Toward Arakelov-Parshin Rigidity

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Fixed Notation.

- $B$ is a smooth (not necessarily projective) curve.

- $f : X \to B$ is a smooth projective family of varieties (of general type) of dimension $n$.

- $X_b = f^{-1}(b)$ is the fiber of $f$ over $b \in B$.

Definition. $f$ is called isotrivial if $X_a \sim X_b$ for $a, b \in B$ general points.
Kodaira-Spencer map.

\[ 0 \to TX/B \to TX \to f^*TB \to 0 \]

induces

\[ \rho_f : TB \to R_1^fT_{X/B}, \]

the Kodaira-Spencer map of \( f \).

**Fact.**

\( f \) is isotrivial \( \iff \rho_f = 0 \), i.e.,

\( f \) is non-isotrivial \( \iff \rho_f \neq 0 \iff \rho_f \) is injective.
First assume that $n = 1$, i.e., $f$ is a family of curves.

**Definition.** Let $g \geq 2$ be fixed. Non-isotrivial families of curves of genus $g$ will be called *admissible*.

**Shafarevich’s Conjecture.** (SC)
For a fixed $g$, there exist only finitely many isomorphism classes of admissible families.

**The Arakelov-Parshin method.**

**Boundedness** (B): There exist only finitely many deformation types of admissible families.

**Rigidity** (R): There exist no non-trivial deformations of admissible families.

Observe, that (B) and (R) together imply (SC).
A word about the proof.

Let $\overline{B}$ be the projective closure of $B$ and $\overline{f} : \overline{X} \to \overline{B}$ a flat extension of $f$ with $\overline{X}$ smooth and projective.

To prove ($B$), one proves that $\deg(\overline{f}_*\omega_X^m/\overline{B})$ is bounded in terms of fixed numerical invariants for $m \gg 0$.

To prove ($R$), one proves that if $f$ is non-isotrivial, then $\omega_X/\overline{B}$ is ample on $\overline{X}$.

By Kodaira vanishing $\omega_X/\overline{B}$ ample implies that

$$H^1(\overline{X}, T_{\overline{X}/\overline{B}}) = H^1(\overline{X}, \omega_{\overline{X}/\overline{B}}^{-1}) = 0.$$

In other words, $f$ has no first order deformations, and hence ($R$) holds.

**Remark.** This proof only works in the case of families of curves, i.e., when $\dim X - \dim B = 1$. 
**GOAL:** Generalize (B) and (R) for higher dimensional families.

- $B$ is a smooth (not necessarily projective) curve.

- $f : X \to B$ is a smooth projective family of varieties of general type of dimension $n$. ($n$ is arbitrary).

**Definition.** Fix a polynomial $h$. Non-isotrivial families of smooth projective varieties of general type with Hilbert polynomial $h$ will be called *admissible*.

**Boundedness (B):** Admissible families are parametrized by a scheme of finite type.

**Rigidity (R):** There exist no non-trivial deformations of admissible families.
**Boundedness** – Holds.

**Theorem.** (Bedulev-Viehweg) Let $\bar{B}$ be the projective closure of $B$ and $\bar{f} : \bar{X} \to \bar{B}$ an extension of $f$. Then for $m \gg 0$, there exists a constant $c$ depending only on $m$, $n$, $K^n_{\bar{X}_b}$, $g(\bar{B})$, $\#(\bar{B} \setminus B)$, such that

$$\deg \left( \bar{f}_* \omega^m_{\bar{X}/\bar{B}} \right) \leq c \cdot \text{rk} \left( \bar{f}_* \omega^m_{\bar{X}/\bar{B}} \right)$$

**Remark.** Generalizations and related work by Oguiso-Viehweg, Viehweg-Zuo and K_____. (cf. Zuo’s lecture)
Rigidity – Fails.

**Example.** Let $Y \to B$ be a non-isotrivial family of smooth projective curves, and $F$ an arbitrary smooth projective curve. Let $f : X = Y \times F \to B$. Deforming $F$ gives a deformation of $f$.

**Problem.** Under what additional condition does (R) hold?

**Notes.**

- Let $B_0 \subseteq B$ be open and $X_0 = f^{-1}(B_0)$. A sufficient condition, if it exists, should be independent of $B$.
  If it holds for $f$, it should also hold for $f|_{X_0}$.

- If (R) fails for $f$, it also fails for $f|_{X_0}$.

**Principle.** When studying (R), we will freely restrict to open subsets of $B$. 
Example. Let $g : Y \to B$ and $h : Z \to B$ be two families of smooth projective curves of genus at least two. Let $X = Y \times_B Z$.

![Diagram](image)

- If either $g$ or $h$ is isotrivial, then by the above principle we may assume that it is actually trivial.

  Hence (R) fails for $f$.

