Proceedings of Symposia in Pure Mathematics

Young person's guide to moduli of higher dimensional varieties

Sándor J. Kovács

Dedicated to János Kollár and Eckart Viehweg

CONTENTS

1. INTRODUCTION

The ultimate goal of algebraic geometry is to classify all algebraic varieties. This is a formidable task that will not be completed in the foreseeable future, but we can (and should) still work towards this goal.

In this paper I will sketch the main idea of the construction of moduli spaces of higher dimensional varieties. In order to make the length of the paper bearable and still touch on the main issues I will make a number of restrictions that are still rather general and hopefully the majority of the readership will consider them to be reasonable for the purposes of a survey.

The idea of the title is shamelessly taken from Miles Reid. His immortal *Young person's guide* [**Rei87**] is an essential read for a modern algebraic geometer especially for anyone interested in reading the present article. To some extent this is a sequel to that, although given how fundamental Miles Reid's YPG is there are many other topics that would allow for making that claim.

The point is that this article, just as the original YPG, was written with an uninitiated reader in mind. Nevertheless, as the reader progresses through the sections they might feel

Supported in part by NSF Grant DMS-0554697 and the Craig McKibben and Sarah Merner Endowed Professorship in Mathematics.

©0000 (copyright holder)

that more and more background is assumed. This is a necessity brought on by the above mentioned boundaries. Hopefully the article will still achieve the desired result and show a glimpse into this exciting, active and beautiful area of research.

The paper starts with reviewing the general philosophy of classification and how it leads to studying moduli problems. I should note that there are other areas of classification that are equally exciting, active and beautiful. In fact, recent results in the Minimal Model Program [**Sho03, Sho04, HM06, HM07, BCHM06**] have a great positive effect on the central problems of the present article even if I will not have the opportunity to do justice to them and explain their influence in detail.

There are many important results one should mention and I will try to list them all, however I fear that that is an impossible goal to live up to. Therefore, I sincerely apologize for any omission I might commit.

The structure of the paper is the following: After the general overview of classification and moduli theory as part of it, moduli problems are reviewed in more detail followed by a quick look at Hilbert schemes. Then the definition and the most important properties of moduli functors are discussed. Throughout it is kept in mind that each observation leads us to reconsider our objective and along the way we have to realize that we cannot escape working with singular varieties. Because of this, the particular type of singularities that one needs to be able to deal with are reviewed and then finally the moduli functors of higher dimensional canonically polarized varieties are defined in the form that is currently believed to be the "right" one.

Last but not least I should mention that this approach is not necessarily the only one producing the desired moduli space. In fact, Abramovich and Hassett recently have proposed a different construction. As their result has not yet appeared, it is not discussed here. However, the reader is urged to take a look at it as soon as possible as it might shed some new light onto the questions discussed here.

DEFINITIONS AND NOTATION 1.1. Let k be an algebraically closed field of characteristic 0. Unless otherwise stated, all objects will be assumed to be defined over k. A *scheme* will refer to a scheme of finite type over k and unless stated otherwise, a *point* refers to a closed point.

For a morphism $Y \to S$ and another morphism $T \to S$, the symbol Y_T will denote $Y \times_S T$. In particular, for $t \in S$ we write $X_t = f^{-1}(t)$. In addition, if $T = \operatorname{Spec} F$, then Y_T will also be denoted by Y_F.

Let X be a scheme and $\mathscr{F}$ an $\mathscr{O}_X$-module. The m^{th} *reflexive power* of $\mathscr{F}$ is the double dual (or reflexive hull) of the m^{th} tensor power of $\mathscr{F}$:

$$\mathscr{F}^{[m]} := (\mathscr{F}^{\otimes m})^{**}.$$

A *line bundle* on X is an invertible $\mathscr{O}_X$-module. A $\mathbb{Q}$-*line bundle* $\mathscr{L}$ on X is a reflexive $\mathscr{O}_X$-module of rank 1 one of whose reflexive power is a line bundle, i.e., there exists an $m \in \mathbb{N}_+$ such that $\mathscr{L}^{[m]}$ is a line bundle. The smallest such m is called the *index* of $\mathscr{L}$.

For the advanced reader: whenever we mention Weil divisors, assume that X is S_2 and think of a *Weil divisorial sheaf*, that is, a rank 1 reflexive $\mathscr{O}_X$-module which is locally free in codimension 1. For flatness issues consult [**Kol08**, Theorem 2].

For the novice: whenever we mention Weil divisors, assume that X is normal and adopt the definition [**Har77**, p.130].

For a Weil divisor D on X, its associated *Weil divisorial sheaf* is the $\mathscr{O}_X$-module $\mathscr{O}_X(D)$ defined on the open set $U \subseteq X$ by the formula

$$\Gamma(U, \mathscr{O}_X(D)) = \left\{ \frac{a}{b} \;\middle|\; a, b \in \Gamma(U, \mathscr{O}_X), b \text{ is not a zero divisor anywhere on } U, \text{ and } D + \operatorname{div}(a) - \operatorname{div}(b) \geq 0 \right\}$$

and made into a sheaf by the natural restriction maps.

A Weil divisor D on X is a *Cartier divisor*, if its associated Weil divisorial sheaf, $\mathscr{O}_X(D)$ is a line bundle. If the associated Weil divisorial sheaf, $\mathscr{O}_X(D)$ is a $\mathbb{Q}$-line bundle, then D is a $\mathbb{Q}$-*Cartier divisor*. The latter is equivalent to the property that there exists an $m \in \mathbb{N}_+$ such that mD is a Cartier divisor.

The symbol $\sim$ stands for *linear* and $\equiv$ for *numerical equivalence* of divisors.

Let $\mathscr{L}$ be a line bundle on a scheme X. It is said to be *generated by global sections* if for every point $x \in X$ there exists a global section $\sigma_x \in H^0(X, \mathscr{L})$ such that the germ σ_x generates the stalk $\mathscr{L}_x$ as an $\mathscr{O}_X$-module. If $\mathscr{L}$ is generated by global sections, then the global sections define a morphism

$$\phi_{\mathscr{L}} \colon X \to \mathbb{P}^N = \mathbb{P}\left(H^0(X, \mathscr{L})\right).$$

$\mathscr{L}$ is called *semi-ample* if $\mathscr{L}^m$ is generated by global sections for $m \gg 0$. $\mathscr{L}$ is called *ample* if it is semi-ample and $\phi_{\mathscr{L}^m}$ is an embedding for $m \gg 0$. A line bundle $\mathscr{L}$ on X is called *big* if the global sections of $\mathscr{L}^m$ define a rational map $\phi_{\mathscr{L}^m} \colon X \dashrightarrow \mathbb{P}^N$ such that X is birational to $\phi_{\mathscr{L}^m}(X)$ for $m \gg 0$. Note that in this case $\mathscr{L}^m$ is not necessarily generated by global sections, so $\phi_{\mathscr{L}^m}$ is not necessarily defined everywhere. I will leave it for the reader the make the obvious adaptation of these notions for the case of $\mathbb{Q}$-line bundles.

