REFLEXIVE PULL-BACKS
AND BASE EXTENSION

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Abstract

We prove that Viehweg’s moduli functor of stable surfaces is locally closed.

1. Introduction

The moduli theory of curves has been studied extensively in the past few decades. A very important and useful feature of the theory is that the moduli space of smooth projective curves of genus $g$ admits a geometrically meaningful compactification as the moduli space of stable curves of genus $g$. The success of moduli theory of curves leads naturally to a desire for a similar theory for higher-dimensional varieties.

In recent years there has been great progress in the moduli theory of surfaces and higher-dimensional varieties by Alexeev, Kollár, Shepherd-Barron, and Viehweg [1], [11], [13], [14]. According to their work, moduli spaces exist for many moduli problems, in particular, for smooth canonically polarized varieties. More generally, it is established that if a moduli problem satisfies certain properties, then a corresponding (coarse) moduli space exists. The most important of these properties are separatedness, boundedness and local closedness. According to the above authors’ work the former two of these hold for the moduli problem of canonically polarized stable surfaces—the candidate for a geometrically meaningful compactification of the moduli space of smooth canonically polarized surfaces. Local closedness, however, has presented a very stubborn problem.

In fact, one of the main problems is that it is not entirely clear what the “right” definition of the moduli functor should be. This is a very delicate
problem as one would like to make the functor large enough to obtain a compact moduli space, but enlarging the class too much could lead to a loss of separatedness and/or boundedness.

In addition, not only the admissible models have to be decided, but also the admissible families of those models. Experts generally agree on what models should be allowed (the semi-log canonical models). However, the right notion of admissible families is still to be decided.

Both Kollár and Viehweg suggest reasonable definitions, but local closedness has yet to be established for either of their moduli functors. At this time it is not even clear whether their definitions differ. However, we should point out that Kollár’s moduli functor is known to satisfy a weak form of local closedness. Precisely, after passage to a formal or étale local ring, local closedness holds provided we restrict to base change morphisms arising from local ring homomorphisms [12], §14.

The goal of this paper is to prove that Viehweg’s moduli functor of canonically polarized varieties is locally closed.

Definitions and notation. Every scheme is considered to be of finite type over an algebraically closed field $k$ unless specifically noted otherwise.

Let $f : X \to S$ be a morphism. Then $X_s$ denotes the fibre of $f$ over the point $s \in S$ and $f_s$ denotes the restriction of $f$ to $X_s$. More generally, for a morphism $\alpha : T \to S$, let $f_T : X_T = X \times_S T \to T$. In particular, one has the following commutative diagram:

$$
\begin{array}{ccc}
X_T = X \times_S T & \xrightarrow{\alpha_X} & X \\
| & | & | \\
T & \xrightarrow{\alpha} & S
\end{array}
$$

For a coherent $\mathcal{O}_X$-module $\mathcal{F}$, $\mathcal{F}_T$ will denote $\alpha_X^* \mathcal{F}$ on $X_T$. Tensor products of $\mathcal{O}_{X_T}$-modules are over $\mathcal{O}_{X_T}$. These conventions will be used through the entire article.

We will write $\mathcal{F}^*$ for the dual $\mathcal{O}_X$-module $\mathcal{H}om_X(\mathcal{F}, \mathcal{O}_X)$ when there is no risk of confusion. The double dual $\mathcal{F}^{**}$ is called the reflexive hull of $\mathcal{F}$ and there is a natural $\mathcal{O}_X$-module homomorphism

$$
\mathcal{F} \to \mathcal{F}^{**};
$$

$\mathcal{F}$ is said to be reflexive if this is an isomorphism. We shall also consider reflexive powers

$$
\mathcal{F}^{[m]} := (\mathcal{F}^{\otimes m})^{**} \quad \mathcal{F}^{[-m]} := (\mathcal{F}^{\otimes m})^*
$$

for $m > 0$. In general, there exist natural maps

$$
(\mathcal{F}^{**})_T \to (\mathcal{F}_T)^{**} \quad \text{and} \quad (\mathcal{F}^{[m]})_T \to (\mathcal{F}_T)^{[m]}
$$
which need not be isomorphisms, even when $F$ is reflexive. Of course, these maps are isomorphisms when $F$ is locally free.

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2. Moduli functors

Definition 2.1. Fix a base scheme $B$. The moduli functor of polarized proper schemes is the contravariant functor

$$
\mathcal{M}: \text{B-schemes} \to \text{Sets}
$$

given by

$$
\mathcal{M}(S/B) := \left\{ \text{pairs } (f: X \to S, \mathcal{L}), \text{where } \begin{array}{l}
\mathcal{L} \text{ is an } f\text{-ample line bundle on } X
\end{array} \right\} / \sim
$$

where two families $(f_1: X_1 \to S, \mathcal{L}_1)$ and $(f_2: X_2 \to S, \mathcal{L}_2)$ are equivalent $[(f_1: X_1 \to S, \mathcal{L}_1) \sim (f_2: X_2 \to S, \mathcal{L}_2)]$ iff there is an isomorphism $h: X_1 \to X_2$ such that $f_1 = f_2 \circ h$ and there is a line bundle $\mathcal{M}$ on $S$ such that $\mathcal{L}_1 \cong h^*\mathcal{L}_2 \otimes f_1^*\mathcal{M}$. For any morphism of $B$-schemes, $\alpha: T \to S$, we have

$$
\mathcal{M}(\alpha)(X \to S, \mathcal{L}) = (X_T \to T, \alpha_X^*\mathcal{L}).
$$

In this article we will restrict to the case of $B = k$, an algebraically closed field. $\text{Sch}_k$ will denote the category of $k$-schemes.

