces of J_M.
- $\boldsymbol{T_J M}$: The ordinary tangent bundle of M equipped with the complex structure J_M, which turns it into a complex vector bundle of rank n.

Now suppose M is an arbitrary smooth manifold. It makes sense to ask whether there is a complex structure on TM, that is, a smooth bundle endomorphism $J : TM \to TM$ satisfying $J \circ J = -\mathrm{Id}$. The existence of such an endomorphism is a necessary condition for the existence of a holomorphic structure on M, but it is not sufficient, as we will see below. For this reason, a manifold whose tangent bundle is endowed with such a complex structure J is called an ***almost complex manifold***, and J is called an ***almost complex structure on*** $\boldsymbol{M}$. Note the potentially confusing shift in terminology: a *complex structure on* TM is called an *almost complex structure on* M (to distinguish it from the traditional use of "complex structure on M" to denote what we are calling a holomorphic structure).

Proposition 1.55 shows that a holomorphic structure on a manifold M determines an almost complex structure on M (that is, a complex structure on TM). The question naturally arises whether the reverse is true: Given an almost complex structure J on a smooth manifold M, is there a holomorphic structure for which J is the canonical almost complex structure as described in Proposition 1.55? In general, the answer is no, because there is a nontrivial necessary condition, as a consequence of the next proposition.

Proposition 1.59. *Suppose M is a complex manifold and $V, W \in \Gamma(T'M)$. Then $[V, W] \in \Gamma(T'M)$.*

Proof. In local holomorphic coordinates, we can write

$$V = V^j \frac{\partial}{\partial z^j}, \qquad W = W^k \frac{\partial}{\partial z^k},$$

and therefore,

$$[V, W] = V^j \left(\frac{\partial W^k}{\partial z^j} \right) \frac{\partial}{\partial z^k} - W^k \left(\frac{\partial V^j}{\partial z^k} \right) \frac{\partial}{\partial z^j}.$$

This last expression takes its values in $T'M$. □

For almost complex structures, it makes sense to ask if the same result holds, by virtue of the following lemma.

Lemma 1.60. *Suppose M is a smooth $2n$-manifold endowed with an almost complex structure J. Then there are smooth rank-n complex subbundles $T'M, T''M \subseteq T_{\mathbb{C}}M$ whose fibers are the i-eigenspaces and $(-i)$-eigenspaces of J, respectively, such that $T_{\mathbb{C}}M = T'M \oplus T''M$.*

► **Exercise 1.61.** Prove this lemma.

An almost complex structure on a smooth manifold M is said to be ***integrable*** if whenever V, W are smooth sections of $T'M$, then $[V, W]$ is also a section of $T'M$. Every almost complex structure on a 2-dimensional real manifold is integrable (see Problem 1-8), but in higher dimensions integrability is a nontrivial condition, as Problems 1-11 and 1-13 illustrate.

The integrability condition looks formally similar to the condition of involutivity for a distribution (subbundle of the tangent bundle) on a smooth manifold, which is a necessary and sufficient condition for the distribution to be tangent to a foliation (see [**LeeSM**, Chap. 19]). But there is no foliation associated with $T'M$ because it is not a subbundle of the (real) tangent bundle of M.

The following corollary is an immediate consequence of Proposition 1.59.

Corollary 1.62. *If J is an almost complex structure on a smooth manifold, a necessary condition for J to be the canonical almost complex structure associated with a holomorphic structure is that J be integrable.* □

The following important converse was proved in 1957 by August Newlander and Louis Nirenberg, showing that integrability is also sufficient.

Theorem 1.63 (Newlander–Nirenberg). *If an almost complex structure on a smooth manifold is integrable, then it arises from a holomorphic structure.*

We will neither prove nor use this theorem (except in Example 1.64 and Problem 7-10 below, which are not essential to our main story). There are several known proofs, all based on deep results from the theory of partial differential equations. Two different proofs can be found in [**Nir73**] and [**Hör90**].

Not every smooth manifold admits an almost complex structure. Two simple requirements are that the manifold must be even-dimensional (Cor. 1.53) and orientable (Problem 1-9). In two real dimensions, these conditions are sufficient, as

Example 1.64 below will show. But in higher dimensions, there are other topological obstructions that are not so easily described. Two spheres that admit almost complex structures are $\mathbb{S}^2$ (by Example 1.64 below) and $\mathbb{S}^6$ (by Problem 1-13). The structure on $\mathbb{S}^2$ is integrable, and turns it into a complex manifold biholomorphic to $\mathbb{CP}^1$ (see Problem 2-4). The known structure on $\mathbb{S}^6$ is not integrable, and it is not known whether $\mathbb{S}^6$ carries a holomorphic structure. It was proved by Armand Borel and Jean-Pierre Serre in 1953 [**BS53**] that $\mathbb{S}^2$ and $\mathbb{S}^6$ are the only spheres that carry almost complex structures; so no other spheres can be made into complex manifolds. A modern proof of this fact can be found in [**May99**, p. 208]. Since our main concern is to study complex manifolds, which already come equipped with canonical almost complex structures, we do not pursue the general question of existence of almost complex structures any further.

