Problem 1. Let $Y, Z$ be non-singular varieties over $k$, $\phi : Y \to Z$ a morphism. Prove that if $\text{char } k = 0$ and $\phi$ is surjective, then there exists a non-empty open set $U \subseteq Y$ such that $T_P \phi$ is also surjective for all $P \in U$.
(difﬁculty 4)

Problem 2. Let $S_1$ be the vector space of linear polynomials in the variables $x_0, \ldots, x_n$ and $P \in Y \subseteq \mathbb{P}^n$ a point on the projective variety $Y$. Let $\phi_P : S_1 \to \mathcal{O}_{P,Y} / \mathfrak{m}_P^2$ be the natural map (deﬁned in class). Let $B_P = \ker \phi_P$ and
$$B = \{(P, H) | P \in Y, H = Z(f) \text{ such that } f \in B_P\}.$$ 
Prove that $B$ is closed and irreducible.
(difﬁculty 3)

Problem 3. Let
$$0 \to \mathcal{F}^* \to \mathcal{G}^* \to \mathcal{H}^* \to 0$$
be a short exact sequence of complexes of objects of an abelian category. Show that for all $i \in \mathbb{Z}$ there exist a natural morphism $\delta^i : h^i(\mathcal{H}^*) \to h^{i+1}(\mathcal{F}^*)$ such that the induced sequence
$$\cdots \to h^i(\mathcal{F}^*) \to h^i(\mathcal{G}^*) \to h^i(\mathcal{H}^*) \to h^{i+1}(\mathcal{F}^*) \to \cdots$$
is exact.
(difﬁculty 2)

Problem 4. Cohomology may be computed using acyclic resolutions. I.e., if
$$0 \to \mathcal{G} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots$$
is exact, and $\mathcal{F}^i$ is acyclic for a left-exact functor $\mathcal{F}$ for every $i \in \mathbb{N}$, then
$$R^i \mathcal{F}(\mathcal{G}) \simeq h^i(\mathcal{F}^*).$$
(difﬁculty 3)

Problem 5. Let $X$ be a topological space, $\mathcal{F}$ a sheaf of abelian groups on $X$, $j : U \hookrightarrow X$ and open set and $i : Z := X \setminus U \hookrightarrow X$ its complement. Let $\mathcal{F}_U := j_!(\mathcal{F}|_U)$ and $\mathcal{F}_Z := i_*(\mathcal{F}|_Z)$. Prove that there exist natural morphisms such that the following is a short exact sequence of sheaves:
$$0 \to \mathcal{F}_U \to \mathcal{F} \to \mathcal{F}_Z \to 0.$$ 
(difﬁculty 3)

Problem 6. Let $X$ be a variety and
$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$
a short exact sequence of $\mathcal{O}_X$-modules. Prove that if $\mathcal{F}$ and $\mathcal{G}$ are flasque, then so is $\mathcal{H}$.
(difﬁculty 2)
Problem 7. Let $X$ be a variety and
\[ 0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \]
a short exact sequence of $\mathcal{O}_X$-modules. Prove that if $\mathcal{F}$ is flasque, then
\[ \Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H}) \]
is surjective.
(difficulty 2)

Problem 8. Let $Y$ be an affine variety and let $A = A(Y)$ be its affine coordinate ring. Let $\phi : X \to Y$ be a morphism from an arbitrary variety. Prove that there exists a coherent $\mathcal{O}_X$-module, $\Omega_{X/Y}$, such that for any open affine subvariety $U \subseteq X$, for which $B = A(U)$ is the affine coordinate ring of $U$, $\Gamma(U, \Omega_{X/Y}) \simeq \Omega_{B/A}$. [Hint: use the condition you need to prove to define the sheaf locally and prove that the local definitions agree on the overlaps.]

Problem 9. Let $Y$ be an arbitrary variety and $\phi : X \to Y$ a morphism from another variety. Prove that there exists a coherent $\mathcal{O}_X$-module, $\Omega_{X/Y}$, such that for any open affine subvariety $V \subseteq Y$, we have $\Omega_{X/Y}|_{\phi^{-1}V} \simeq \Omega_{\phi^{-1}V/V}$, where the latter sheaf is the one defined in the previous exercise. [Hint: use the condition you need to prove to define the sheaf locally and prove that the local definitions agree on the overlaps.]

Problem 10. Let $\phi : X \to Y$ be a morphism of varieties. Prove that then there exists an exact sequence,
\[ \phi^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0. \]

Problem 11. Let $\phi : X \to Y$ be a morphism of varieties and $y \in Y$ a point. Consider the ideal sheaf $\mathcal{I}_y := \mathfrak{m}_y \cdot \mathcal{O}_X \subseteq \mathcal{O}_X$. Assume that $\mathcal{I}_y$ is a radical ideal (i.e., for any open subset $U \subseteq X$, $\mathcal{I}_y(U) \subseteq \mathcal{O}_X(U)$ is a radical ideal). Let $X_y \subseteq X$ be the vanishing set of the ideal $\mathcal{I}_y$, in other words let $X_y := \operatorname{supp} \mathcal{O}_X/\mathcal{I}_y$. Prove that then $\Omega_{X/Y} \otimes \mathcal{O}_{X_y} \simeq \Omega_{X_y}$. 