- It is easy to prove that

$$H^1(X, T_{X/B}) \simeq H^1(Y, T_{Y/B}) \oplus H^1(Z, T_{Z/B}).$$

Recall that if $g$ and $h$ are both non-isotrivial, then

$$H^1(Y, T_{Y/B}) = H^1(Z, T_{Z/B}) = 0.$$

Hence (R) holds for $f$. 
Simple Question. Let \( g : Y \rightarrow B \) and \( h : X \rightarrow Y \) be non-isotrivial families of curves of genus at least two.

![Diagram](image)

Does (R) hold for \( f \)?

**Expectation.** YES.

**Answer.** YES.

**Reason.** Later.
Back to the product.

\[ X = Y \times_B Z \]

\[ f \quad = \quad g \times h \]

\[ B \quad B \quad B \]

**Kodaira-Spencer maps.**

\[ \rho_g : T_B \to R^1 g_* T_{Y/B} \]

\[ \rho_h : T_B \to R^1 h_* T_{Z/B} \]

\[ \sim \sim \]

\[ \rho_g \otimes \rho_h : T_B^\otimes 2 \to R^1 g_* T_{Y/B} \otimes R^1 h_* T_{Z/B} \]

**Corollary.** \( \rho_g \otimes \rho_h \neq 0 \) implies that \((R)\) holds for \( f \).
Notation. $\wedge^m T_X$ will be denoted by $T^m_X$.

Observe. By the Künneth formula,

$$R^1 g_* T_{Y/B} \otimes R^1 h_* T_{Z/B} \simeq R^2 f_* T^2_{X/B}.$$ 

Iterated Kodaira-Spencer maps.

$$0 \to T_{X/B} \to T_X \to f^* T_B \to 0 \quad \sim$$

$$0 \to T^2_{X/B} \to T^2_X \to T_{X/B} \otimes f^* T_B \to 0 \quad \sim$$

$$\rho_f^{(2)} : R^1 f_* T_{X/B} \otimes T_B \to R^2 f_* T^2_{X/B}$$

$$0 \to T_{X/B} \otimes f^* T_B \to T_X \otimes f^* T_B \to f^* T_B^{\otimes 2} \to 0$$

$$\rho_f^{(1)} : T_B^{\otimes 2} \to R^1 f_* T_{X/B} \otimes T_B$$
Iterated Kodaira-Spencer maps (continued).

\[ \rho^{(1)}_f : T_B \otimes 2 \rightarrow R^1 f_* T_{X/B} \otimes T_B \]

\[ \rho^{(2)}_f : R^1 f_* T_{X/B} \otimes T_B \rightarrow R^2 f_* T^2_{X/B} \]

\[ \rho^{(2)}_f \circ \rho^{(1)}_f : T_B \otimes 2 \rightarrow R^2 f_* T^2_{X/B} \]

\[ \rho_g \otimes \rho_h : T_B \otimes 2 \rightarrow R^1 g_* T_{Y/B} \otimes R^1 h_* T_{Z/B} \simeq R^2 f_* T^2_{X/B} \]

**Proposition.** \( \rho^{(2)}_f \circ \rho^{(1)}_f = \rho_g \otimes \rho_h \).

**Corollary.** \( \rho^{(2)}_f \circ \rho^{(1)}_f \neq 0 \Rightarrow (R) \) holds for \( f \).

**Remark.** This statement no longer makes reference to the product structure.
General case: $X$ is no longer a product. $1 \leq p \leq n$,

\[ 0 \to T^p_{X/B} \otimes f^*T_B^{(n-p)} \to T^p_X \otimes f^*T_B^{(n-p)} \to \]

\[ \to T^{p-1}_{X/B} \otimes f^*T_B^{(n-p+1)} \to 0 \]

\[ \rho_f^{(p)} : R^{p-1}f_*(T^{p-1}_{X/B} \otimes T_B^{(n-p+1)}) \to R^pf_*T^p_{X/B} \otimes T_B^{(n-p)} \]

**Definition.** $\rho_f := \rho_f^{(n)} \circ \rho_f^{(n-1)} \circ \cdots \circ \rho_f^{(1)}$

\[ \rho_f : T^n_B \longrightarrow R^n f_*T^n_{X/B} \]

**Definition.** $f$ is called *strongly non-isotrivial* if $\rho_f \neq 0$.

**Example.** Let $Y_i \to B$ be non-isotrivial families of smooth projective curves for $i = 1, \ldots, r$. Then $X = Y_1 \times_B \cdots \times_B Y_r \to B$ is strongly non-isotrivial.

**Remark.** Since $T_B$ is a line bundle and $R^n f_*T^n_{X/B}$ is locally free, $\rho_f \neq 0$ if and only if it is injective.
The case of $\dim B > 1$.

Let (again) $f: X \to B$ be a smooth projective family of varieties (of general type) of dimension $n$, $B$ a smooth (not necessarily projective) variety.

