FAMILIES OF VARIETIES
OF GENERAL TYPE:
THE SHAFAREVICh CONJECTURE
AND RELATED PROBLEMS

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HIGHER DIMENSIONAL VARIETIES
AND RATIONAL POINTS

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• **Everything is defined over** $k$, with char $k = 0$, and (most of the time) $\overline{k} = k$.

**NOTATION.**

• $B$ is a smooth curve of genus $g$

• $\Delta \subset B$ is a finite subset

**DEFINITION.** A family (**over** $B$),

$f : X \to B$, consists of

• $X$, a smooth projective variety, and

• $f : X \to B$, a flat projective morphism with connected fibres.

**NOTATION.** For $b \in B$, $X_b = f^{-1}(b)$.

Similarly, for $Z \to B$, $X_Z = X \times_B Z$.

**DEFINITION.** A family $f : X \to B$ is isotrivial if $X_a \simeq X_b$ for general $a, b \in B$. 
**SHAFAREVICH CONJECTURE (1962).**
Function Field Case

**Fix** \( B, \Delta \) **and an integer** \( q \geq 2 \). **Let** \( g = g(B) \).

(I) **There exists only finitely many isomorphism classes of smooth non-isotrivial families of curves of genus** \( q \) **over** \( B \setminus \Delta \). (These will be called "admissible families").

(II) **If** \( 2g - 2 + \# \Delta \leq 0 \), **then** \( \nexists \) **no such families.**

Note:

\[
2g - 2 + \# \Delta \leq 0 \quad \Leftrightarrow \quad \begin{cases} 
g = 0 & \& \# \Delta \leq 2 
g = 1 & \& \Delta = \emptyset 
\end{cases}
\]

- **The Shaferovich Conjecture in the function field case was confirmed by**

**Parshin** **for** \( \Delta = \emptyset \) (1968),

**Arakelov in general** (1971).
REFORMULATION.

(B) Boundedness
Admissible families are parametrized by $\mathbb{T}$, a scheme of finite type.
($\exists$ only finitely many deformation types.)

(R) Rigidity
$\dim \mathbb{T} = 0$.
($\nexists$ non-trivial deformations of these families.)

(H) Hyperbolicity
$\mathbb{T} \neq \emptyset \Rightarrow 2g - 2 + \#\Delta > 0$.
(No admissible families exist if $2g - 2 + \#\Delta \leq 0$).

Remark. $(B) + (R) \Rightarrow (I)$
$(H) \Leftrightarrow (II)$
INTERMEZZO: THE NUMBER FIELD CASE

- **LET** \((R, m)\) **BE** A DVR, \(F = \text{Frac}(R)\) **AND**
  \(C\) A SMOOTH PROJECTIVE CURVE OVER \(F\).

**DEFINITION.** \(C\) **HAS A** **GOOD REDUCTION OVER** \(R\)
**IF** \(\exists Z\) SMOOTH PROPER VARIETY OVER \(\text{Spec } R\)
**SUCH THAT** \(Z_F \simeq C\).

- **LET** \(R\) **BE** A DEDEKIND RING, \(F = \text{Frac}(R)\) **AND**
  \(C\) A SMOOTH PROJECTIVE CURVE OVER \(F\).

**DEFINITION.** \(C\) **HAS A** **GOOD REDUCTION AT THE**
**CLOSED POINT** \(m \in \text{Spec } R\)
**IF** **IT HAS A** **GOOD REDUCTION OVER** \(R_m\).
SHAFAREVICH CONJECTURE (1962). Number Field Case

• Let $q \geq 2$ be an integer,

• $F$ a number field (a finite extension of $\mathbb{Q}$),

• $R \subset F$ the ring of integers of $F$, and

• $\Delta \subset \text{Spec } R$ a finite set.

Then there exists only finitely many smooth projective curves over $F$ of genus $q$ that have good reduction outside $\Delta$.

• The Shafarevich Conjecture in the number field case was confirmed by FALTINGS (1983)

• HOMEWORK. Reformulate the Shafarevich Conjecture in the function field case to resemble the above statement.
NOTATION. IF $C$ IS A SMOOTH PROJECTIVE CURVE OVER $F$ (AN ARBITRARY FIELD), THEN $C(F)$ DENOTES THE $F$-RATIONAL POINTS OF $C$.

MORDELL CONJECTURE (1922).