The *canonical divisor* of a scheme X is denoted by K_X and the *canonical sheaf* of X is denoted by ω_X.

A smooth projective variety X is of *general type* if ω_X is big. It is easy to see that this condition is invariant under birational equivalence between smooth projective varieties. An arbitrary projective variety is of *general type* if so is a desingularization of it.

A projective variety is *canonically polarized* if ω_X is ample. Notice that if a smooth projective variety is canonically polarized, then it is of general type.

ACKNOWLEDGEMENT. I would like to thank Christopher Hacon, Max Lieblich and Zsolt Patakfalvi for useful discussions and for pointing out errors and misprints in an early draft of this manuscript.

2. CLASSIFICATION

2.A. *Q&A*

As mentioned in the introduction, our ultimate goal is to classify all algebraic varieties.

We will approach the classification problem through birational geometry, that is, our plan for classification can be summarized as follows.

PLAN 2.1.

(2.1.1) Choose a "nice" representative from every birational class.
(2.1.2) Give a well-defined way of obtaining this nice representative. (I.e., given an arbitrary variety, provide an algorithm to find this representative).
(2.1.3) Classify the representatives.

As we try to execute this plan, we will face many questions that will guide our journey. The plan itself raises the first question.

QUESTION 2.2. What should we consider "nice"?

Before answering that, let us see how we might approach (2.1.3).

First, one looks for discrete invariants, preferably such that are invariant under deformation. For instance, dimension, degree, genus, etc.

Once many discrete invariants are found, consider a class of varieties that share the same discrete invariants. One expects that these will be parametrized by continuous invariants, or as Riemann called them, *moduli*.

EXAMPLE 2.3 *Plane Curves.* Let $X \subseteq \mathbb{P}^2$ be a projective plane curve. The discrete invariant we need is the degree. Let us suppose that we fix that and we are only considering curves of degree d. It is easy to see that plane curves of degree d are parametrized by $\mathbb{P}^{\frac{d(d+3)}{2}}$. The continuous parameters are the coefficients of the defining equation of the curve.

Still, before answering Question 2.2, let us ask another one:

QUESTION 2.4. What discrete invariants should we consider?

The first one seems obvious: *dimension*. The next that comes to mind is perhaps *degree*, but this leads to another issue: Degree depends on the embedding and so do many other invariants. So the next question to answer is:

QUESTION 2.5. Is there a natural way to embed our varieties?

Embeddings correspond to sets of generating global sections of very ample line bundles, or if we forget about automorphisms of the ambient space for a moment, then to very ample line bundles.

This brings up another question:

QUESTION 2.6. How do we find ample line bundles on a variety?

The problem is that our variety may not be given with an embedding, or even if it is given as a subvariety of a projective space, that given embedding may not be the natural one (if there is such).

If a variety X, even if it is smooth, is given without additional information, it is really hard to find non-trivial ample line bundles, or for that matter, any non-trivial line bundles. There is practically only one that we can expect to find, the *canonical line bundle*[1], i.e., ω_X, the determinant of the cotangent bundle Ω_X. (Of course there is also the determinant of the tangent bundle as well, but that is simply the inverse of the canonical bundle and so doesn't give an independent line bundle. Obviously, if we find one line bundle, we will have all of its powers, positive and negative included.)

So we could ask ourselves:

QUESTION 2.7. Is the canonical bundle ample?

Most likely the readers know the answer to this one: No, not necessarily. So perhaps the better question is,

QUESTION 2.8. How likely is it that the canonical bundle is ample?

Let us consider the case of curves. In this case, the right answer (2.1.1) is very simple (not its proof however!): In each birational class there exists exactly one smooth projective

[1]"The canonical bundle is not called canonical for nothing" – Joe Harris

curve. It is known and well documented how one obtains this representative (one possibility is explained in [**Har77**, §I.6]) so (2.1.2) is also covered.

With respect to (2.1.3) and more particularly Question 2.8, there are three different types of behavior:

- $X = \mathbb{P}^1$: $\omega_X \simeq \mathscr{O}_{\mathbb{P}^1}(-2)$ is anti-ample,
- X an elliptic curve: $\omega_X \simeq \mathscr{O}_X$ is trivial.
- X any other smooth projective curve: ω_X is ample.

This suggests that we may expect that most birational classes will contain a member with an ample canonical bundle.

Let us take a moment to examine the third case, that is, when X is a smooth projective curve and ω_X is ample. In this case $\omega_X^{\otimes 3}$ is always very ample and thus X can be embedded by the global sections of that line bundle:

$$H^0(X, \omega_X^{\otimes 3}) : X \hookrightarrow \mathbb{P}^N.$$

The obvious discrete invariant to consider now is the degree of this embedding, i.e.,

$$d = 3 \deg \omega_X = 6\chi(X, \omega_X).$$

Using Kodaira vanishing, Riemann-Roch and Serre duality we can compute N:

$$N + 1 = h^0(X, \omega_X^{\otimes 3}) = \chi(X, \omega_X^{\otimes 3}) = d + \chi(X, \mathscr{O}_X) = 5\chi(X, \omega_X).$$

Therefore, we are interested in classifying smooth projective curves of degree 6χ in $\mathbb{P}^{5\chi-1}$, where $\chi = \chi(X, \omega_X)$. In this case the discrete invariant we needed, that is, in addition to $\dim = 1$, was the degree of the third pluricanonical embedding. However, in order to make this work in higher dimensions we will need more invariants to get a reasonable moduli space. The right invariant will be the Hilbert polynomial of ω_X, which in the above example contains equivalent information as the dimension and the degree combined:

$$h_X(m) := h^0(X, \omega_X^{\otimes m}) = 2\chi m - \chi.$$

REMARK 2.9. The reader has probably noticed that I am going to great lengths to avoid using the *genus* of the curves involved. The reason behind this is that knowing the genus is equivalent to knowing $\chi(X, \omega_X)$, or even $h_X(m)$, and the latter is the invariant that generalizes well to higher dimensions. So why not start getting used to it?

Now we can make our first attempt to decide what we would like to call "nice":

DEFINITION 2.10. Let X be *nice* if X is smooth, projective and ω_X is ample.

ISSUE 2.11. This is not going to fulfill all of our requirements because there are varieties that are not birational to a nice variety as defined in (2.10). For instance, let X be a smooth minimal surface of general type that contains a (-2)-curve (a smooth proper rational curve with self-intersection (-2)). Then ω_X is not ample. Since X is not rational or ruled, it is the only minimal surface in its birational class and hence cannot be birational to a surface with an ample canonical bundle.

This is however not a huge setback. It only means that the above definition of "nice" is not the right one yet.