Any subfunctor of this moduli functor is called a moduli functor of polarized proper schemes.

Definition 2.2. A subfunctor $\mathfrak{F} \subset \mathcal{M}$ is called locally closed iff the following condition is satisfied:

For every $(f: X \to S, \mathcal{L}) \in \mathcal{M}(S)$ there is a locally closed subscheme $j: S^u \hookrightarrow S$ such that if $\alpha: T \to S$ is any morphism, then

$$
(f_T: X_T \to T, \alpha_X^*\mathcal{L}) \in \mathfrak{F}(T) \quad \text{iff there is a factorization } T \xrightarrow{\alpha} S^u \xrightarrow{j} S.
$$

We say that $\mathfrak{F} \subset \mathcal{M}$ is open iff $S^u \subset S$ is open for every $S$.

For the definition of bounded, separated, and complete moduli functors the reader is referred to [14, 1.15].
**Definition 2.3.** Fix a polynomial $h \in \mathbb{Q}[t]$ such that $h(\mathbb{Z}) \subseteq \mathbb{Z}$. The moduli functor of polarized schemes with Hilbert polynomial $h$ is the subfunctor $\mathcal{MP}_h$ of $\mathcal{MP}$ given by

$$\mathcal{MP}_h(S) = \{(f : X \to S, \mathcal{L}) \in \mathcal{MP}(S) | \chi(\mathcal{L}_{X_s}^\vee) = h(\nu) \text{ for all } \nu \in \mathbb{Z} \text{ and } s \in S\}.$$  

This is an open and closed subfunctor.

**Definition 2.4.** A subfunctor $\mathcal{M}^{[N]} \subset \mathcal{MP}$ is called a moduli functor of canonically polarized $\mathbb{Q}$-Gorenstein schemes of index $N$ if, for each $(f : X \to S, \mathcal{L}) \in \mathcal{M}^{[N]}(S)$,

(2.4.1) $X_s$ is connected, Cohen-Macaulay, and Gorenstein outside a closed subscheme of codimension at least two for each $s \in S$;

(2.4.2) $f$ is equivalent to a family of the form $(f : X \to S, \omega_X^{[N]} / S)$.

**Remark 2.5.** Assumption (2.4.1) implies that the fibers are equidimensional projective schemes. One can show that the relative Cohen-Macaulay condition is open (see [5, IV, 12.2.1]), as is the locus where the relative dualizing sheaf is locally free (the relative Gorenstein locus). Since the complement to the relative Gorenstein locus intersects the fibers in codimension $\leq 2$ over an open subset of the base, it follows that Assumption (2.4.1) is open. Note that the singularity assumptions may also be expressed as a condition on the morphism $f$: its relative dualizing complex is supported in degree $-\dim(X/S)$ and the resulting dualizing sheaf is locally free over an open subset whose complement meets each fiber in codimension two. We refer the reader to [9] for a recent account of relative duality.

We emphasize that for families of canonically polarized $\mathbb{Q}$-Gorenstein schemes of index $N$, $\omega_X^{[N]} / S$ is invertible by definition. This is a condition on the morphism, not just a condition on the fibers. Indeed, $\omega_X^{[N]} / S$ may fail to be invertible even when $\omega_X^{[N]}$ is invertible for each $s \in S$ (see [13]). Also, it is not entirely obvious that Assumption (2.4.2) actually yields a subfunctor, i.e., that families of canonically polarized schemes pull back to families of canonically polarized schemes. This is proved in the following lemma:

**Lemma 2.6.** Given a family of canonically polarized $\mathbb{Q}$-Gorenstein schemes of index $N$, $f : X \to S$, and a morphism $\alpha : T \to S$, we have

$$\alpha_X^* \omega_X^{[N]} / S \simeq \omega_{X_T / T}^{[N]}.$$ 

**Proof.** Let $U \subset X$ be the relative Gorenstein locus of $f$, i.e., the largest open subset $U$ of $X$ such that $U_s$ is Gorenstein for all $s \in S$ or equivalently
the largest open subset $U$ of $X$ such that $\omega_{X/S}|U$ is a line bundle. Then $\omega_{X/S}|U \simeq \omega_{U/S}^N$ and hence

$$\alpha_X^\ast \omega_{X/S}|_{\alpha_X^{-1}U} \simeq \alpha_X^\ast \omega_{U/S}^N \simeq \omega_{\alpha_X^{-1}U/T}^N \simeq \omega_{X_T/T}|_{\alpha_X^{-1}U}^N.$$  

Now $\operatorname{codim}(U_s, X_s) \geq 2$ for all $s \in S$, so $\operatorname{codim}((\alpha_X^{-1}U)_t, (X_T)_t) \geq 2$ for all $t \in T$. Finally, $\alpha_X^\ast \omega_{X/S}^N$ and $\omega_{X_T/T}^N$ are reflexive, and since they are isomorphic on $\alpha_X^{-1}U$, they are isomorphic on $X_T$ (cf. (3.6.2)).

If $\mathcal{M}^{[N]}_s$ is a functor of canonically polarized $\mathbb{Q}$-Gorenstein schemes of index $N$, then we can consider $\mathcal{M}^{[N]}_h$ as well. An argument using Proposition 3.6 (and very similar to Lemma 2.6) implies

$$\mathcal{M}^{[N]}_h(S) = \{(f : X \to S) \in \mathcal{M}^{[N]}_s(S) \mid \chi(\omega_{X_s}^{[\nu,N]}) = h(\nu)$$

for all $\nu \in \mathbb{Z}$ and $s \in S$.