LET $F$ BE A NUMBER FIELD AND $C$ A SMOOTH PROJECTIVE CURVE OF GENUS $q \geq 2$ DEFINED OVER $F$. THEN $C(F)$ IS FINITE.

- THE MORDELL CONJECTURE WAS CONFIRMED BY FALTINGS (1983)
MORDELL CONJECTURE for function fields.

Let $F$ be a function field (i.e., the function field of a variety over $k$, where $k$ is an algebraically closed field of characteristic 0).

Let $C$ be a smooth projective non-isotrivial curve over $F$ of genus $q \geq 2$.

Then $C(F)$ is finite.

- The Mordell Conjecture for function fields was confirmed by MANIN (1963).

REMARKS.

- The essential case is $\text{tr. deg}_k F = 1$, i.e., $F = K(B)$ where $B$ is a smooth projective curve over $k$.

- $X = \bar{C}$, $f : X \to B$ “closure” of $C \to \text{Spec} F$.

- $P \in C(F) \iff C \supseteq \text{Spec} F \iff X \supset B

F$-RTL POINT $\quad \text{Spec} F$-SECTION $\quad B$-SECTION
Parshin’s Trick. Shafarevich ⇒ Mordell.
(In both function/number field case)

Covering Trick

For all $P \in C(F)$ or equivalently for all sections $X \overset{\sigma_P}{\rightarrow} B$, there exists a finite cover of $X$, $W_P \overset{\pi_P}{\rightarrow} X$ such that

- $\deg \pi_P$ is bounded in terms of $q$
- $W_P \rightarrow B$ is smooth over $B \setminus \Delta$
- $\pi_P$ is ramified exactly over the image of $\sigma_P$
- Genus of the fibers of $W_P \rightarrow B$ is bounded

- The Shafarevich Conjecture implies that there are only finitely many different $W_P$’s.
DE FRANCHIS THEOREM

Let $C$ be a curve of genus at least 2, and $Z$ an arbitrary (fixed) variety.
Then there exists only finitely many dominant rational maps $Z \to C$.

- Combined with the previous observation this implies that there are only finitely many different maps $W_P \to X$.
- Since those maps are ramified exactly over the image of $\sigma_P$, this means that there are only finitely many $\sigma_P$’s, i.e., $C(F)$ is finite.
(H) HYPERBOLICITY

DEFINITION. A complex analytic space $X$ is called **Brody hyperbolic** if $\forall \mathbb{C} \to X$ holomorphic map is constant.

REMARK. $X$ Brody hyperbolic implies that

- $\forall \mathbb{C}^* \to X$ holomorphic map is constant,
- $\forall T \to X$ holomorphic map is constant,

where $T$ is an arbitrary complex torus.

COROLLARY. $X$ is Brody hyperbolic

\[ \uparrow \]
\[
\left\{ \begin{array}{l}
\forall \mathbb{C}^* \to X \text{ holomorphic map is constant,} \\
\forall T \to X \text{ holomorphic map is constant.}
\end{array} \right. 
\]

DEFINITION. An algebraic variety $X$ is called **algebraically hyperbolic** if

- $\forall \mathbb{A}^1 \setminus \{0\} \to X$ regular map is constant,
- $\forall A \to X$ regular map is constant,

where $A$ is an arbitrary abelian variety.
**DEFINITION.** (Reminder)

- A line bundle $\mathcal{L}$ on $X$ is called **ample** if the global sections of $\mathcal{L}^\otimes m$ define a morphism $\phi_\mathcal{L} : X \to \mathbb{P}^N$ such that $X \sim \phi(X)$.

- A line bundle $\mathcal{L}$ on $X$ is called **big** if the global sections of $\mathcal{L}^\otimes m$ define a rational map $\phi_\mathcal{L} : X \dashrightarrow \mathbb{P}^N$ such that $X$ is birational to $\phi(X)$.

- $X$ is of **general type** if $\omega_X$ is big.

**LANG’S CONJECTURE.**

Let $X$ be a projective variety. Then

$X$ is algebraically hyperbolic $\iff$ Every subvariety of $X$ is of general type.
MODULI SPACES OF CURVES. (SKETCHY)

\[ \mathcal{M}_g = \{\text{smooth projective curves of genus } g\} / \sim \]

- **MUMFORD**: For \( g \geq 2 \) there exists a quasi-projective coarse moduli scheme \( \mathcal{M}_g \) for \( \mathcal{M}_g \), i.e., a scheme \( \mathcal{M}_g \) and a natural bijection between \( \mathcal{M}_g \) and the set of closed points of \( \mathcal{M}_g \).