2.B. *Curves*

The first invariant we want to fix is the dimension and so let us start to get more serious by considering the case of $\dim = 1$ systematically. We have seen that our second important invariant is the Hilbert polynomial of ω_X. Fixing that is equivalent to fixing $\chi = \chi(X, \omega_X)$. We have the following cases for nice varieties of dimension 1 (cf. (2.10)):

2.12. TRICHOTOMY.

- $\chi < 0$: $X \simeq \mathbb{P}^1$,
- $\chi = 0$: X is an elliptic curve,
- $\chi > 0$: X is a curve with ω_X ample.

In this case, we are able to answer our previous questions. The classes with $\chi \leq 0$ are reasonably understood from a classification point of view.

For $\chi < 0$ we have only one smooth projective curve, while for $\chi = 0$ we have the elliptic curves which are classified by their j-invariant. There are of course plenty of things to still understand about elliptic curves, but those belong to a different study.

For $\chi > 0$ the definition of "nice" in (2.10) works well as there is a unique nice curve in each birational class. The moduli part of the classification was first accomplished by Mumford. There are many excellent sources on moduli of curves. Perhaps the two most frequently used ones are [**MFK94**] and [**HMo98**].

To study higher dimensional varieties we need some preparations.

2.C. *Fano varieties*

ISSUES 2.13. In the cases of surfaces and higher dimensional varieties we encounter two new issues that we will have to deal with:

(2.13.1) There will be a lot more varieties that are not birational to a "nice" variety according to the current definition of "nice".

(2.13.2) There are smooth projective varieties that are birational to each other.

EXAMPLE 2.14. As we have seen in the case of the projective line, we similarly have that for $X = \mathbb{P}^n$ ω_X is not ample and this is true for any X' birational to $\mathbb{P}^n$.

EXAMPLE 2.15. More generally, we have the class of smooth *Fano varieties*, i.e., smooth projective varieties X with ω_X^{-1} ample, that are not birational to nice varieties.

EXAMPLE 2.16. Finally, in this series of examples, a variety X admitting a *Fano fibration*, i.e., a flat morphism $X \to T$ such that X_t is a Fano variety for general $t \in T$, is also not birational to a nice variety.

CONCLUSION 2.17. We will have to deal with Fano varieties differently. However, they, too, have a natural ample line bundle; namely ω_X^{-1}, so all is not lost.

2.D. *Kodaira dimension*

In order to mirror the trichotomy of the curve case, we need to introduce another invariant. This is very similar and very close to χ in the curve case.

Let X be a smooth projective variety and consider the rational map induced by a set of generators of $H^0(X, \omega_X^{\otimes m})$:

$$\phi_m : X \dashrightarrow \mathbb{P}^N.$$

It is relatively easy to see that for $m \gg 0$, the birational class of the image $\phi_m(X)$ is independent of m.

DEFINITION 2.18. The *Kodaira dimension* of X is denoted by $\kappa(X)$ and defined as

$$\kappa(X) := \dim \phi_m(X) \text{ for } m \gg 0.$$

DEFINITION 2.19. X is of *general type*, if $\kappa(X) = \dim X$. In particular, if ω_X is ample, then X is of general type.

EXAMPLE 2.20. $\kappa(\mathbb{P}^n) = \dim \emptyset < 0$. In fact, for any Fano variety X, $\kappa(X) < 0$.

EXAMPLE 2.21. For curves we have (again) three cases:

- $\kappa < 0:$ $X \simeq \mathbb{P}^1$ ($\chi < 0$),
- $\kappa = 0:$ X is an elliptic curve ($\chi = 0$),
- $\kappa = 1:$ X is a curve with ω_X ample ($\chi > 0$).

EXAMPLE 2.22. Let X be a uniruled variety. Then X does not admit any global pluricanonical forms and hence $\kappa(X) < 0$. It is conjectured that this characterizes uniruled varieties.

For more on the classification of uniruled varieties see [**Mor87**, §11] and [**Kol96**].

2.E. *Fibrations*

As in the case of curves Kodaira dimension gives us a powerful tool to separate varieties into classes with differently behaving canonical classes in arbitrary dimensions. In fact, we will see that even though there are more possibilities for the possible values of the Kodaira dimension as the actual dimension grows, there will still be only three important classes to consider.

Our next step is to adopt the following principle.

PRINCIPLE 2.23. Let $\varphi : X \to Y$ be a fibration between smooth projective varieties, i.e., a dominant morphism with connected fibers. Motivated by our rough birational classification point of view, we will rest once we can classify Y and the general fiber of φ. Of course, this leaves many questions unanswered, but then again, I never promised to answer all questions.

2.24. MORI FIBRATION [**KM98**, §2]. Let X be a smooth projective variety such that $\kappa(X) < 0$. Then it is conjectured that there exists a birational model $X^\natural$ for X and a fibration $\varphi^\natural : X^\natural \to Y^\natural$ such that $Y^\natural$ is a smooth projective variety with $\dim Y^\natural < \dim X^\natural$ and $F^\natural$ is a Fano variety, where $F^\natural$ is the generic geometric fiber of $\varphi^\natural$.

This is known for $\dim X \leq 3$ by [**Mor88**] (cf. [**Mor82, Rei83, Kaw84, Kol84, Sho85**]). In fact, here I am skipping the mentioning of the Minimal Model Program, which is a beautiful and very deep theory. In particular, one could (or perhaps should?) discuss extremal contractions and flips. However, since the focus of this article is on moduli theory I will leave this topic for the reader to discover. A good place to start is [**Kol87**] and the standard reference is [**KM98**].

It follows that iterating (2.24) will exhibit X to be birational to a tower of Fano fibrations over a base that is either itself a Fano variety or has non-negative Kodaira dimension. In the latter case we appeal to (2.25) and in general apply Principle 2.23.

2.25. IITAKA FIBRATION [**Iit82**, §11.6], [**Mor87**, 2.4]. Let X be a smooth projective variety with $\kappa(X) \geq 0$. Then there exists a birational model $X^\flat$ for X and a fibration $\varphi^\flat : X^\flat \to Y^\flat$ such that $Y^\flat$ is a smooth projective variety with $\dim Y^\flat = \kappa(X)$ and $\kappa(F^\flat) = 0$, where $F^\flat$ is the generic geometric fiber of $\varphi^\flat$. Furthermore, the birational class of $Y^\flat$ is uniquely determined by these properties.

Appealing to Principle 2.23 we conclude that we may restrict our attention to three types of varieties (cf. (2.12)) that are the building blocks of all varieties:

2.26. TRICHOTOMY.

- Fano varieties. These include $\mathbb{P}^n$. (For $\dim X = 1 : \chi < 0$).
- Varieties with Kodaira dimension 0. These include Abelian and Calabi-Yau varieties. (For $\dim X = 1 : \chi = 0$).
- Varieties with maximal Kodaira dimension, i.e., varieties of general type. These include varieties with an ample canonical bundle. (For $\dim X = 1 : \chi > 0$).