Remark 2.7. Note that we speak of "a" moduli functor and not "the" moduli functor. The reason is that in order to obtain a relatively nice moduli space one has to restrict to a smaller class than all the canonically polarized $\mathbb{Q}$-Gorenstein schemes of index $N$. On the other hand, one could consider "the" moduli space of smooth varieties, but in that case one would not obtain a compact moduli space. The "right" class of schemes will be somewhere between these two and part of the difficulty is to identify that class.

Assumptions 2.8.

(2.8.1) $\mathcal{M}^{[N]}_h$ is locally closed;
(2.8.2) $\mathcal{M}^{[N]}_h$ is bounded;
(2.8.3) $\mathcal{M}^{[N]}_h$ is separated;
(2.8.4) $\mathcal{M}^{[N]}_h$ is complete;
(2.8.5) for all smooth projective curves $S$, and for all $(f : X \to S) \in \mathcal{M}^{[N]}_h(S)$, the sheaf $f_\ast \omega_{X/S}^{[\nu,N]}$ is semi-positive for all $\nu$ sufficiently large and divisible.

Theorem 2.9 ([10, 4.2.1], [13, 5.7], [11, 5.6], [14, 9.23, 9.30]). Assume that $k$ has characteristic zero. We retain the notation introduced above and assume that $\mathcal{M}^{[N]}_h$ satisfies the conditions of (2.8).

Let $\nu > 0$ be a fixed integer such that $\omega_{X}^{[\nu,N]}$ is very ample and without higher cohomology for all $X \in \mathcal{M}^{[N]}_h(k)$.

Then there exists a coarse algebraic moduli space $\mathcal{M}^{[N]}_h$ for $\mathcal{M}^{[N]}_h$ which is a projective scheme and for all $\mu > 0$ there exist a $p > 0$ and an ample invertible sheaf $\lambda_{\mu,[\nu,N]}^{(p)}$ on $\mathcal{M}^{[N]}_h$, such that for all $(f : X \to S) \in \mathcal{M}^{[N]}_h(S)$ and for the induced morphism $\phi : S \to \mathcal{M}^{[N]}_h$ one has $\phi_\ast \lambda_{\mu,[\nu,N]}^{(p)} = \left(\det f_\ast \omega_{X/S}^{[\nu,N]}\right)^p$. 


**Viehweg’s Functor** (Property $V[N]$). Consider a family of polarized varieties, $f : X \to S$, satisfying Assumption (2.4.1). We say that $f$ satisfies property $V[N]$ if $\omega_{X/S}^{[N]}$ is invertible.

Note that families of canonically polarized $\mathbb{Q}$-Gorenstein schemes of index $N$ automatically satisfy property $V[N]$ (by Assumption (2.4.2)).

**Definition 2.10.** Let $\mathcal{Y}^{N,d}_h$ be the moduli functor of canonically polarized $\mathbb{Q}$-Gorenstein schemes of index $N$ and Hilbert polynomial $h$ satisfying the following:

(2.10.1) for each $s \in S$, $X_s$ is a reduced scheme of dimension $d$ and has semi-log canonical singularities.

We emphasize that we are retaining Assumptions (2.4.1) and (2.4.2). Note that each fiber $X_s$ automatically has index $N$.

Let $N' = mN$ be a positive multiple of $N$ and $h'(t) = h(mt)$. There is a natural transformation,

$$\mathcal{Y}^{N,d}_h \to \mathcal{Y}^{N',d}_{h'},$$

induced by taking the $m$th power of the canonical polarization.

**Kollár’s Functor** (Property K). Consider a family of polarized varieties, $f : X \to S$, satisfying Assumption (2.4.1). We say that $f$ satisfies property K if

$$\alpha^* X_{X/S}^{[j]} \simeq \omega_{X'/T}^{[j]}$$

for any morphism, $\alpha : T \to S$, and each $j \in \mathbb{Z}$.

For canonically polarized $\mathbb{Q}$-Gorenstein schemes of index $N$, it suffices to verify this for $j = 1, \ldots, N - 1$. Indeed, Proposition 3.6 yields

$$\omega_{X/S}^{[j+\sigma N]} = \omega_{X/S}^{[j]} \otimes (\omega_{X/S}^{[N]})^\nu.$$

**Definition 2.11.** Let $\mathcal{R}^{N,d}_h$ be the moduli functor of canonically polarized $\mathbb{Q}$-Gorenstein schemes of index $N$ and Hilbert polynomial $h$ satisfying the following:

(2.11.1) for each $s \in S$, $X_s$ is a reduced scheme of dimension $d$ and has semi-log canonical singularities;

(2.11.2) each family $(f : X \to S, \omega_{X/S}^{[N]}) \in \mathcal{R}^{N,d}_h(S)$ satisfies property K.

If a family satisfies property K, then the family is in $\mathcal{R}^{N,d}_h$ if the indices of the fibers all divide $N$. Let $N' = mN$ be a positive multiple of $N$ and $h'(t) = h(mt)$. Then the natural transformation,

$$\mathcal{R}^{N,d}_h \to \mathcal{R}^{N',d}_{h'},$$

induced by taking the $m$th power of the canonical polarization, is an open immersion.
These conditions are stronger than those of Viehweg’s functor, so there is
a natural transformation of moduli functors,
\[ R_h^{[N],d} \to \mathcal{V}_h^{[N],d}, \]
inducing a bijection between \( R_h^{[N],d}(k) \) and \( \mathcal{V}_h^{[N],d}(k) \).

