Preface

Why study geometry? Those who have progressed far enough in their mathematical education to read this book can probably come up with lots of answers to that question:

- *Geometry is useful.* It's hard to find a branch of mathematics that has more practical applications than geometry. A precise understanding of geometric relationships is prerequisite for making progress in architecture, astronomy, computer graphics, engineering, mapmaking, medical imaging, physics, robotics, sewing, or surveying, among many other fields.
- *Geometry is beautiful.* Because geometry is primarily about spatial relationships, the subject comes with plenty of illustrations, many of which have an austere beauty in their own right. On a deeper level, the study of geometry uncovers surprising and unexpected relationships among shapes, the contemplation of which can inspire an exquisitely satisfying sense of beauty. And, of course, to some degree, geometric relationships underlie almost all visual arts.
- *Geometry comes naturally.* Along with counting and arithmetic, geometry is one of the earliest areas of intellectual inquiry to have been systematically pursued by human societies. Similarly, children start to learn about geometry (naming shapes) as early as two years old, about the same time they start learning about numbers. Almost every culture has developed some detailed understanding of geometrical relationships.
- *Geometry is logical.* As will be explored in some detail in this book, very early in Western history geometry became the paradigm for logical thought and analysis, and students have learned the rudiments of logic and proof in geometry courses for more than two millennia.

All of these are excellent reasons to devote serious study to geometry. But there is a more profound consideration that animates this book: the story of geometry is the story of mathematics itself. There is no better way to understand what modern mathematics is, how it is done, and why it is the way it is than by undertaking a thorough study of the roots of geometry.

Mathematics occupies a unique position among the fields of human intellectual inquiry. It shares many of the characteristics of science: like scientists, mathematicians strive for precision, make assertions that are testable and refutable, and seek to discover universal laws. But there is at least one way in which mathematics is markedly different from most sciences, and indeed from virtually every other subject: in mathematics, it is possible to have a degree of certainty about the truth of an assertion that is usually impossible to attain in any other field. When mathematicians assert that the area of a unit circle is exactly π, we know this is true as surely as we know the facts of our direct experience, such as the fact that I am currently sitting at a desk looking at a computer screen. Moreover, we know that we could calculate π to far greater precision than anyone will ever be able to measure actual physical areas.

Of course, this is not to claim that mathematical knowledge is absolute. No human knowledge is ever 100% certain—we all make mistakes, or we might be hallucinating or dreaming. But mathematical knowledge is as certain as anything in the human experience. By contrast, the assertions of science are always approximate and provisional: Newton's law of universal gravitation was true enough to pass the experimental tests of his time, but it was eventually superseded by Einstein's general theory of relativity. Einstein's theory, in turn, has withstood most of the tests it has been subjected to, but physicists are always ready to admit that there might be a more accurate theory.

How did we get here? What is it about mathematics that gives it such certainty and thus sets it apart from the natural sciences, social sciences, humanities, and arts? The answer is, above all, the concept of *proof*. Ever since ancient times, mathematicians have realized that it is often possible to demonstrate the truth of a mathematical statement by giving a logical argument that is so convincing and so universal that it leaves essentially no doubt. At first, these arguments, when they were found, were mostly ad hoc and depended on the reader's willingness to accept other seemingly simpler truths: if we grant that such-and-such is true, then the Pythagorean theorem logically follows. As time progressed, more proofs were found and the arguments became more sophisticated. But the development of mathematical knowledge since ancient times has been much more than just amassing ever more convincing arguments for an ever larger list of mathematical facts.

There have been two decisive turning points in the history of mathematics, which together are primarily responsible for leading mathematics to the special position it occupies today. The first was the appearance, around 300 BCE, of Euclid's *Elements*. In this monumental work, Euclid organized essentially all of the mathematical knowledge that had been developed in the Western world up to that time (mainly geometry and number theory) into a systematic logical structure. Only a few simple, seemingly self-evident facts (known as *postulates* or *axioms*) were stated without proof, and all other mathematical statements (known as *propositions* or *theorems*) were proved in a strict logical sequence, with the proofs of the simplest theorems based only on the postulates and with proofs of more complicated theorems based on theorems that had already been proved. This structure, known today as the *axiomatic method*, was so effective and so convincing that it became the model for justifying all further mathematical inquiry.