In this article we will concentrate on the third case: varieties of general type. Similarly to the case of curves, this is indeed the "general" case.

It has been a long standing conjecture and only proven recently (cf. [**HM06, HM07, BCHM06**]) that every variety of general type is birational to a canonically polarized variety, its *canonical model*, i.e., a variety with an ample canonical bundle. The only trouble is that unfortunately this canonical model may be singular. On the other hand, in order to gain a good understanding of moduli, one needs to study degenerations as well, so we will be forced to consider singular varieties in our moduli problem anyway. Fortunately, the singularities forced by the canonical model are not worse than the ones we must allow in order to have a compact moduli space.

Now we are at a point that we can form a reasonable plan that was called for in (2.1).

PLAN 2.27. The discrete invariant we need to fix is the Hilbert polynomial of the canonical bundle of the canonical model. Then we plan to do the following:

(2.27.1) Let "nice" be defined (for now) as in Definition 2.10. We will later replace "smooth" with something else.

(2.27.2) Starting with an arbitrary variety X, perform the following procedure to obtain a "nice" model.

- apply Nagata's Theorem [**Nag62**] to get a proper closure of X: $\hat{X}$,
- apply Chow's Lemma [**Har77**, Ex.II.4.10] to obtain a projectivization of $\hat{X}$ if necessary: $\bar{X}$
- apply Hironaka's Theorem [**Hir64**] to get a resolution of singularities of $\bar{X}$: $\widetilde{X}$
- apply the Minimal Model Program [**KM98**] and Mori fibrations (2.24) to restrict to the case $\kappa(X) \geq 0$,
- apply Iitaka fibrations (2.25) to restrict to the case $\kappa(X) = \dim X$,
- form the canonical model: $\operatorname{Proj} \bigoplus_{m=0}^{\infty} H^0(X, \omega_X^{\otimes m})$ [**HM06, HM07, BCHM06**].

(2.27.3) Classify the canonical models.

Now the "only" thing left is the last step, classifying the canonical models. In other words, we need to construct a moduli space for them.

3. Moduli problems

3.A. *Representing functors*

Let Sets denote the category of sets and Cat an arbitrary category. Further let

$$\mathcal{F} : \mathsf{Cat} \to \mathsf{Sets}$$

be a contravariant functor. Recall that $\mathcal{F}$ is *representable* if there is an object $\mathfrak{M} \in \operatorname{Ob} \mathsf{Cat}$ such that $\mathcal{F} \simeq \operatorname{Hom}_{\mathsf{Cat}}(\_\_, \mathfrak{M})$. If such an $\mathfrak{M}$ exists, it is called a *universal object* or a *fine moduli space* for $\mathcal{F}$.

3.B. *Moduli functors*

NOTATION 3.1. Let $f : X \to B$ be a morphism and $\mathscr{K}$ and $\mathscr{L}$ two line bundles on X. Then

$$\mathscr{K} \sim_B \mathscr{L}$$

will mean that there exists a line bundle $\mathscr{N}$ on B such that $\mathscr{K} \simeq \mathscr{L} \otimes f^*\mathscr{N}$.

REMARK 3.1.1. If B and X_b for all $b \in B$ are integral of finite type, f is flat and projective, then $\mathscr{K} \sim_B \mathscr{L}$ is equivalent to the condition that $\mathscr{K}\big|_{X_b} \simeq \mathscr{L}\big|_{X_b}$ for all $b \in B$ [**Har77**, Ex. III.12.6].

DEFINITION 3.2. Let S be a scheme and Sch_S the category of S-schemes. Let

$$\mathcal{MP} : \mathsf{Sch}_S \to \mathsf{Sets}$$

be the *moduli functor of polarized proper schemes over* S:

(3.2.1) For an object $B \in \operatorname{Ob}\mathsf{Sch}_S$,

$$\mathcal{MP}(B) := \big\{(f : X \to B, \mathscr{L}) \mid f \text{ is a flat, projective morphism and } \mathscr{L} \text{ is an } f\text{-ample line bundle on } X\big\}\big/\simeq$$

where "$\simeq$" is defined as follows: $(f_1 : X_1 \to B, \mathscr{L}_1) \simeq (f_2 : X_2 \to B, \mathscr{L}_2)$ if and only if there exists a B-isomorphism $\phi : X_1/B \xrightarrow{\simeq} X_2/B$ such that $\mathscr{L}_1 \sim_B \phi^*\mathscr{L}_2$.

(3.2.2) For a morphism $\alpha \in \operatorname{Hom}_{\mathsf{Sch}_S}(A, B)$,

$$\mathcal{MP}(\alpha) := (\_\_) \times_B \alpha,$$

i.e.,

$$\begin{array}{cccc} \mathcal{MP}(\alpha) : & \mathcal{MP}(B) & \longrightarrow & \mathcal{MP}(A) \\ & (f : X \to B, \mathscr{L}) & \longmapsto & (f_A : X_A \to A, \mathscr{L}_A). \end{array}$$

REMARK 3.2.1. This definition has the disadvantage that it does not satisfy faithfully flat descent cf. [**BLR90**, 6.1]. This is essentially caused by similar problems with the naive definition of the relative Picard functor [**Gro62a**, 232] or [**BLR90**, 8.1]. This problem may be dealt with by appropriate sheafification of $\mathcal{MP}$. The notion of canonical polarization below also provides a natural solution in many cases. For details see [**Vie95**, §1].

Considering our current aim, we leave these worries behind for the rest of the article, but warn the reader that they should be addressed.

In any case, unfortunately, the functor $\mathcal{MP}$ is too big to handle, so we need to study some of its subfunctors that are more reasonable. In the context of the previous section, $\mathcal{MP}$ does not take into account any discrete invariants. If we follow our plan and start by fixing certain discrete invariants, then we are led to study natural subfunctors of $\mathcal{MP}$.

DEFINITION 3.3. Let k be an algebraically closed field of characteristic 0 and Sch_k the category of k-schemes. Let $h \in \mathbb{Q}[t]$ and $\mathcal{M}_h^{\text{smooth}} : \mathsf{Sch}_k \to \mathsf{Sets}$ the following functor:

(3.3.1) For an object $B \in \operatorname{Ob}\mathsf{Sch}_k$,

$$\mathcal{M}_h^{\text{smooth}}(B) := \big\{ f : X \to B \mid f \text{ is a smooth projective family such that } \forall b \in B,\ \omega_{X_b} \text{ is ample and } \chi(X_b, \omega_{X_b}^{\otimes m}) = h(m)\big\}\big/\simeq$$

where "$\simeq$" is defined as follows: $(f_1 : X_1 \to B) \simeq (f_2 : X_2 \to B)$ if and only if there exists a B-isomorphism $\phi : X_1/B \xrightarrow{\simeq} X_2/B$.