- "**NATURAL**" here means, that all flat families of smooth projective curves of genus \( g \) induces a morphism from the base of the family to \( \mathcal{M}_g \) such that each point is mapped to the isomorphism class of the fibre over the point:

\[ \forall f : X \to B \leadsto \exists \eta_f : B \to \mathcal{M}_g \quad \forall b \in B \quad \eta_f(b) = [X_b]. \]
• **Projective compactification:** \( \mathcal{M}_g \subseteq \overline{\mathcal{M}}_g \)

• \( \overline{\mathcal{M}}_g \) **should also be “natural”**.

**DEFINITION.**

A **reduced projective curve** \( C \) is **stable** if

- \( C \) **has only normal crossing singularities,**
- \( \omega_C \) **is ample.**

\[
\overline{\mathcal{M}}_g = \{ \text{stable curves of genus } g \}/\sim
\]

• **MUMFORD:** For \( g \geq 2 \) there exists a coarse moduli scheme \( \overline{\mathcal{M}}_g \) for \( \overline{\mathcal{M}}_g \), such that \( \mathcal{M}_g \subseteq \overline{\mathcal{M}}_g \) is an open subscheme.

\[
\forall f : X \to B \quad \text{family of stable curves of genus } g \quad \overset{\sim}{\Rightarrow} \quad \exists \eta_f : B \to \overline{\mathcal{M}}_g \quad \forall b \in B \quad \eta_f(b) = [X_b].
\]
BACK TO THE SHAFAREVICH CONJECTURE

(H) \[ 2g - 2 + \#\Delta \leq 0 \quad \Rightarrow \quad \exists f : X \to B \text{ admissible} \]

\[ \sqcup \]

\[ B \setminus \Delta = \begin{cases} 
\mathbb{P}^1 \\
\mathbb{A}^1 \\
\mathbb{A}^1 \setminus \{0\} \\
\text{elliptic curve}
\end{cases} \]

\[ \exists \mathbb{B} \setminus \Delta \to \mathcal{M}_g \text{ non-constant coming from a family} \]

Therefore

\( (H) \iff \exists \mathbb{A}^1 \setminus \{0\} \to \mathcal{M}_g \text{ non-constant coming from a family} \)

DEFINITION. \( \mathcal{M}_g \) is modular hyperbolic if

\( \forall S \to \mathcal{M}_g \text{ regular map coming from a family is constant for } S = \mathbb{A}^1 \setminus \{0\}, \text{ and } S \text{ an abelian variety.} \)

(\( H \) \( \iff \) \( \mathcal{M}_g \) is modular hyperbolic.)
RECALL:

(B) **The admissible families are parametrized by** $T$, a scheme of finite type.

The naturality of moduli spaces implies that a family, $f : X \to B$, corresponds to a map

$\eta_f : B \setminus \Delta \to \mathcal{M}_g$, (and then a map $\bar{\eta}_f : B \to \overline{\mathcal{M}}_g$),

so one can try to find the parameter space $T$ as a subscheme of $\text{Mor}((B, B \setminus \Delta), (\overline{\mathcal{M}}_g, \mathcal{M}_g))$.

Let $\mathcal{L}$ be an ample line bundle on $\overline{\mathcal{M}}_g$. Then

(B) $\iff$ deg $\eta_f^* \mathcal{L}$ is bounded on $B$.

(independent of $f$)
The construction of $\overline{M}_g$ produces natural ample line bundles:

$\exists \lambda_m^{(p)}$ line bundles on $\overline{M}_g$ such that $\forall f : X \to B$ family of stable curves, if $\eta_f : B \to \overline{M}_g$ is the induced map, then

$$\det(f_*\omega_X^{m/B})^p = \eta_f^*\lambda_m^{(p)}$$

It follows from the (independent) works of Kawamata, Kollár and Viehweg, that $\lambda_m^{(p)}$ is ample for $m > 1$.

Therefore

(B) $\iff$ $\deg f_*\omega_X^{m/B}$ is bounded on $B$

for some $m > 1$. 
(WB) **Weak Boundedness**

For an admissible family $f : X \to B$, $\deg f_* \omega^m_{X/B}$ is bounded in terms of $g = g(B), \# \Delta, q = g(X_{\text{gen}}), m$.