But Euclid's Eden was not without its snake. Beginning soon after Euclid's time, mathematicians raised objections to one of Euclid's geometric postulates (the famous fifth postulate, which we describe in Chapter 1) on the ground that it was too complicated to be considered truly self-evident in the way that his other postulates were. A consensus developed among mathematicians who made a serious study of geometry that the fifth postulate was really a theorem masquerading as a postulate and that Euclid had simply failed to find the proper proof of it. So for about two thousand years, the study of geometry was largely dominated by attempts to find a proof of Euclid's fifth postulate based only on the other four postulates. Many proofs were offered, but all were eventually found to be fatally flawed, usually because they implicitly assumed some other fact that was also not justified by the first four postulates and thus required a proof of its own.

The second decisive turning point in mathematics history occurred in the first half of the nineteenth century, when an insight occurred simultaneously and independently to three different mathematicians (Nikolai Lobachevsky, János Bolyai, and Carl Friedrich Gauss). There was a good reason nobody had succeeded in proving that the fifth postulate followed from the other four: such a proof is logically impossible! The insight that struck these three mathematicians was that it is possible to construct an entirely self-consistent kind of geometry, now called *non-Euclidean geometry*, in which the first four postulates are true but the fifth is false. This geometry did not seem to describe the physical world we live in, but the fact that it is logically just as consistent as Euclidean geometry forced mathematicians to undertake a radical reevaluation of the very nature of mathematical truth and the meaning of the axiomatic method.

Once that insight had been achieved, things progressed rapidly. A new paradigm of the axiomatic method arose, in which the axioms were no longer thought of as statements of self-evident truths from which reliable conclusions could be drawn, but rather as arbitrary statements that were to be *accepted* as truths in a given mathematical context, so that the theorems following from them must be true if the axioms are true.

Paradoxically, this act of unmooring the axiomatic method from any preconceived notions of absolute truth eventually allowed mathematics to achieve previously undreamed of levels of certainty. The reason for this paradox is not hard to discern: once mathematicians realized that axioms are more or less arbitrary assumptions rather than self-evident truths, it became clear that proofs based on the axioms can use *only* those facts that have been explicitly stated in the axioms or previously proved, and the steps of such proofs must be based on clear and incontrovertible principles of logic. Thus it is no longer permissible to base arguments on geometric or numerical intuition about how things behave in the "real world." If it happens that the axioms are deemed to be a useful and accurate description of something, such as a scientific phenomenon or a class of mathematical objects, then every conclusion proved within that axiomatic system is exactly as certain and as accurate as the axioms themselves. For example, we can all agree that Euclid's geometric postulates are extremely accurate descriptions of the geometry of a building on the scale at which humans experience it, so the conclusions of Euclidean geometry can be relied upon to describe that geometry as accurately as we might wish. On the other hand, it might turn out to be the case that the axioms of non-Euclidean geometry are much more accurate descriptions of the geometry of the universe as a whole (on a scale at which galaxies can be treated as uniformly scattered dust), which would mean that the theorems of non-Euclidean geometry could be treated as highly accurate descriptions of the cosmos.

If mathematics occupies a unique position among fields of human inquiry, then geometry can be said to occupy a similarly unique position among subfields of mathematics. Geometry was the first subject to which Euclid applied his logical analysis, and it has been taught to students for more than two millennia as a model of logical thought. It was the subject that engendered the breakthrough in the nineteenth century that led to our modern conception of the axiomatic method (and thus of the very nature of mathematics). Until very recently, virtually every high-school student in the United States took a geometry course based on some version of the axiomatic method, and that was the only time in the lives of most people when they learned about rigorous logical deduction.

The primary purpose of this book is to tell this story: not exactly the historical story, because it is not a book about the history of mathematics, but rather the intellectual story of how one might begin with Euclid's understanding of the axiomatic method in geometry and progress to our modern understanding of it. Since the axiomatic method underlies the way modern mathematics is universally done, understanding it is crucial to understanding what mathematics is, how we read and evaluate mathematical arguments, and why mathematics has achieved the level of certainty it has.