(3.3.2) For a morphism $\alpha \in \operatorname{Hom}_{\mathsf{Sch}_k}(A, B)$,

$$\mathcal{M}_h^{\text{smooth}}(\alpha) := (\_\_) \times_B \alpha.$$

REMARK 3.4. For $S = \operatorname{Spec} k$, $\mathcal{M}_h^{\text{smooth}}$ is a subfunctor of $\mathcal{MP}$.

EXAMPLE 3.5.

$$\mathcal{M}_h^{\text{smooth}}(\operatorname{Spec} k) = \big\{X | X \text{ is a smooth projective variety with } \omega_X \text{ ample and } \chi(\omega_X^{\otimes m}) = h(m)\big\}.$$

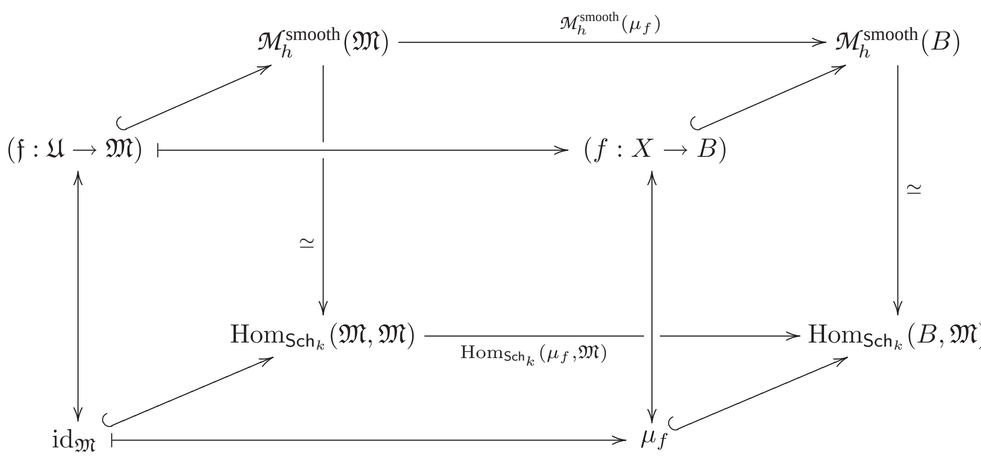
QUESTION 3.6. So, what would it mean exactly that $\mathcal{M}_h^{\text{smooth}}$ is representable?

OBSERVATIONS 3.7. Suppose(!) that $\mathcal{M}_h^{\text{smooth}}$ is representable, i.e., assume (but do not believe) that there exists an $\mathfrak{M} \in \operatorname{Ob} \mathsf{Sch}_k$ such that $\mathcal{M}_h^{\text{smooth}} \simeq \operatorname{Hom}_{\mathsf{Sch}_k}(\_\_, \mathfrak{M})$. Then one makes the following observations.

(3.7.1) First let $B = \operatorname{Spec} k$. Then $\mathcal{M}_h^{\text{smooth}}(\operatorname{Spec} k) \simeq \operatorname{Hom}_{\mathsf{Sch}_k}(\operatorname{Spec} k, \mathfrak{M}) = \mathfrak{M}(k)$, the set of k-points of $\mathfrak{M}$. In other words, the set of closed points of $\mathfrak{M}$ are in one-to-one correspondence with smooth projective varieties X with ω_X ample and $\chi(X, \omega_X^{\otimes m}) = h(m)$. For such a variety X its corresponding point in $\mathfrak{M}(k)$ will be denoted by $[X]$.

(3.7.2) Next let $B = \mathfrak{M}$. Then one obtains that $\mathcal{M}_h^{\text{smooth}}(\mathfrak{M}) \simeq \operatorname{Hom}_{\mathsf{Sch}_k}(\mathfrak{M}, \mathfrak{M})$. Now let $(\mathfrak{f} : \mathfrak{U} \to \mathfrak{M}) \in \mathcal{M}_h^{\text{smooth}}(\mathfrak{M})$ be the element corresponding to the identity $\mathrm{id}_{\mathfrak{M}} \in \operatorname{Hom}_{\mathsf{Sch}_k}(\mathfrak{M}, \mathfrak{M})$. For a closed point $x : \operatorname{Spec} k \to \mathfrak{M}$ one has by functoriality that $x = [\mathfrak{U}_x]$, where $\mathfrak{U}_x = \mathfrak{U} \times_{\mathfrak{M}} x$. Therefore, $(\mathfrak{f} : \mathfrak{U} \to \mathfrak{M})$ is a *tautological family*.

(3.7.3) Finally, let B be arbitrary. Then by the definition of representability one has that $\mathcal{M}_h^{\text{smooth}}(B) \simeq \operatorname{Hom}_{\mathsf{Sch}_k}(B, \mathfrak{M})$, i.e., every family $(f : X \to B) \in \mathcal{M}_h^{\text{smooth}}(B)$ corresponds in a one-to-one manner to a morphism $\mu_f : B \to \mathfrak{M}$. Applying the functor $\mathcal{M}_h^{\text{smooth}}(\_\_) \simeq \operatorname{Hom}_{sch_k}(\_\_, \mathfrak{M})$ to μ_f leads to the following:



By (3.3.2) this implies that $(f : X \to B) \simeq (\mathfrak{f} \times_{\mathfrak{M}} \mu_f : \mathfrak{U} \times_{\mathfrak{M}} B \to B)$, so $(\mathfrak{f} : \mathfrak{U} \to \mathfrak{M})$ is actually a *universal family*.

(3.7.4) Let $(f : X \to B) \in \mathcal{M}_h^{\text{smooth}}(B)$ be a non-trivial family, all of whose members are isomorphic. For an example of such a family see (3.9) below. Let F denote the variety to which the fibers of f are isomorphic, i.e., $F \simeq X_b$ for all $b \in B$. Then by (3.7.2) $\mu_f(b) = [F] \in \mathfrak{M}$ for all $b \in B$. However, for this f then $(\mathfrak{f} \times_{\mathfrak{M}} \mu_f : \mathfrak{U} \times_{\mathfrak{M}} B \to B) \simeq (B \times F \to B)$, which is a contradiction.

CONCLUSION 3.8. Our original assumption led to a contradiction, so we have to conclude that $\mathcal{M}_h^{\text{smooth}}$ is *not* representable.

EXAMPLE 3.9. Let B and C be two smooth projective curves admitting non-trivial double covers $\widetilde{B} \to B \simeq \widetilde{B} \big/ \mathbb{Z}_2$ and $\widetilde{C} \to C \simeq \widetilde{C} \big/ \mathbb{Z}_2$. Consider the diagonal $\mathbb{Z}_2$-action on $\widetilde{B} \times \widetilde{C}$: $\sigma(b, c) := (\sigma(b), \sigma(c))$ for $\sigma \in \mathbb{Z}_2$ and let $X = \widetilde{B} \times \widetilde{C} \big/ \mathbb{Z}_2$ and $f : X \to B$ the induced morphism $[(b, c) \sim (\sigma(b), \sigma(c))] \mapsto [b \sim \sigma(b)]$. It is easy to see that the fibers of

f are all isomorphic to $\widetilde{C}$, but $X \not\simeq B \times \widetilde{C}$. Similar examples may be constructed as soon as there exists a non-trivial representation $\pi_1(B) \to \operatorname{Aut} C$.