Combined with the existence of moduli spaces:

$$\text{(WB)} \Rightarrow \text{(B)}$$

Also, by a simple argument:

$$\text{(WB)} \Rightarrow \text{(H)}$$

(WB) is the “right” statement

**Generalizations.** (Function Field Case)

- Higher dimensional fibers
- Higher dimensional base
- Singular fibers
HIGHER DIMENSIONAL FIBERS.

**First Task:**

GENERALIZE STATEMENT/CONDITIONS.

- $q = g(X_{\text{gen}}) \geq 2 \iff \omega_{X_{\text{gen}}} \text{ IS AMPLE.}$

- FIX $q \iff \text{FIX } h_{\omega_{X_{\text{gen}}}}$

  $h_{\omega_{X_{\text{gen}}}} \text{ IS THE HILBERT POLYNOMIAL OF } X_{\text{gen}}.$

- $B$ IS A SMOOTH CURVE OF GENUS $g$

- $\Delta \subset B$ IS A FINITE SUBSET

- $h$ IS A POLYNOMIAL

**Definition.** An admissible family is a non-isotrivial family, $f : X \to B$ such that $X$ is a smooth projective variety and for all $b \in B \backslash \Delta$, $X_b$ is a smooth projective variety with $\omega_{X_b}$ ample and $h_{\omega_{X_b}} = h$.

Two such families are **equivalent** if they are isomorphic over $B \backslash \Delta$. 
SHAFAREVICH CONJECTURE.

(B) **Boundedness**
Admissible families are parametrized by $\mathcal{T}$, a scheme of finite type.

(R) **Rigidity**
$\dim \mathcal{T} = 0$.

(H) **Hyperbolicity**
$\mathcal{T} \neq \emptyset \Rightarrow 2g - 2 + \#\Delta > 0$.

(WB) **Weak Boundedness**
For an admissible family $f : X \to B$,
$\deg f_{\ast}\omega^m_{X/B}$ is bounded in terms of $g, \#\Delta, h, m$.

**Compactified**
- Moduli spaces $+ (\text{WB}) \Rightarrow (\text{B})$
  (Conjectural)

- $(\text{WB}) \Rightarrow (\text{H})$

- $(\text{WB})$ is again the “right” statement
What about \((R)\)?

**Example.**

- **Let** \(Y \to B\) **be an arbitrary non-isotrivial family of curves of genus \(\geq 2\), and**
- **\(C\) a smooth projective curve of genus \(\geq 2\).**

**Then**

- \(f : X = Y \times C \to B\) **is an admissible family, and**
- **A deformation of \(C\) gives a deformation of \(f\).**

**Hence** \((R)\) **fails.**

**Question:** Under what additional conditions does \((R)\) hold?
(WB) **WEAK BOUNDEDNESS**

**Contributors:**

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**KOVÁCS**

**Theorem.**

Let $f : X \to B$ be an admissible family with $B, \Delta, h$ fixed. Let $\delta = \# \Delta$, $g = g(B)$, and $n = \dim X_{\text{gen}} = \dim X - 1$.

If $f_* \omega^m_{X/B} \neq 0$, then $\exists \ e = e(m, h)$ such that

$$\deg f_* \omega^m_{X/B} \leq m \cdot e \cdot \rk f_* \omega^m_{X/B} \cdot (n(2g - 2 + \delta) + \delta)$$
THEOREM.

A similar statement holds if

- $X_{\text{gen}}$ is of general type,
  \( (\text{Viehweg-Zuo/Kovács}) \)
  or
- $X_{\text{gen}}$ has a good minimal model and $\kappa(X_{\text{gen}}) \geq 0$,
  \( (\text{Viehweg-Zuo}) \)
  or
- $X_{\text{gen}}$ has rational Gorenstein singularities.
  \( (\text{Kovács}) \)
Examples for non-isotrivial families of curves over $\mathbb{P}^1$ with exactly 3 singular fibers.

Example #1.

$$x^3y + ty^3z + (t - 1)xz^3 = 0$$

Example #2 - $\infty$.

$$x^n y + ty^n z + (t - 1)xz^n = 0$$

Example # $(\infty + 1) - (2\cdot \infty)$.

$$y^2 = x^n - nt x + (n - 1)t$$
POSITIVITY OF PUSH-FORWARDS.

DEFINITION. A locally free sheaf, $\mathcal{E}$, is ample if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on $\mathbb{P}(\mathcal{E})$ is ample.