**Moduli of Surfaces: Smoothability and Boundedness.**

**Definition 2.12.** Let \( \mathcal{V}_{h,\text{sm}}^{[N],2}(k) \) denote the following subset of \( \mathcal{V}_h^{[N],2}(k) \):

\[
\mathcal{V}_{h,\text{sm}}^{[N],2}(k) = \{ X \mid X \in \mathcal{V}_h^{[N],2}(k), \text{ and } \exists (g : Y \to C) \in \mathcal{V}_h^{[N],2}(C), \text{ such that } \\
C \text{ is an irreducible curve, } X \simeq X_c \text{ for some } c \in C, \text{ and } \\
X_{\text{gen}} \text{ is a normal surface with at most rational double points.} \}
\]

We define \( R_{h,\text{sm}}^{[N],2}(k) \) analogously.

Once we construct the moduli schemes \( \mathcal{V}_{h,\text{sm}}^{[N],2} \) and \( \mathcal{K}_{h,\text{sm}}^{[N],2} \), we may realize \( \mathcal{V}_{h,\text{sm}}^{[N],2}(k) \) and \( \mathcal{K}_{h,\text{sm}}^{[N],2}(k) \) as the closed points of certain subvarieties. The points satisfying the smoothability condition form a union of irreducible components, and this union forms a closed subvariety. However, it is not known whether this admits a functorial scheme structure.

**Remark 2.13.** Assume that \( k \) has characteristic zero for the remainder of
this subsection. \[1, 5.11\] implies that there exists an \( N \in \mathbb{N} \) such that
\[
\mathcal{V}_h(k) := \bigcup_{m \in \mathbb{N}} \mathcal{V}_h^{[m],2}(k) = \mathcal{V}_h^{[N],2}(k)
\]
and
\[
\mathcal{V}_{h,\text{sm}}(k) := \bigcup_{m \in \mathbb{N}} \mathcal{V}_{h,\text{sm}}^{[m],2}(k) = \mathcal{V}_{h,\text{sm}}^{[N],2}(k).
\]

In order to construct moduli spaces for \( \mathcal{V}_h^{[N],d} \) and \( \mathcal{K}_h^{[N],d} \), one has to verify
the assumptions of \[.8\] All the properties listed in \[.8\] except \( .8.3 \), are the same for both \( \mathcal{V}_h^{[N],d} \) and \( \mathcal{K}_h^{[N],d} \).

- \[10, 2.1.2\] implies \( .8.2 \).
- \[13, 5.1\] implies \( .8.3 \) and \( .8.4 \), at least for the irreducible
components satisfying the smoothability condition. For the general
case, one has to construct a unique stable limit for a one-parameter
family of nonnormal stable surfaces. Consider its normalization
as a family of stable log surfaces with boundary equal to the con-
ductor. Apply the log minimal model program and the results of
\[7\] to obtain a unique limiting stable log surface. We glue back
together along the conductor to recover the stable limit of our
original family.
- \[11, 4.12\] implies \( .8.5 \).
That leaves us to verify (2.8.1), and in the rest of the article we will concentrate on this property.

Proof of Local Closedness. To prove that $\Psi_h^{[N], d}$ is locally closed, one would naturally list the properties of the functor and prove one by one that all of them are locally closed. However, this requires special attention. A potentially tricky part is that the order of this procedure matters. For instance, the requirement that $\omega_X^{[N]}|_{X/S}$ be invertible should not be considered until only open properties remain, because it may very well happen that $\omega_X^{[N]}|_{X/S}$ is not invertible along an admissible fiber $X_s$ but $\omega_X^{[N]}|_{X_T/T}$ becomes invertible after restricting to some locally closed $T \subset S$ containing $s$. In particular, the locus where $\omega_X^{[N]}|_{X/S}$ is invertible does not coincide with the locus where $\omega_X^{[N]}|_{X_T/T}$ is invertible. The key problem is: taking reflexive powers does not generally commute with base extension.

The next theorem is the main result of this article. Here we reduce local closedness to a rather technical statement which will be proved in the next section.

**Theorem 2.14.** The moduli functor of canonically polarized $\mathbb{Q}$-Gorenstein schemes of index $N$ is locally closed.

**Proof.** In proving local closedness, we address the conditions imposed on the fibers of $f : X \to S$ separately from the conditions imposed on the morphism $f$ itself. We have already observed in Remark 2.5 that condition (2.4.1) is open. Now we turn to condition (2.4.2), i.e., $\omega_X^{[N]}|_{X/S}$ is locally free. Suppose we are given an arbitrary family of polarized varieties $(f : X \to S, \mathcal{L})$ with fibers satisfying (2.4.1). We apply Theorem 3.11 with $F = \omega_X^{[N]}|_{X/S}$. This sheaf may be terribly singular, perhaps even with torsion along certain fibers. However, $\omega_X^{[N]}|_{X/S}$ does have one salient property; it commutes with arbitrary base extensions $\alpha : T \to S$, i.e.,

$$\alpha^* \omega_X^{[N]}|_{X/S} = \omega_T^{[N]}|_{X_T/T}.$$

By Theorem 3.11 there exists a locally closed subscheme $S_u \subset S$ with the following universal property. Given a morphism $\alpha : T \to S$, there exists an invertible sheaf $\mathcal{N}$ on $T$ and an isomorphism

$$(\omega_X^{[N]}|_{X/T})^{**} \cong \mathcal{L}_T \otimes f_T^* \mathcal{N}$$

if and only if $\alpha$ factors through $S_u$. By definition we have

$$\omega_X^{[N]}|_{X_T/T} = (\omega_X^{[N]}|_{X_T/T})^{**},$$

so the proof that condition (2.4.2) is locally closed is complete. \[\square\]