3.C. *Coarse moduli spaces*

Since we cannot expect our moduli functors to be representable, we have to make do with something weaker.

DEFINITION 3.10. A functor $\mathcal{F} : \mathsf{Sch}_k \to \mathsf{Sets}$ is *coarsely representable* if there exists an $\mathfrak{M} \in \operatorname{Ob} \mathsf{Sch}_k$ and a natural transformation

$$\eta : \mathcal{F} \to \operatorname{Hom}_{\mathsf{Sch}_k}(\underline{\quad}, \mathfrak{M})$$

such that

(3.10.1) $\eta_{\operatorname{Spec} k} : \mathcal{F}(\operatorname{Spec} k) \xrightarrow{\simeq} \operatorname{Hom}_{\mathsf{Sch}_k}(\operatorname{Spec} k, \mathfrak{M}) = \mathfrak{M}(k)$ is an isomorphism, and

(3.10.2) given an arbitrary $\mathfrak{N} \in \operatorname{Ob} \mathsf{Sch}_k$ and a natural transformation

$$\zeta : \mathcal{F} \to \operatorname{Hom}_{\mathsf{Sch}_k}(\underline{\quad}, \mathfrak{N})$$

there exists a unique natural tranformation

$$\nu : \operatorname{Hom}_{\mathsf{Sch}_k}(\underline{\quad}, \mathfrak{M}) \to \operatorname{Hom}_{\mathsf{Sch}_k}(\underline{\quad}, \mathfrak{N})$$

such that

$$\nu \circ \eta = \zeta.$$

If such an $\mathfrak{M}$ exists, it is called a *coarse moduli space* for $\mathcal{F}$.

Let us now reconsider the question and observations we made in (3.6) and (3.7) with regard to this new definition.

QUESTION 3.11. What would it mean that $\mathcal{M}_h^{\text{smooth}}$ is coarsely representable?

OBSERVATIONS 3.12. Assume that there exists an $\mathfrak{M}_h \in \operatorname{Ob} \mathsf{Sch}_k$ satisfying the conditions listed in Definition 3.10 above, i.e., assume that $\mathcal{M}_h^{\text{smooth}}$ is coarsely represented by $\mathfrak{M}_h$. Then one makes the following observations.

(3.12.1) Let $B = \operatorname{Spec} k$. Then by (3.10.1) we still have $\mathcal{M}_h^{\text{smooth}}(\operatorname{Spec} k) \simeq \mathfrak{M}_h(k)$, the set of k-points of $\mathfrak{M}_h$. In other words, the set of closed points of $\mathfrak{M}_h$ are in one-to-one correspondence with smooth projective varieties X with ω_X ample and $\chi(X, \omega_X^{\otimes m}) = h(m)$. For such a variety X its corresponding point in $\mathfrak{M}_h(k)$ will be denoted by $[X]$.

(3.12.2) Let $B = \mathfrak{M}_h$. Then there exists a map

$$\eta_{\mathfrak{M}_h} : \mathcal{M}_h^{\text{smooth}}(\mathfrak{M}_h) \to \operatorname{Hom}_{\mathsf{Sch}_k}(\mathfrak{M}_h, \mathfrak{M}_h),$$

but there is no guarantee that $\mathrm{id}_{\mathfrak{M}_h} \in \operatorname{Hom}_{\mathsf{Sch}_k}(\mathfrak{M}_h, \mathfrak{M}_h)$ is in the image of $\eta_{\mathfrak{M}_h}$, and hence a tautological family $(\mathfrak{f}_h : \mathfrak{U}_h \to \mathfrak{M}_h)$ may not exist.

(3.12.3) Let B be arbitrary. Then there exists a map $\eta_B : \mathcal{M}_h^{\text{smooth}}(B) \to \operatorname{Hom}_{\mathsf{Sch}_k}(B, \mathfrak{M})$, i.e., every family $(f : X \to B) \in \mathcal{M}_h^{\text{smooth}}(B)$ corresponds to a morphism $\mu_f : B \to \mathfrak{M}$, which still has some useful properties. Since it is given by a natural transformation, we have that for all $b \in B$,

$$\mu_f(b) = [X_b].$$

Applying the functors $\mathcal{M}_h^{\text{smooth}}(\underline{\quad})$ and $\operatorname{Hom}_{sch_k}(\underline{\quad}, \mathfrak{M})$ to μ_f leads to the following:

We have observed in (3.12.2) that there may not be a tautological family

$$(\mathfrak{f}_h : \mathfrak{U}_h \to \mathfrak{M}_h) \in \mathcal{M}_h^{\text{smooth}}(\mathfrak{M}_h)$$

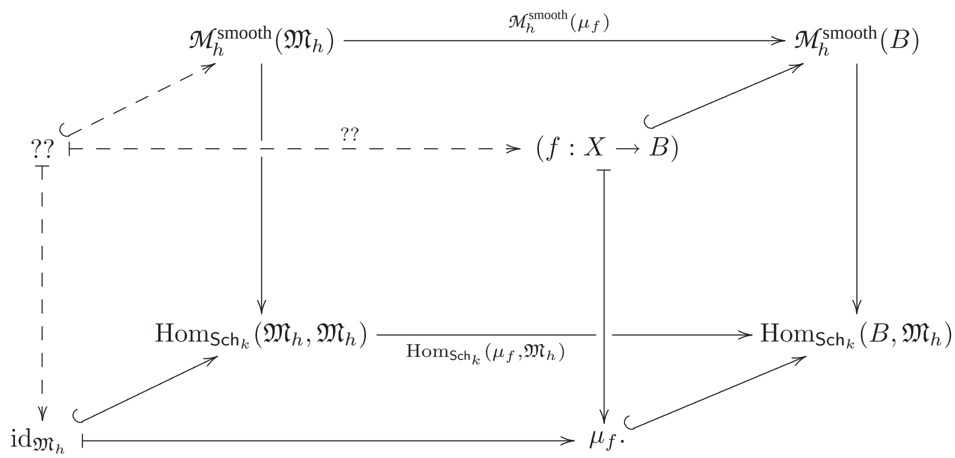


FIGURE 3.1.

that maps to $\mathrm{id}_{\mathfrak{M}_h}$. However, even if such a family existed, we could not conclude that it maps to $(f : X \to B)$ via $\mathcal{M}_h^{\text{smooth}}(\mu_f)$, because the vertical arrows in Figure 3.1 are not necessarily one-to-one. In other words, even if we find a tautological family, it is not necessarily a *universal family*.