It should be emphasized, though, that while the axiomatic method is our only tool for ensuring that our mathematical knowledge is sound and for communicating that soundness to others, it is not the only tool, and probably not even the most important one, for *discovering* or *understanding* mathematics. For those purposes we rely at least as much on examples, diagrams, scientific applications, intuition, and experience as we do on proofs. But in the end, especially as our mathematics becomes increasingly abstract, we cannot justifiably claim to be sure of the truth of any mathematical statement unless we have found a proof of it.

This book has been developed as a textbook for a course called *Geometry for Teachers*, and it is aimed primarily at undergraduate students who plan to teach geometry in a North American high-school setting. However, it is emphatically *not* a book about how to teach geometry; that is something that aspiring teachers will have to learn from education courses and from hands-on practice. What it offers, instead, is an opportunity to understand on a deep level how mathematics works and how it came to be the way it is, while at the same time developing most of the concrete geometric relationships that secondary teachers will need to know in the classroom. It is not only for future teachers, though: it should also provide something of interest to anyone who wishes to understand Euclidean and non-Euclidean geometry better and to develop skills for doing proofs within an axiomatic system.

In recent years, sadly, the traditional proof-oriented geometry course has sometimes been replaced by different kinds of courses that minimize the importance of the axiomatic method in geometry. But regardless of what kinds of curriculum and pedagogical methods teachers plan to use, it is of central importance that they attain a deep understanding of the underlying mathematical ideas of the subject and a mathematical way of thinking about them—a version of what the mathematics education specialist Liping Ma [**Ma99**] calls "profound understanding of fundamental mathematics." It is my hope that this book will be a vehicle that can help to take them there.

Many books that treat axiomatic geometry rigorously (such as [**Gre08**], [**Moi90**], [**Ven05**]) pass rather quickly through Euclidean geometry, with a major goal of developing non-Euclidean geometry and its relationship to Euclidean geometry. In this book, by contrast, the primary focus is on Euclidean geometry, because that is the subject that future secondary geometry teachers will need to understand most deeply. I do make a serious excursion into hyperbolic geometry at the end of the book, but it is meant to round out the study of Euclidean geometry and place it in its proper perspective, not to be the main focus of the book.

Organization

The book is organized into twenty chapters. The first chapter introduces Euclid. It seeks to familiarize the reader with what Euclid did and did not accomplish and to explain why the axiomatic method underwent such a radical revision in the nineteenth century. The first book of Euclid's *Elements* should be read in parallel with this chapter. It is a good idea for an aspiring geometry teacher to have a complete copy of the *Elements* in his or her library, and I strongly recommend the edition [**Euc02**], which is a carefully edited edition containing all thirteen volumes of the *Elements* in a single book. But for those who are not ready to pay for a hard copy of the book, it is easy to find translations of the *Elements* on the Internet. An excellent source is Dominic Joyce's online edition [**Euc98**], which includes interactive diagrams.

The second chapter of this book constitutes a general introduction to the modern axiomatic method, using a "toy" axiomatic system as a laboratory for experimenting with the concepts and methods of proof. This system, called *incidence geometry*, contains only a few axioms that describe the intersections of points and lines. It is a complete axiomatic system, but it describes only a very small part of geometry so as to make it easier to concentrate on the logical features of the system. It is not my invention; it is adapted from the axiomatic system for Euclidean geometry introduced by David Hilbert at the turn of the last century (see Appendix A), and many other authors (e.g., [**Gre08**], [**Moi90**], [**Ven05**]) have also used a similar device to introduce the principles of the axiomatic method. The axioms I use are close to those of Hilbert but are modified slightly so that each axiom focuses on only one simple fact, so as to make it easier to analyze how different interpretations do or do not satisfy all of the axioms. At the end of the chapter, I walk students through the process of constructing proofs in the context of incidence geometry.

The heart of the book is a modern axiomatic treatment of Euclidean geometry, which occupies Chapters 3 through 16. I have chosen a set of axioms for Euclidean geometry that is based roughly on the SMSG axiom system developed for high-school courses in the 1960s (see Appendix C), which in turn was based on the system proposed by George Birkhoff [**Bir32**] in 1932 (see Appendix B). In choosing the axioms for this book, I've endeavored to keep the postulates closely parallel to those that are typically used in high-school courses, while maintaining a much higher standard of rigor than is typical in such courses. In particular, my postulates differ from the SMSG ones in several important ways:

- I restrict my postulates to plane geometry only.
- I omit the redundant ruler placement and supplement postulates. (The first is a theorem in Chapter 3, and the second is a theorem in Chapter 4.)
- I rephrase the plane separation postulate to refer only to the elementary concept of intersections between line segments and lines, instead of building convexity into the postulate. (Convexity of half-planes is proved in Chapter 3.)
- I replace the three SMSG postulates about angle measures with a single "protractor postulate," much closer in spirit to Birkhoff's original angle measure postulate, more closely parallel to the ruler postulate, and, I think, more intuitive. (Theorems equivalent to the three SMSG angle measure postulates are proved in Chapter 4.)