AFTER KAWAMATA, KOLLÁR and VIEHWEG:

THEOREM 1. Let $m > 1.$
If $f_*\omega^m_{X/B} \neq 0$ then $f_*\omega^m_{X/B}$ is ample on $B$.

COROLLARY 1. $\deg f_*\omega^m_{X/B} > 0$.

THEOREM 2. Let $f : X \to B$ be an admissible family, $m > 1$, $r = \text{rk} f_*\omega^m_{X/B}$, and $\mathcal{M}$ a line bundle on $B$ such that

$$\deg \mathcal{M} < \deg f_*\omega^m_{X/B}.$$ 

Then $\exists e = e(m, h)$, such that

$$(f_*\omega^m_{X/B})^{\otimes e \cdot r} \otimes \mathcal{M}^{-1}$$

is ample on $B$. 
COROLLARY 2. (for $\Delta = \emptyset$)

Let $\mathcal{N}$ be a line bundle on $B$ such that

$$\deg \mathcal{N}^{m \cdot e \cdot r} < \deg f_* \omega^m_{X/B}.$$ 

Then $\omega_{X/B} \otimes f^* \mathcal{N}^{-1}$ is ample on $X$.

PROOF. (Very Sketchy)

$$(f_* (\omega^m_{X/B} \otimes f^* \mathcal{N}^{-m})) \otimes e \cdot r \simeq (f_* \omega^m_{X/B}) \otimes e \cdot r \otimes \mathcal{N}^{-m \cdot e \cdot r}$$

- **Theorem 2** $\Rightarrow$ $f_* (\omega^m_{X/B} \otimes f^* \mathcal{N}^{-m})$ is ample on $B$.
- $\omega^m_{X/B} \otimes f^* \mathcal{N}^{-m}|_{X_{\text{gen}}} \simeq \omega^m_{X_{\text{gen}}}$ is ample on $X_{\text{gen}}$. \(\square\)

To prove (WB):

- **Find an $\mathcal{N}$ depending only on $B, \Delta$ such that**

  $\omega_{X/B} \otimes f^* \mathcal{N}^{-1}$ is **not** ample on $X$.

- **Then**

  $$\deg \mathcal{N}^{m \cdot e \cdot r} \nless \deg f_* \omega^m_{X/B},$$

  i.e.,

  $$\deg f_* \omega^m_{X/B} \leq m \cdot e \cdot r \cdot \deg \mathcal{N}.$$
VANISHING THEOREM. Let $f : X \to B$ be a family, ($B$ a smooth projective curve), $n = \dim X_{\text{gen}}$, and $\mathcal{L}$ an ample line bundle on $X$ such that $\mathcal{L} \otimes f^* \omega_B(\Delta)^{-n}$ is also ample. Then

$$H^{n+1}(X, \mathcal{L} \otimes f^* \omega_B) = 0$$

- Corollaries 1 and 2 with $\mathcal{N} = \mathcal{O}_B$

  imply that $\omega_{X/B}$ is ample.

- $\dim X = n + 1$, so $H^{n+1}(X, \omega_{X/B} \otimes f^* \omega_B) \neq 0$

  $\downarrow$

- $\omega_{X/B} \otimes f^* \omega_B(\Delta)^{-n}$ cannot be ample.

  (Otherwise the Vanishing Theorem would apply).

- $\mathcal{N} = f^* \omega_B(\Delta)^n$ will work. $\rightsquigarrow$

  $$\deg f_* \omega_{X/B}^m \leq \deg f^* \omega_B(\Delta)^{n \cdot m \cdot e \cdot r}$$

  $$= m \cdot e \cdot r \cdot \dim X_{\text{gen}} \cdot (2g - 2 + \# \Delta)$$
SMOOTH FIBERS THAT ARE MINIMAL OF GENERAL TYPE/SINGULAR FIBERS.

DEFINITION. A line bundle, $\mathcal{L}$, is semi-ample if $\mathcal{L}^\otimes m$ is generated by global sections for $m \gg 0$.

PRINCIPLE.