**Theorem 2.15.** If $k$ is a field of characteristic zero, then $\Psi_h^{[N], 2}$ is a locally closed moduli functor. In particular, $\Psi_h^{[N], 2, \text{sm}}(k)$ is locally closed.
Proof. It remains to verify that condition (2.10.1) is locally closed once conditions (2.4.1) and (2.4.2) are imposed. In particular, we may assume that we have families of canonically polarized $\mathbb{Q}$-Gorenstein varieties of index $N$.

The condition that the geometric fibers $X_s$ are reduced is open by [5, IV, 12.2.1]. The locus where the fibers have semi-log canonical singularities is open by [8, 2.6] (see also [13, §5]).

Remark 2.16 (characteristic zero). If one assumes the existence of minimal models in dimension $d + 1$, the results of [8] imply that having semi-log canonical singularities is an open condition for families of canonically polarized $\mathbb{Q}$-Gorenstein varieties of index $N$. It follows that $\mathcal{Z}_{N,d}$ is locally closed.

3. Local closedness of reflexive pull-backs

We first recall the following result from [5, IV, §6.3]:

**Proposition 3.1.** Let $A$ and $B$ be noetherian local rings, $k$ the residue field of $A$, $\phi : A \rightarrow B$ a local homomorphism, $M$ an $A$-module of finite type, and $N$ a $B$-module of finite type. If $N$ is flat and nonzero as an $A$-module, then

$$\text{depth}_B(M \otimes_A N) = \text{depth}_A(M) + \text{depth}_{B \otimes_A k}(N \otimes_A k).$$

We assume that all schemes are noetherian.

Let $f : X \rightarrow S$ be a flat morphism of schemes and $\mathcal{E}$ a coherent $\mathcal{O}_X$-module flat over $S$. We say $\mathcal{E}$ is $S_r$ relative to $f$ if the following holds: for each $x \in X, s = f(x), F = X_s$ we have

$$\text{depth}_{\mathcal{O}_{F,x}}(\mathcal{E}|_F) \geq \min(r, \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{S,s}).$$

In other words, the restriction of $\mathcal{E}$ to each fiber is $S_r$. By Proposition 3.1, this is equivalent to

$$\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{E}) \geq \text{depth}_{\mathcal{O}_{S,x}}(\mathcal{O}_{S,s}) + \min(r, \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{S,s}).$$

Two special cases deserve further attention. If $\mathcal{E} = \mathcal{O}_X$, then our definition coincides with the ordinary definition of an $S_r$ morphism (see [5, IV, §6]). If $S = \text{Spec}(K)$, where $K$ is a field, then we recover a notion of an $S_r$ sheaf. For example, a sheaf is $S_1$ provided it has no imbedded points. We can translate the definition of $S_r$-sheaves using the cohomological interpretation of depth (see [5, §III.3 Ex. 4]).

**Proposition 3.2.** Let $f : X \rightarrow S$ be a flat morphism of schemes and $\mathcal{E}$ a coherent $\mathcal{O}_X$-module flat over $S$. Then $\mathcal{E}$ is $S_r$ relative to $f$ if and only if, for each $x \in X, s = f(x)$, we have

$$\min\{i : H^i_s(\mathcal{E}) \neq 0\} \geq \text{depth}_{\mathcal{O}_{S,s}}(\mathcal{O}_{S,s}) + \min(r, \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{S,s}).$$
Here $H^i_x(E)$ denotes cohomology on $\text{Spec}(O_{X,x})$ with support at the closed point and coefficients in $E_x$. If $Z \subset X$ is closed, we use $H^i_Z$ to denote the local cohomology sheaf associated to cohomology on $X$ with support along $Z$.

**Proposition 3.3.** Let $f : X \to S$ be a flat morphism of schemes and $E$ a coherent $O_X$-module flat over $S$ and $S_2$ relative to $f$. Let $Z \subset X$ be a closed subscheme with ideal sheaf $I_Z$. Assume that $\text{codim}(Z_s, X_s) \geq r$ for each $s \in S$. Then we have $\text{depth}_{O_X}(I_Z, E) \geq r$, or equivalently, $H^k_Z(E) = 0$ for $k = 0, \ldots, r - 1$.

**Proof.** The cohomological interpretation of depth gives the equivalence of the two conclusions.

For each point $x \in Z$ we have

$$H^0_x = H^0_x \circ H^0_Z$$

which induces a spectral sequence

$$E_2^{p,q} := H^p_x \circ H^q_Z \Rightarrow H^{p+q}_Z.$$

The proof is by induction on $k$, starting with $k = 0$. Assume that $H^0_Z(E) \neq 0$ and its support contains a point $x \in Z$. We have $H^0_x(H^0_Z(E)) \neq 0$ and thus $H^0_x(E) \neq 0$. Writing $s = f(x)$, we obtain a contradiction to Proposition 3.2.