For an integer $p$, $1 \leq p \leq n$,

$$T^p_X = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^p \supset \mathcal{F}^{p+1} = 0$$

$$\mathcal{F}^i / \mathcal{F}^{i+1} \sim T^i_{X/B} \otimes f^*T^{p-i}_B$$

In particular, $\mathcal{F}^p \sim T^p_{X/B}$ and $\mathcal{F}^{p-1} / \mathcal{F}^p \sim T^{p-1}_{X/B} \otimes f^*T_B$.

$$\Rightarrow$$

$$0 \to T^p_{X/B} \otimes f^*T_B^{\otimes(n-p)} \to \mathcal{F}^{p-1} \otimes f^*T_B^{\otimes(n-p)} \to \cdots \to T^{p-1}_{X/B} \otimes f^*T_B^{\otimes(n-p+1)} \to 0$$

$$\rho_f^{(p)}: R^{p-1} f_*T^{p-1}_{X/B} \otimes T_B^{\otimes(n-p+1)} \to R^p f_*T^p_{X/B} \otimes T_B^{\otimes(n-p)}$$
Definition. \( \rho_f \) is called strongly non-isotrivial (everywhere) over \( B \) if \( \rho_f \) is injective.

Example. Let \( Y_i \to B \) be non-isotrivial families of smooth projective curves for \( i = 1, \ldots, r \). Then \( X = Y_1 \times_B \cdots \times_B Y_r \to B \) is strongly non-isotrivial over \( B \).

Remark. One can consider various refinements:

- Considering maps for which the composition of fewer \( \rho^{(p)} \)'s is injective or non-zero. This is important in particular to study moduli spaces of varieties that products with one rigid term.

- Combining this condition with \( \text{Var}(f) \), the variation of \( f \) in (birational) moduli.
THEOREM. Let $f$ be a smooth projective family of varieties of general type. If $f$ is strongly non-isotrivial over $B$, then $(R)$ holds for $f$.

To do. Find more examples of strongly non-isotrivial families.
“Simple Question” revisited. Let \( g : Y \to B \) and \( h : X \to Y \) be non-isotrivial families of curves, \( \dim B = 1 \).

\[
\begin{array}{c}
X \xrightarrow{h} Y \xrightarrow{g} B \\
\phantom{X} \underset{f}{\xrightarrow{}} \phantom{B}
\end{array}
\]

Does (R) hold for \( f \)?

Remarks.

- The assumption that \( h \) is non-isotrivial is not the most natural condition in this situation.

- Over bases of dimension \( \geq 1 \) one usually requires that the variation of the family is maximal. In this case that means \( \text{Var}(h) = \dim Y = 2 \).

- However, if \( X \) is the product of two non-isotrivial families of curves over \( B \), then \( \text{Var}(h) = 1 \).
The Kodaira-Spencer map, $\rho_h : T_Y \to R^1 h_* T_{X/Y}$, measures the variation of the family over $Y$, but we are only interested in variation over $B$.

**Definition.** $h$ is *non-isotrivial with respect to $B$* if

$$\ker \rho_h \subset T_{Y/B}.$$

**Lemma.** $h$ is *non-isotrivial with respect to $B$* if and only if $g^* T_B \to R^1 h_* T_{X/B}$ is injective.

**Proposition.** If $g$ and $h$ are non-isotrivial with respect to $B$, then $f$ is strongly non-isotrivial over $B$.

**Corollary.**
- $(R)$ holds for $f$.
- The answer to the “Simple Question” is indeed yes.
Let $f : X \to B$ be a smooth projective family,

$$n = \dim X - \dim B.$$ 

**(Weak de Jong) Procedure.**

**Step One.** Take a general projection onto $\mathbb{P}^1_B$ and use Stein factorization. This produces

$$X' \overset{\text{birational}}{\longrightarrow} X \overset{f}{\longrightarrow} B \overset{f_1}{\longrightarrow} X_1 : \text{a family of curves}$$
Step Two. Iterating Step One produces $X_0 = B, X_1, \ldots, X_n$ such that there exists a birational morphism $\sigma : X_n \to X$ and for every $i = 1, \ldots, n$, $f_i : X_i \to X_{i-1}$ is a family of curves.
CONJECTURE

\[ f \text{ is strongly non-isotrivial} \]
\[ \updownarrow \]
\[ f_i \text{ are non-isotrivial with respect to } B \text{ for } i = 1, \ldots, n. \]

Corollary of conjecture

\[ f_i \text{ is non-isotrivial} \]
\[ \text{with respect to } B \text{ for } i = 1, \ldots, n \]
\[ \downarrow \]
\[ (R) \text{ holds for } f \]

Lemma.  \( f \) is strongly non-isotrivial if and only if \( f \circ \sigma \) is strongly non-isotrivial.

THEOREM.  \( f_i \) is smooth and non-isotrivial
\[ \text{with respect to } B \text{ for } i = 1, \ldots, n \]
\[ \downarrow \]
\[ f \text{ is strongly non-isotrivial} \]