(3.12.4) Finally, let $(f : X \to B) \in \mathcal{M}_h^{\text{smooth}}(B)$ be a non-trivial family all of whose members are isomorphic. Let F denote the fiber of f, i.e., $F \simeq X_b$ for all $b \in B$. Then by (3.12.3) $\mu_f(b) = [F] \in \mathfrak{M}$ for all $b \in B$. However, this does not lead to a contradiction now (see the remark at the end of (3.12.3)).

4. HILBERT SCHEMES

We saw in the previous section that moduli functors are usually not representable. In this section we will see an example for a representable functor.

Let $g : Y \to Z$ be a projective morphism, $\mathscr{L}$ a g-ample line bundle on Y and $\mathscr{F}$ a coherent g-flat sheaf on Y. Then for $m \gg 0$ one has that $g_*(\mathscr{F} \otimes \mathscr{L}^{\otimes m})$ is locally free and $R^i g_*(\mathscr{F} \otimes \mathscr{L}^{\otimes m}) = 0$ for $i > 0$. By the Riemann-Roch theorem there exists a polynomial $h_{Y/Z,\mathscr{F},\mathscr{L}}$ such that

$$h_{Y/Z,\mathscr{F},\mathscr{L}}(m) = \operatorname{rk} g_*(\mathscr{F} \otimes \mathscr{L}^{\otimes m}).$$

We will call this the *Hilbert polynomial* of g with respect to $\mathscr{F}$ and $\mathscr{L}$. If there is no danger of confusion then we will use the notation $h_{\mathscr{L}} := h_{Y/Z,\mathscr{O}_Y,\mathscr{L}}$ and will call $h_{\mathscr{L}}$ the Hilbert polynomial of $\mathscr{L}$.

Let S be a scheme and $X \in \operatorname{Ob} \mathsf{Sch}_S$. We define the *Hilbert functor*,

$$\mathcal{H}ilb(X/S) : \mathsf{Sch}_S \to \mathsf{Sets}$$

as follows. For a $Z \in \operatorname{Ob} \mathsf{Sch}_S$,

$$\begin{aligned} \mathcal{H}ilb(X/S)(Z) := & \big\{ V \mid V \subseteq X \times_S Z \text{ flat and proper subscheme over } Z \big\} \\ \simeq & \big\{ \mathscr{F} \mid \mathscr{F} \simeq \mathscr{O}_{X \times_S Z} \big/ \mathscr{J} \text{ flat with proper support over } Z \big\}, \end{aligned}$$

and for a $\phi \in \operatorname{Hom}_{\mathsf{Sch}_S}(Z, Y)$,

$$\begin{aligned} \mathcal{H}ilb(X/S)(\phi) : \mathcal{H}ilb(X/S)(Y) &\to \mathcal{H}ilb(X/S)(Z) \\ V &\mapsto V \times_Y Z \subseteq (X \times_S Y) \times_Y Z \simeq X \times_S Z \end{aligned}$$

If $\mathscr{L}$ is a relatively ample line bundle on X/S and $p \in \mathbb{Q}[z]$, then we define

$$\mathcal{H}ilb_p(X/S)(Z) := \left\{ \mathscr{F} \in \mathcal{H}ilb(X/S)(Z) \mid h_{X_Z/Z, \mathscr{F}, \mathscr{L}_Z} = p \right\}.$$

Notice that if Z is connected, then

$$\mathcal{H}ilb(X/S)(Z) = \bigcup_p \mathcal{H}ilb_p(X/S)(Z).$$

Theorem 4.1 [**Gro62b, Gro95**] [**Kol96**, I.1.4]. *Let X/S be a projective scheme, $\mathscr{L}$ a relatively ample line bundle on X/S and p a polynomial. Then the functor $\mathcal{H}ilb_p(X/S)$ is represented by a projective S-scheme* $\operatorname{Hilb}_p(X/S)$, *called the* Hilbert scheme *of X/S with respect to p.*

REMARK 4.2. Similarly to (3.7.2-3), one observes that by the definition of representability, $\operatorname{id}_{\operatorname{Hilb}_p(X/S)} \in \operatorname{Hom}_{\mathsf{Sch}_S}(\operatorname{Hilb}_p(X/S), \operatorname{Hilb}_p(X/S))$ corresponds to a *universal object*, or *universal family*, $\operatorname{Univ}_p(X/S) \in \mathcal{H}ilb_p(X/S)(\operatorname{Hilb}_p(X/S))$. By the definition of $\mathcal{H}ilb_p(X/S)$, one sees that $\operatorname{Univ}_p(X/S) \subseteq X \times_S \operatorname{Hilb}_p(X/S)$ is flat and proper over $\mathcal{H}ilb_p(X/S)$ with Hilbert polynomial p.

DEFINITION 4.3. We define the *Hilbert scheme* of X/S as follows:

$$\operatorname{Hilb}(X/S) := \coprod_p \operatorname{Hilb}_p(X/S).$$

5. INTRODUCTION TO THE CONSTRUCTION OF THE MODULI SPACE

5.A. *Boundedness*

There are several properties a moduli functor needs to satisfy in order for it to admit a (coarse) moduli space. We will discuss some of these in more detail. The first one is *boundedness*.

DEFINITION 5.1. Let $\mathcal{F}$ be a subfunctor of $\mathcal{MP}$. Then we say that $\mathcal{F}$ is *bounded* if there exists a scheme of finite type T and a family $(\pi : U \to T, \mathscr{L}) \in \mathcal{MP}(T)$ with the following property:

For any $(\sigma : X \to B, \mathscr{N}) \in \mathcal{F}(B)$ there exists an étale cover $\cup B_i \to B$ and morphisms $\nu_i : B_i \to T$ such that for all i,

$$(\sigma : X \to B, \mathscr{N})\big|_{B_i} \simeq \nu_i^*(\pi : U \to T, \mathscr{L}).$$

In this case we say that $(\pi : U \to T, \mathscr{L})$ is a *bounding family* for $\mathcal{F}$.

If in addition $(\pi : U \to T, \mathscr{L}) \in \mathcal{F}(T)$, then $(\pi : U \to T, \mathscr{L})$ is called a *locally versal family* for $\mathcal{F}$.

REMARK 5.1.1. When using canonical polarizations, then one may restrict to open covers in the definition. See [**Vie95**, 1.15] and [**Kol94**].

The first major general theorem about boundedness is Matsusaka's Big Theorem. Here we only cite a special case. For the more general statement please refer to the original article.

Theorem 5.2 [**Mat72**]. *Fix a polynomial $h \in \mathbb{Q}[t]$. Then $\mathcal{M}_h^{smooth}$ is bounded.*

In fact, in order to prove boundedness of $\mathcal{M}_h^{\text{smooth}}$, it is enough to prove the following:

Theorem 5.3. *Fix $h \in \mathbb{Q}[t]$. Then there exists an integer $m > 1$ such that $\omega_X^{\otimes m}$ is very ample for all $X \in \mathscr{M}_h^{smooth}(\operatorname{Spec} k)$.*

DEFINITION 5.4. Let the smallest integer m satisfying the condition in (5.3) be denoted by $m(h)$.