SEMI-AMPLE & BIG LINE BUNDLE ON SMOOTH

$\approx$

AMPLE LINE BUNDLE ON SINGULAR

USUAL SITUATION

Goal: A statement for $(X, \mathcal{L})$

$X$ possibly singular, and $\mathcal{L}$ ample on $X$

Trick: Work with $(Y, \mathcal{K})$

$f : Y \to X$ desingularization (i.e., $Y$ smooth)

$\mathcal{K} = f^* \mathcal{L}$ semi-ample & big on $Y$
Upside-Down Situation

Goal: A statement for \((Y, \mathcal{K})\)

\(Y\) smooth, \(\mathcal{K}\) semi-ample & big on \(Y\)

Trick: Work with \((X, \mathcal{L})\)

given by \(f: Y \to X\) induced by \(\mathcal{K}^\otimes m\)

\(\mathcal{K}^\otimes m = f^* \mathcal{L},\ \mathcal{L}\) ample on \(X\)

Plan:

1. First prove the statement for

\[ Y \text{ smooth, } \mathcal{K} \text{ ample.} \]

2. Extend the argument for

\[ X \text{ singular, } \mathcal{L} \text{ ample.} \]

3. Try to “pull back” the result.
**DERIVED CATEGORY \((D(X))\)**

“short exact sequence” \(\leadsto\) “distinguished triangle”

\[ 0 \to A \to B \to C \to 0 \leadsto A \to B \to C \xrightarrow{+1} \]

“cohomology” \(\leadsto\) “hypercohomology”

\[ H^i(X, A) \leadsto \mathbb{H}^i(X, A) \]

\[ R^i \varphi_* A \leadsto R \varphi_* A \]

“sheaf” \(\leadsto\) “complex”

\[ \mathcal{L} \leadsto \cdots \to 0 \to \mathcal{L} \to 0 \to \ldots \]
DU BOIS.

$\exists \Omega^p_X$ FILTERED COMPLEX IN $D(X)$ WITH ALL THE REQURED PROPERTIES.

NAVARRO AZNAR et al.

KODAIRA-AKIZUKI-NAKANO VANISHING HOLDS, i.e.,

LET $\Omega^p_X := G r^p_{\text{filt}} \Omega_X[p]$ AND $\mathcal{L}$ AN AMPLE LINE BUNDLE ON $X$. THEN

$$\mathbb{H}^q(X, \Omega^p_X \otimes \mathcal{L}) = 0$$

FOR $p + q > \text{dim } X$

KOVÁCS.

$\forall p \exists \Omega^p_{X/B}$ COMPLEX IN $D(X)$ WITH ALL THE REQUIRED PROPERTIES.
HIGHER DIMENSIONAL BASES.

- $B$ is a smooth projective variety
- $\Delta \subset B$ is a divisor with normal crossings
- $h$ is a polynomial

DEFINITION. An admissible family is a non-isotrivial family, $f : X \to B$ such that $X$ is a smooth projective variety and for all $b \in B \setminus \Delta$, $X_b$ is a smooth projective variety with $\omega_{X_b}$ ample and $h_{\omega_{X_b}} = h$.

Two such families are equivalent if they are isomorphic over $B \setminus \Delta$. 
There exists a coarse moduli scheme, $\mathcal{M}_h$, parametrizing such $X_b$’s.

\[\exists \eta : B \setminus \Delta \longrightarrow \mathcal{M}_h\]

\[b \longmapsto [X_b]\]

Isotrivial $\iff \eta$ is constant.

**Definition.** $\text{Var } f = \dim(\text{Im } \eta) \leq \dim B$

Interested in the case $\text{Var } f = \dim B$

**Shafarevich Conjecture:**

(B) and (R) are interesting for $\dim B = 1$.

(H) $\leadsto \mathcal{M}_h$ is modular hyperbolic

$\leadsto$ Need:

$\forall A \rightarrow X$ regular map coming from a family is constant, where $A$ is an arbitrary abelian variety.
THEOREM. $\mathcal{M}_h$ IS MODULAR HYPERBOLIC

(H) IMPLIES THAT IF $f : X \to \mathbb{P}^1$ IS AN ADMISSIBLE FAMILY, THEN $\#\Delta > 2$.

WHAT ABOUT $\mathbb{P}^m$?

$f : X \to \mathbb{P}^m$ WITH $\text{Var } f = m$

$\Delta \subset \mathbb{P}^m$ A NORMAL CROSSING DIVISOR

FROM ABOVE: $\deg \Delta > 2$

THEOREM. $\deg \Delta > m + 1$,

I.E.,

$\omega_{\mathbb{P}^n}(\Delta)$ IS AMPLE.

VIEHWEG’S CONJECTURE. IF $f : X \to B$ IS AN ADMISSIBLE FAMILY, THEN $\omega_B(\Delta)$ IS BIG.