Example 1.64 (Holomorphic Structures on 2-Manifolds). Suppose M is an orientable real 2-manifold. We can always endow M with a Riemannian metric and an orientation. With that data, we can define an almost complex structure on M by letting J be "counterclockwise rotation by 90°." More precisely, for each nonzero $v \in T_pM$, we let Jv be the unique vector w such that $\langle v, w \rangle = 0$, $|w| = |v|$, and (v, w) is an oriented basis for T_pM. If (b_1, b_2) is any smooth oriented orthonormal local frame, then we have $Jb_1 = b_2$ and $Jb_2 = -b_1$, which shows that J is smooth. This almost complex structure is integrable by the result of Problem 1-8, so it arises from a holomorphic structure by the Newlander–Nirenberg theorem. Thus every orientable smooth real 2-manifold can be given a holomorphic structure. A real 2-manifold endowed with a particular holomorphic structure is called a ***Riemann surface***. Be careful about the terminology: a Riemann surface is a *complex curve* (1-dimensional complex manifold), while a *complex surface* is a 2-dimensional complex manifold. (It is possible for the same real 2-manifold to have different holomorphic structures that are not biholomorphic to each other, however; see Problem 1-4.) //

Problems

1-1. With $G \subseteq \mathrm{GL}(3, \mathbb{C})$ as in Example 1.20, let $\Gamma \subseteq G$ be the subgroup consisting of matrices whose entries are Gaussian integers. Prove that Γ is cocompact by showing that every coset in G/Γ has at least one representative lying in the unit cube $[0, 1]^6 \subseteq \mathbb{C}^3$.

1-2. Suppose $U \subseteq \mathbb{C}^n$ is open and $f : U \to \mathbb{C}$ is a holomorphic function that is nonzero on $U \smallsetminus S$, where $S \subseteq \mathbb{C}^n$ is a complex-linear subspace of codimension at least 2. Show that f is nonzero everywhere in U.

1-3. Prove that every 1-dimensional Hopf manifold is biholomorphic to a complex torus $\mathbb{C}/\Lambda$, and determine an explicit lattice Λ.

1-4. For any two vectors $v, w \in \mathbb{C}$ that are linearly independent over $\mathbb{R}$, let $T_{v,w} = \mathbb{C}/\Lambda(v, w)$ denote the 1-dimensional complex torus obtained as a quotient of $\mathbb{C}$ by the lattice $\Lambda(v, w)$ generated by v and w.

(a) For any such v, w, show that there exists $\tau \in \mathbb{C}$ with $\operatorname{Im} \tau > 0$ such that $T_{v,w}$ is biholomorphic to $T_{1,\tau}$.

(b) Let $\mathrm{SL}(2, \mathbb{Z})$ denote the group of integer matrices with determinant 1. Suppose $\tau, \tau' \in \mathbb{C}$ satisfy $\operatorname{Im} \tau > 0$ and $\operatorname{Im} \tau' > 0$. Show that $T_{1,\tau}$ is biholomorphic to $T_{1,\tau'}$ if and only if there exists $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ such that $\tau' = (a\tau + b)/(c\tau + d)$. [Hint: Show that any biholomorphism $T_{1,\tau} \to T_{1,\tau'}$ lifts to an automorphism of $\mathbb{C}$.]

1-5. Show that the Lie group $\mathrm{U}(n)$ acts continuously and transitively on the Grassmannian $\mathrm{G}_k(\mathbb{C}^n)$ by $A \cdot S = A(S)$ for $A \in \mathrm{U}(n)$ and $S \subseteq \mathbb{C}^n$ a subspace of dimension k. Use this to show that $\mathrm{G}_k(\mathbb{C}^n)$ is compact for every k and n.

1-6. Suppose $E \to M$ is a complex vector bundle. Show that there exists a conjugation operator, that is, a conjugate-linear bundle homomorphism $c\colon E \to E$ satisfying $c \circ c = \mathrm{Id}$, if and only if E is isomorphic (over $\mathbb{C}$) to the complexification of a real bundle.

1-7. Prove Proposition 1.58 ($T_J M$ is a smooth complex vector bundle isomorphic to $T'M$).

1-8. Prove that every almost complex structure on a real 2-manifold is integrable.

1-9. Suppose M is a smooth manifold that admits an almost complex structure. Prove that M is orientable.

1-10. Let M be a smooth manifold and J be an almost complex structure on M. Define a map $N\colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ by

$$N(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y].$$

(a) Show that N is bilinear over $C^\infty(M)$, and therefore defines a $(1, 2)$-tensor field on M, called the ***Nijenhuis tensor of J***.

(b) Show that J is integrable if and only if $N \equiv 0$. [Hint: Extend N to act on complex vector fields, and take X and Y to be smooth sections of $T'M$ or $T''M$.]