Now assume that $H^i_x(E) = 0$ for $i = 0, \ldots, k - 1$ where $k < r$, but that $H^k_x(E) \neq 0$ and its support contains $x \in Z$. It follows that $H^k_x(E) = H^0_x(H^0_Z(E)) \neq 0$, which again contradicts Proposition 3.2.

**Corollary 3.4.** Let $f : X \to S$ be a flat $S_2$ morphism, and $Z \subset X$ a subscheme such that $\text{codim}(Z_s, X_s) \geq r$ for each $s \in S$. Then $\text{grade}(I_Z) \geq r$.

**Proposition 3.5.** Let $f : X \to S$ be a flat morphism of schemes, $E$ a coherent sheaf flat over $S$ and $S_2$ relative to $f$, and $j : U \hookrightarrow X$ an open subscheme with complement $Z$. Assume that $\text{codim}(Z_s, X_s) \geq 2$ for each $s \in S$. Then the natural map

$$E \to j_* (E|_U)$$

is an isomorphism.

**Proof.** The long exact sequence

$$(3.5.1) \quad 0 \to H^0_Z(E) \to E \to j_* (E|_U) \to H^1_Z(E) \to 0$$

yields the isomorphism. □

**Proposition 3.6.** Let $f : X \to S$ be a flat $S_2$ morphism, $F$ a reflexive coherent $O_X$-module. Let $Z \subset X$ be a closed subscheme so that $\text{codim}(Z_s, X_s) \geq 2$ for each $s \in S$, and let $U$ be the complement of $Z$.

$$(3.6.1) \quad H^k_Z(F) = 0 \text{ for } k = 0, 1 \text{ and the natural map } F \to j_* (F|_U)$$

is an isomorphism.
(3.6.2) Let $F'$ be another coherent $O_X$-module which is either $S_2$ relative to $f$ or reflexive. If $F|_U \simeq F'|_U$, then $F \simeq F'$.

Proof. Consider a presentation of $F^*$ by locally free sheaves

$$E_2 \to E_1 \to F^* \to 0.$$ 

On dualizing we obtain

$$0 \to F \to E_1^* \to E_2^*.$$ 

Since the $E_i^*$ are locally free and $f$ is $S_2$, Proposition 3.3 yields

$$H^k(Z_i(E_i)) = 0$$ for $k = 0, 1$.

Taking the associated long exact sequences, we obtain the desired vanishing for $F$. The long exact sequence in local cohomology (cf. (3.5.1)) yields the first isomorphism. The isomorphism between $F$ and $F'$ is obtained by pushing forward. \qed

We obtain a criterion for when push-forwards of reflexive sheaves are reflexive:

**Corollary 3.7.** Let $f : X \to S$ be a flat $S_2$ morphism and $j : U \hookrightarrow X$ an open subscheme with complement $Z$. Assume that $\text{codim}(Z_s, X_s) \geq 2$ for each $s \in S$. If $G$ is a reflexive coherent sheaf on $U$, then $j_* G$ is also reflexive and coherent.

Proof. Choose a coherent subsheaf $E \subset j_* G$ so the induced map $E|_U \to G$ is an isomorphism [6, II.5 Ex.15]. The reflexive hull $E^{**}$ is also coherent and we have $E^{**}|_U \simeq G$. An application of (3.6.2) implies

$$E^{**} \to j_*(E^{**}|_U) \simeq j_* G$$

is an isomorphism. \qed

We also obtain the following (cf. [2, 1.4.1]):

**Corollary 3.8.** Let $f : X \to S$ be a flat $S_2$ morphism, $E$ a coherent sheaf flat over $S$ and $S_2$ relative to $f$, and $j : U \hookrightarrow X$ an open subscheme with complement $Z$. Assume that $\text{codim}(Z_s, X_s) \geq 2$ for each $s \in S$ and $E|_U$ is reflexive. Then $E$ is reflexive, and furthermore, for each $\alpha : T \to S$ the pull-back $E_T$ is reflexive.

Proof. We apply (3.6.2) to show that the natural map $E \to E^{**}$ is an isomorphism. Our hypotheses are preserved under base extension, so $E_T$ is reflexive for each $\alpha : T \to S$. \qed

Suppose that $f : X \to S$ is a flat projective Cohen-Macaulay morphism of relative dimension $d$. Then the relative dualizing sheaf $\omega_{X/S}$ exists, commutes with base extension, and is $S_d$ relative to $f$ [9, §9.21], [3, 3.6.1], [1, 21.8]. In light of our previous results, it is natural to compare the relative dualizing sheaf with the reflexive hull of a coherent sheaf.
Theorem 3.9. Let \( f : X \to S \) be a flat projective Cohen-Macaulay morphism of relative dimension \( d \) with geometrically connected fibers. Let \( G \) be a coherent sheaf on \( X \), and \( U \hookrightarrow X \) an open subset with complement \( Z \) so that \( \text{codim}(Z_s, X_s) \geq 2 \) for each \( s \in S \). Assume that \( \omega_{X/S} \) and \( G \) are locally free on \( U \). Then there exists a locally closed subscheme \( S^u \subset S \) with the following property. Given a morphism \( \alpha : T \to S \), there exists an invertible sheaf \( N \) on \( T \) and an isomorphism

\[
(G_T)^* \cong_{\alpha} \omega_{X_{T}/T} \otimes f^*_T N
\]

if and only if \( \alpha \) factors through \( S^u \).

