508 Spring 2012 – HOMEWORKS

Problem 1. *Let Y, Z be non-singular varieties over k, $\phi : Y \to Z$ a morphism. Prove that if $\operatorname{char} k = 0$ and ϕ is surjective, then there exists a non-empty open set $U \subseteq Y$ such that $T_P\phi$ is also surjective for all $P \in U$.*
(difficulty 4)

Problem 2. *Let S_1 be the vector space of linear polynomials in the variables $x_0, \dots, x_n$ and $P \in Y \subseteq \mathbb{P}^n$ a point on the projective variety Y. Let $\phi_P : S_1 \to \mathscr{O}_{P,Y} \big/ \mathfrak{m}_{P,Y}^2$. be the natural map (defined in class). Let $B_P = \ker \phi_P$ and*

$$B = \{(P, H) | P \in Y, H = Z(f) \text{ such that } f \in B_P\}.$$

Prove that B is closed and irreducible.
(difficulty 3)

Problem 3. *Let*

$$0 \to \mathscr{F}^\bullet \to \mathscr{G}^\bullet \to \mathscr{H}^\bullet \to 0$$

be a short exact sequence of complexes of objects of an abelian category. Show that for all $i \in \mathbb{Z}$ there exist a natural morphism $\delta^i : h^i(\mathscr{H}^\bullet) \to h^{i+1}(\mathscr{F}^\bullet)$ such that the induced sequence

$$\dots \to h^i(\mathscr{F}^\bullet) \to h^i(\mathscr{G}^\bullet) \to h^i(\mathscr{H}^\bullet) \to h^{i+1}(\mathscr{F}^\bullet) \to \dots$$

is exact.
(difficulty 2)

Problem 4. *Cohomology may be computed using acyclic resolutions. I.e., if*

$$0 \to \mathscr{G} \to \mathscr{F}^0 \to \mathscr{F}^1 \to \dots$$

is exact, and $\mathscr{F}^i$ is acyclic for a left-exact functor F for every $i \in \mathbb{N}$, then

$$\mathscr{R}^i\mathsf{F}(\mathscr{G}) \simeq h^i(\mathsf{F}(\mathscr{F}^\bullet)).$$

(difficulty 3)

Problem 5. *Let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups on X, $j : U \hookrightarrow X$ and open set and $i : Z := X \setminus U \hookrightarrow X$ its complement. Let $\mathscr{F}_U := j_! (\mathscr{F}|_U)$ and $\mathscr{F}_Z := i_* (\mathscr{F}|_Z)$. Prove that there exist natural morphisms such that the following is a short exact sequence of sheaves:*

$$0 \to \mathscr{F}_U \to \mathscr{F} \to \mathscr{F}_Z \to 0.$$

(difficulty 3)

Problem 6. *Let X be a variety and*

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

a short exact sequence of $\mathscr{O}_X$-modules. Prove that if $\mathscr{F}$ and $\mathscr{G}$ are flasque, then so is $\mathscr{H}$.
(difficulty 2)

Problem 7. *Let X be a variety and*

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

a short exact sequence of $\mathscr{O}_X$-modules. Prove that if $\mathscr{F}$ is flasque, then

$$\Gamma(X, \mathscr{G}) \to \Gamma(X, \mathscr{H})$$

is surjective.
(difficulty 2)

Problem 8. *Let Y be an affine variety and let $A = A(Y)$ be its affine coordinate ring. Let $\phi : X \to Y$ be a morphism from an arbitrary variety. Prove that there exists a coherent $\mathscr{O}_X$-module, $\Omega_{X/Y}$, such that for any open affine subvariety $U \subseteq X$, for which $B = A(U)$ is the affine coordinate ring of U, $\Gamma(U, \Omega_{X/Y}) \simeq \Omega_{B/A}$.*
[Hint: use the condition you need to prove to define the sheaf locally and prove that the local definitions agree on the overlaps.]

Problem 9. *Let Y be an arbitrary variety and $\phi : X \to Y$ a morphism from another variety. Prove that there exists a coherent $\mathscr{O}_X$-module, $\Omega_{X/Y}$, such that for any open affine subvariety $V \subseteq Y$, we have $\Omega_{X/Y}|_{\phi^{-1}V} \simeq \Omega_{\phi^{-1}V/V}$, where the latter sheaf is the one defined in the previous exercise.*
[Hint: use the condition you need to prove to define the sheaf locally and prove that the local definitions agree on the overlaps.]

Problem 10. *Let $\phi : X \to Y$ be a morphism of varieties. Prove that then there exists an exact sequence,*

$$\phi^* \Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0.$$

Problem 11. *Let $\phi : X \to Y$ be a morphism of varieties and $y \in Y$ a point. Consider the ideal sheaf $\mathscr{I}_y := \mathfrak{m}_y \cdot \mathscr{O}_X \subseteq \mathscr{O}_X$. Assume that $\mathscr{I}_y$ is a radical ideal (i.e., for any open subset $U \subseteq X$, $\mathscr{I}(U) \subset \mathscr{O}_X(U)$ is a radical ideal). Let $X_y \subseteq X$ be the vanishing set of the ideal $\mathscr{I}_y$, in other words let $X_y := \operatorname{supp} \mathscr{O}_X / \mathscr{I}_y$. Prove that then $\Omega_{X/Y} \otimes \mathscr{O}_{X_y} \simeq \Omega_{X_y}$.*