1-11. For $n \geq 2$, define an almost complex structure on $\mathbb{C}^n$ as follows:

$$J\frac{\partial}{\partial x^1} = (1 + (x^2)^2)\frac{\partial}{\partial y^1}, \qquad J\frac{\partial}{\partial y^1} = -\frac{1}{(1 + (x^2)^2)}\frac{\partial}{\partial x^1},$$
$$J\frac{\partial}{\partial x^k} = \frac{\partial}{\partial y^k}, \qquad J\frac{\partial}{\partial y^k} = -\frac{\partial}{\partial x^k}, \qquad k = 2, \dots, n.$$

Show that J is not integrable.

1-12. Let M be a $2n$-dimensional smooth manifold. Suppose ζ is a smooth closed complex n-form on M that is ***locally decomposable*** (i.e., can locally be written as a wedge product of complex 1-forms), and satisfies $\zeta \wedge \bar{\zeta} \neq 0$ everywhere on M. Show that there is a unique integrable almost complex structure on M for which $T'M = \{v \in T_{\mathbb{C}}M : v \lrcorner \bar{\zeta} = 0\}$ (where $\lrcorner$ denotes ***interior multiplication*** with a vector field, defined by $(v \lrcorner \bar{\zeta})(\dots) = \bar{\zeta}(v, \dots)$; see [**LeeSM**, p. 358]).

1-13. An almost complex structure on $\mathbb{S}^6$: Let $\mathbb{O}$ denote the algebra of ***octonions***, which is an 8-dimensional nonassociative algebra over $\mathbb{R}$ defined as follows. Start with the ***quaternions***, the 4-dimensional associative algebra $\mathbb{H}$ over $\mathbb{R}$ with basis $(\mathbb{1}, \mathbb{i}, \mathbb{j}, \mathbb{k})$ and bilinear multiplication defined by

$$\mathbb{i}^2 = \mathbb{j}^2 = \mathbb{k}^2 = -\mathbb{1}, \quad \mathbb{1}q = q\mathbb{1} = q \text{ for all } q \in \mathbb{H},$$
$$\mathbb{i}\mathbb{j} = -\mathbb{j}\mathbb{i} = \mathbb{k}, \quad \mathbb{j}\mathbb{k} = -\mathbb{k}\mathbb{j} = \mathbb{i}, \quad \mathbb{k}\mathbb{i} = -\mathbb{i}\mathbb{k} = \mathbb{j}.$$

Then define $\mathbb{O} = \mathbb{H} \times \mathbb{H}$, with the bilinear product defined by

$$(p, q)(r, s) = (pr - sq^*,\ p^*s + rq),$$

where the conjugate of a quaternion is

$$(w\mathbb{1} + x\mathbb{i} + y\mathbb{j} + z\mathbb{k})^* = w\mathbb{1} - x\mathbb{i} - y\mathbb{j} - z\mathbb{k}.$$

Define the conjugate of an octonion $P = (p, q)$ by $P^* = (p^*, -q)$. Let $\mathbb{R} = \{P \in \mathbb{O} : P^* = P\}$ denote the set of real octonions, identified with the real numbers in the natural way, and $\mathbb{E} = \{P \in \mathbb{O} : P^* = -P\}$ the set of imaginary octonions. Define an inner product on $\mathbb{O}$ by $\langle P, Q\rangle = \frac{1}{2}(P^*Q + Q^*P) \in \mathbb{R}$. Let $\mathbb{S} = \{P \in \mathbb{E} : |P| = 1\}$ be the unit sphere in $\mathbb{E}$, and for each $P \in \mathbb{S}$, define a map $J_P : T_P\mathbb{S} \to \mathbb{O}$ by $J_P(Q) = QP$, where we identify $T_P\mathbb{S}$ with the real-linear subspace $P^\perp \cap \mathbb{E} \subseteq \mathbb{O}$. Although the multiplication in $\mathbb{O}$ is not associative, it is the case that $(PQ)^* = Q^*P^*$ and $(PQ)P = P(QP)$ for all $P, Q \in \mathbb{O}$, and you may use these facts without proof. (See also Problem 8-7 in [**LeeSM**].)

(a) Show that J_P maps $T_P\mathbb{S}$ to itself, and defines an almost complex structure on $\mathbb{S}$.

(b) Show that this almost complex structure is not integrable.

[Remark: It is still unknown whether $\mathbb{S}^6$ admits an integrable almost complex structure. Many well-known and respected mathematicians have written papers purporting to answer this question one way or the other, but all the proofs have been found to be wrong or incomplete.]

1-14. Let (M, g) and (N, h) be Riemannian manifolds of the same dimension. A smooth map $F : M \to N$ is said to be ***conformal*** if $F^*h = \lambda g$ for some smooth, positive function λ on M.

(a) Suppose (M, g) and (N, h) are oriented Riemannian 2-manifolds, and give M and N the holomorphic structures described in Example 1.64. Suppose $F : M \to N$ is a local diffeomorphism. Show that F is holomorphic if and only if it is conformal and orientation-preserving.

(b) Give examples of diffeomorphisms $F, G : \mathbb{C}^2 \to \mathbb{C}^2$ such that F is holomorphic but not conformal, and G is conformal and orientation-preserving but not holomorphic.