Proof. We produce a subscheme \( S^u \subset S \) over which the isomorphism (3.9.1) exists. Let \( W' \subset S \) denote the subscheme supporting \( R^d f_* G \). Note that \( W' \) has a naturally defined scheme structure. Indeed, let \( P^* \to G \) be a presentation of \( G \) by locally free \( \mathcal{O}_X \)-modules. The proof of the cohomology and base change theorem [5, III, §7.7] produces a complex of locally free \( \mathcal{O}_S \)-modules,

\[
0 \to E_0 \to E_1 \to \cdots \to E_{d-1} \xrightarrow{\psi} E_d \to 0
\]

computing the direct image sheaves \( R^j(f_T)^* P^*_T \) for any base extension \( T \to S \). Since the maximal fibre dimension of \( f \) is \( d \), \( R^j(f_T)^* = 0 \) for all \( j > d \). Furthermore, \( G \) is the highest nonzero cohomology sheaf of \( P^* \), so

\[
R^d f_* P^* \cong R^d f_* G.
\]

In particular, \( R^d f_* G \) is the cokernel of \( \psi \) and we define \( W' \) using the rank(\( E_d \))-minors of \( \psi \). It also follows that the formation of \( R^d f_* G \) commutes with base extension, i.e.,

\[
(R^d f_* G)_s \to H^d(X_s, G_s)
\]

is an isomorphism for each \( s \in S \).

Let \( W' \subset W' \) be the locally closed subset obtained by removing points where \( \text{rank}(\psi) < \text{rank}(E_d) - 1 \) and \( f_W : X_W \to W \) the corresponding morphism. Hence \( \mathcal{M} := R^d(f_W)_* G_W = (R^d f_* G)_W \) is locally free of rank one. The relative duality theorem of Kleiman [9, §10,21] gives an isomorphism of \( \mathcal{O}_W \)-modules

\[
\text{Hom}_{f_W}(G_W, \omega_{X_{W}/W} \otimes f^*_W \mathcal{M}) \cong \text{Hom}_W(\mathcal{M}, \mathcal{M}) = \mathcal{O}_W.
\]

The identity \( 1 \in \mathcal{O}_W \) gives a natural homomorphism \( \phi : G_W \to \omega_{X_{W}/W} \otimes f^*_W \mathcal{M} \) which factors

\[
G_W \xrightarrow{\phi} \omega_{X_{W}/W} \otimes f^*_W \mathcal{M}
\]

\[
(\mathcal{G}_W)^{**} \xrightarrow{\phi^{**}} \omega_{X_{W}/W} \otimes f^*_W \mathcal{M}
\]

because \( \omega_{X_{W}/W} \) is reflexive.
Let $S^u \subset W$ be the open subset over which $\phi|_U$ is an isomorphism. The map $\phi^{**}$ is an isomorphism over $S^u$ by Proposition 3.6. For any $S^u$-scheme $T$, the map
\[(\phi^{**})_T : ((\mathcal{G}_W)^{**})_T \to (\omega_{X_W/W})_T \otimes f_T^* \mathcal{M}\]
induced by base extension is also an isomorphism. Since $(\omega_{X_W/W})_T = \omega_{X_T/T}$ is flat and reflexive, the same holds for $((\mathcal{G}_W)^{**})_T$. Hence Proposition 3.6 guarantees that the natural map $((\mathcal{G}_W)^{**})_T \to (\mathcal{G}_T)^{**}$ is an isomorphism.

It remains to show that $S^u$ satisfies the universal property. Let $T$ be an $S$-scheme, $\mathcal{N}$ an invertible sheaf on $T$, and
\[(3.9.4) \quad \rho^{**} : (\mathcal{G}_T)^{**} \to \omega_{X_T/T} \otimes f_T^* \mathcal{N}\]
an isomorphism. For each $t \in T$, the natural map $\mathcal{G}_t \to ((\mathcal{G}_T)^{**})_t$ is an isomorphism over $U_t$, a subset with codimension $\geq 2$ complement. We therefore obtain an isomorphism of cohomology groups
\[H^d(X_t, \mathcal{G}_t) \to H^d(X_t, ((\mathcal{G}_T)^{**})_t)\]
and the base-change isomorphism (3.9.2) yields
\[\mathbb{R}^d(f_T)_* \mathcal{G}_T \to \mathbb{R}^d(f_T)_*[([\mathcal{G}_T])^{**}].\]
The composed morphism
\[\rho : \mathcal{G}_T \to (\mathcal{G}_T)^{**} \to \omega_{X_T/T} \otimes f_T^* \mathcal{N}\]
induces
\[\mathbb{R}^d(f_T)_* \mathcal{G}_T \to \mathbb{R}^d(f_T)_*[\omega_{X_T/T} \otimes f_T^* \mathcal{N}].\]

Since $f_T$ is Cohen-Macaulay, relative duality yields an isomorphism
\[\mathbb{R}^d(f_T)_* \omega_{X_T/T} \simeq \mathcal{E}xt^d f_T_* (\mathcal{O}_{X_T}, \omega_{X_T/T}) \simeq \mathcal{H}om_T([f_T], \mathcal{O}_{X_T}, \mathcal{O}_T) \simeq \mathcal{O}_T,\]
where the last isomorphism follows from the fact that $f_T$ has geometrically connected fibers. It follows that
\[\mathbb{R}^d(f_T)_* \mathcal{G}_T \simeq \mathbb{R}^d(f_T)_*[\omega_{X_T/T} \otimes f_T^* \mathcal{N}] \simeq \mathbb{R}^d(f_T)_* \omega_{X_T/T} \otimes \mathcal{N} \simeq \mathcal{N},\]
hence $T \to S$ factors as $T \to W \to S$ and $\mathcal{N} \simeq \mathcal{M}_T$. Applying duality again, we may regard $\rho$ as an element of $\mathcal{H}om_T(\mathcal{N}, \mathcal{N}) \simeq (\mathcal{H}om_W(\mathcal{M}, \mathcal{M}))_T$ and compare $\rho$ and $\phi_T$ over $T$. The identification in the isomorphism (3.9.3) is functorial, hence $\phi_T$ corresponds to $1 \in \mathcal{H}om_T(\mathcal{N}, \mathcal{N})$, $\rho$ corresponds to some $r \in \mathcal{O}_T$, and $\rho = r \phi_T$.