- I replace the SMSG postulate about the area of a triangle with a postulate about the area of a unit square, which is much more fundamental.

After the treatment of Euclidean geometry comes a transitional chapter (Chapter 17), which summarizes the most important postulates that have been found to be equivalent to the Euclidean parallel postulate. In so doing, it sets the stage for the study of hyperbolic geometry by introducing such important concepts as the angle defect of a polygon. Then Chapters 18 and 19 treat hyperbolic geometry, culminating in the classification of parallel lines into asymptotically parallel and ultraparallel lines.

Chapter 20 is a look forward at some of the directions in which the study of geometry can be continued. I hope it will whet the reader's appetite for further advanced study in geometry.

After Chapter 20 come a number of appendices meant to supplement the main text. The first four appendices (A through D) are reference lists of axioms for the axiomatic systems of Hilbert, Birkhoff, SMSG, and this book. The next two appendices (E and F) give brief descriptions of the conventions of mathematical language and proofs; they can serve either as introductions to these subjects for students who have not been introduced to rigorous proofs before or as review for those who have. Appendices G and H give a very brief summary of background material on sets, functions, and the real number system, which is presupposed by the axiom system used in this book. Finally, Appendix I outlines an alternative approach to the axioms based on the ideas of transformations and rigid motions; it ends with a collection of challenging exercises that might be used as starting points for independent projects.

There is ample material here for a full-year course at a reasonable pace. For shorter courses, there are various things that can be omitted, depending on the tastes of the instructor and the needs of the students. Of course, if your interest is solely in Euclidean geometry, you can always stop after Chapter 16 or 17. On the other hand, if you want to move more quickly through the Euclidean material and spend more time on non-Euclidean geometry, there are various Euclidean topics that are not used in an essential way later in the book and can safely be skipped, such as the material on nonconvex polygons at the end of Chapter 8 and some or all of Chapters 14, 15, or 16. Don't worry if the course seems to move slowly at first—it has been my experience that it takes a rather long time to get through the first six chapters, but things tend to move more quickly after that.

I adhere to some typographical conventions that I hope will make the book easier to use. Mathematical terms are typeset in ***bold italics*** when they are officially introduced, to reflect the fact that definitions are just as important as theorems and proofs but fit better into the flow of paragraphs rather than being called out with special headings. The symbol □ is used to mark the ends of proofs, and it is also used at the end of the statement of a corollary that follows so easily that it is not necessary to write down a proof. The symbol // marks the ends of numbered examples.

The exercises at the ends of the chapters are essential for mastering the type of thinking that leads to a deep understanding of mathematical concepts. Most of them are proofs. Almost all of them can be done by using techniques very similar to ones used in the proofs in the book, although a few of the exercises in the later chapters might require a bit more ingenuity.

Prerequisites

Because this book is written as a textbook for an advanced undergraduate course, I expect readers to be conversant with most of the subjects that are typically treated at the beginning and intermediate undergraduate levels. In particular, readers should be comfortable at least with one-variable calculus, vector algebra in the plane, mathematical induction, and elementary set theory. It would also be good to have already seen the least upper bound principle and the notions of injective and surjective functions.

Students who have had a course that required them to construct rigorous mathematical proofs will have a distinct advantage in reading the book and doing the exercises. But for students whose experience with proofs is limited, a careful study of Appendices E and F should provide sufficient opportunities to deepen their understanding of what proofs are and how to read and write them.

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Additional Resources

I plan to post some supplementary materials, as well as a list of corrections, on the website www.ams.org/bookpages/amstext-21. For the sake of future readers, I encourage all readers to make notes of any mistakes you find and any passages you think could use improvement, whether major or trivial, and send them to me at the email address posted on the website above. Happy reading!

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