To complete the proof it suffices to check that $r \in \mathcal{O}_T^*$. Consider the restriction of the isomorphism (3.9.3) to the open subset $U_T$ where the sheaves are all locally free
\[\rho|_{U_T} = \rho^{**}|_{U_T} : (\mathcal{G}_T)|_{U_T} \to (\omega_{X_T/T} \otimes f_T^* \mathcal{N})|_{U_T}.\]
If \( r \) were not invertible at some \( t \in T \), then \( \rho|_{U_t} \) would have nontrivial cokernel over \( t \), a contradiction. \( \square \)

**Corollary 3.10.** Retain all the notation and hypotheses of Theorem 3.9. Assume, in addition, that \( G \) is \( S_2 \)-relative to \( f \). Then there exists a locally closed subscheme \( S^u \subset S \) with the following property. Given a morphism \( \alpha : T \to S \), there exists an invertible sheaf \( \mathcal{N} \) on \( T \) and an isomorphism

\[
  G_T \cong \omega_{X_T/T} \otimes f_T^* \mathcal{N}
\]

if and only if \( \alpha \) factors through \( S^u \).

**Proof.** This follows from Theorem 3.9 and Corollary 3.8. \( \square \)

**Theorem 3.11.** Let \( f : X \to S \) be a flat projective Cohen-Macaulay morphism of relative dimension \( d \) with geometrically connected fibers. Let \( \mathcal{L} \) be an invertible sheaf on \( X \), \( \mathcal{F} \) a coherent sheaf on \( X \), and \( U \hookrightarrow X \) an open subset with complement \( Z \) so that \( \text{codim}(Z_s, X_s) \geq 2 \) for each \( s \in S \). Assume that \( \omega_{X/S} \) and \( \mathcal{F} \) are locally free on \( U \). Then there exists a locally closed subscheme \( S^u \subset S \) with the following property. Given a morphism \( \alpha : T \to S \), there exists an invertible sheaf \( \mathcal{N} \) on \( T \) and an isomorphism

\[
  (\mathcal{F}_T)^* \cong \mathcal{L}_T \otimes f_T^* \mathcal{N}
\]

if and only if \( \alpha \) factors through \( S^u \).

**Proof.** Without loss of generality, we may assume that \( \mathcal{L} \) is trivial (replace \( \mathcal{F} \) by \( \mathcal{F} \otimes \mathcal{L}^{-1} \)). Apply Theorem 3.9 with \( G = \mathcal{F} \otimes \omega_{X/S} \), so that \( (G_T)^* \cong \omega_{X_T/T} \otimes f_T^* \mathcal{N} \) iff \( T \) factors through \( S^u \). On the other hand, \( (G_T)^* \cong \omega_{X_T/T} \otimes f_T^* \mathcal{N} \) if and only if \( G_T|_{U_T} \cong (\omega_{X_T/T} \otimes f_T^* \mathcal{N})|_{U_T} \), which is the case exactly when \( \mathcal{F}_T|_{U_T} \cong f_T^* \mathcal{N}|_{U_T} \), or equivalently, when \( (\mathcal{F}_T)^* \cong f_T^* \mathcal{N} \) (by Proposition 3.6). \( \square \)

**Remark 3.12.** It is natural to try to generalize this result for more general sheaves \( \mathcal{L} \). The above argument is still valid provided \( \mathcal{L} \) satisfies the following:

1. \( \mathcal{L} \) is \( S_2 \) relative to \( f \);
2. \( \mathcal{L}|_U \) is invertible.

**Proof.** We apply Theorem 3.9 with \( G = \mathcal{F} \otimes \mathcal{L}^* \otimes \omega_{X/S} \). We obtain a locally closed subset \( S^u \subset S \) such that \( (G_T)^* \cong \omega_{X_T/T} \otimes f_T^* \mathcal{N} \) iff \( T \) factors through \( S^u \). Again, this is the case if and only if

\[
  (\mathcal{F} \otimes \mathcal{L}^* \otimes \omega_{X_T/T})|_{U_T} \cong f_T^* \mathcal{N}|_{U_T},
\]

which by Assumption 3.12.2 is equivalent to

\[
  \mathcal{F}|_{U_T} \cong \mathcal{L} \otimes f_T^* \mathcal{N}|_{U_T},
\]

which in turn is equivalent to \( (\mathcal{F}_T)^* \cong (\mathcal{L}_T)^* \otimes f_T^* \mathcal{N} \). Applying Corollary 3.8 along with Assumptions 3.12.1 and 3.12.2 yields that \( \mathcal{L}_T \) is reflexive for each \( T \), so the last isomorphism exists iff \( (\mathcal{F}_T)^* \cong \mathcal{L}_T \otimes f_T^* \mathcal{N} \). \( \square \)
References


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