Assume that we know that (5.3) holds. Then by the Kodaira Vanishing Theorem $\omega_X^{\otimes m}$ has no higher cohomology for all $X \in \mathscr{M}_h^{\text{smooth}}(\operatorname{Spec} k)$, and so

$$h^0(X, \omega_X^{\otimes m}) = \chi(X, \omega_X^{\otimes m}) = h(m).$$

Let $N = h(m) - 1$. Then for all $X \in \mathscr{M}_h^{\text{smooth}}(\operatorname{Spec} k)$ the m-th pluricanonical map

$$H^0(X, \omega_X^{\otimes m}) : X \hookrightarrow \mathbb{P}^N$$

is an embedding. Now let $T = \operatorname{Hilb}_h(\mathbb{P}^N/k)$, $U = \operatorname{Univ}_h(\mathbb{P}^N/k)$ and consider the two projections $\pi_1 : \mathbb{P}^N \times \operatorname{Hilb}_h(\mathbb{P}^N/k) \to \mathbb{P}^N$ and $\pi_2 : \mathbb{P}^N \times \operatorname{Hilb}_h(\mathbb{P}^N/k) \to \operatorname{Hilb}_h(\mathbb{P}^N/k)$. Let $\pi = \pi_2\big|_U : U \to T$ and $\mathscr{L} = \pi_1^*\mathscr{O}_{\mathbb{P}^N/k}(1)\big|_U$. Then $(\pi : U \to T, \mathscr{L})$ gives a bounding family for $\mathscr{M}_h^{\text{smooth}}$. Therefore (5.3) implies (5.2).

REMARK 5.5. We will see later that it is necessary to allow singular objects in our moduli functors. This will lead to many difficulties, among them the unfortunate fact that Matsusaka's Big Theorem will not be strong enough for our purposes.

5.B. *Plan*

The success of using the Hilbert scheme in order to obtain boundedness might make one believe that the Hilbert scheme itself might work as a moduli space. However, unfortunately this is not the case as the points of $\operatorname{Hilb}_h(\mathbb{P}^N/k)(k)$ also parametrize subschemes that are not in the moduli functor $\mathscr{M}_h^{\text{smooth}}$. For example, they maybe horribly singular and the polarizing line bundle is not necessarily the canonical bundle.

The next guess maybe taking the locus of Hilbert points that corresponds to such subvarieties of $\mathbb{P}^N$ that are in $\mathscr{M}_h^{\text{smooth}}(\operatorname{Spec} k)$, i.e., smooth with canonical polarization. This is a much better guess, but still not perfect. There are two fundamental problems. First, it is not at all clear that this locus is a subscheme of $\operatorname{Hilb}_h(\mathbb{P}^N/k)$, or even if its support is a subscheme, then whether there is a natural scheme structure that is compatible with the functor $\mathscr{M}_h^{\text{smooth}}$. This actually turns out to be a difficult technical problem referred to as *local closedness* and we will return to it later. The second problem is that a single object of $\mathscr{M}_h^{\text{smooth}}(\operatorname{Spec} k)$ will appear several times in $\operatorname{Hilb}_h(\mathbb{P}^N/k)$; any subscheme of $\mathbb{P}^N$ appears as a potentially different subscheme after acting with an element of $\operatorname{Aut}(\mathbb{P}^N)$, but in the moduli functor we only want a single copy of each isomorphism class.

The way to proceed is "obvious". Assume that we can solve the local closedness problem and indeed we can find a subscheme that consists of exactly the points that belong to $\mathscr{M}_h^{\text{smooth}}(\operatorname{Spec} k)$. (Actually we need to worry about more than that, but let's not get all gloomy just yet). Then we get a natural action of $\operatorname{Aut}(\mathbb{P}^N)$ on this subscheme and taking the quotient by $\operatorname{Aut}(\mathbb{P}^N)$ should yield our desired moduli space. I should mention that taking this quotient is not entirely obvious, but fortunately possible [**Vie91, Kol97a, KeM97**].

5.C. *Local closedness*

We have already observed that in order to carry out the the plan laid out in 5.B we need to identify the set of Hilbert points corresponding to the moduli functor and find a (natural) scheme structure on this set. The technical condition to allow doing this is the following.

DEFINITION 5.6. A subfunctor $\mathcal{F} \subseteq \mathcal{MP}$ is *locally closed* (resp. *open, closed*) if the following condition holds: For every $(f : X \to B, \mathscr{L}) \in \mathcal{MP}(B)$ there exists a locally closed (resp. open, closed) subscheme $\iota : B' \hookrightarrow B$ such that if $\tau : T \to B$ is any morphism then

$$(f_T : X_T \to T, \mathscr{L}_T) \in \mathcal{F}(T) \quad \Leftrightarrow \quad \tau \text{ factors through } \iota \quad \begin{array}{ccc} T & \xrightarrow{\tau} & B \\ {\scriptstyle\exists}\,\downarrow & \nearrow_{\iota} & \\ B' & & \end{array} .$$

OBSERVATION 5.7. There are two main ingredients of proving that $\mathcal{M}_h^{\text{smooth}}$ is locally closed. Let $m = m(h)$ as defined in (5.4). Suppose that $(f : X \to B, \mathscr{L}) \in \mathcal{MP}(B)$. Note that in the construction of the moduli space this $\mathscr{L}$ comes from $\mathscr{O}_{\mathbb{P}^N}(1)$ where $N = h(m) - 1$. Now one needs to prove that:

(5.7.1) the set $\{b \in B \mid X_b \in \mathcal{M}_h^{\text{smooth}}(\operatorname{Spec} k)\}$ is a locally closed subset of B, and

(5.7.2) the condition $\mathscr{L}\big|_{X_b} \simeq \omega_{X_b}^m$ is locally closed on B.

At this point these conditions are not too hard to satisfy. To prove (5.7.1) one observes that being smooth is open, being projective is assumed. The canonical bundle, ω_{X_b} being ample is open, but this we actually do not even need as it will follow from (5.7.2). The requirement on the Hilbert polynomial will also follow from (5.7.2). In turn, (5.7.2) follows from the following lemma.

Lemma 5.8. [**Vie95**, 1.19] *Let $f : X \to B$ be a flat projective morphism and $\mathscr{K}$ and $\mathscr{L}$ two line bundles on X. Assume that $h^0(X_b, \mathscr{O}_{X_b}) = 1$ for all $b \in B$. Then there exists a locally closed subscheme $\iota : B' \hookrightarrow B$ such that if $\tau : T \to B$ is any morphism then*

$$\mathscr{K}_T \sim_T \mathscr{L}_T \quad \Leftrightarrow \quad \tau \textit{ factors through } \iota \quad \begin{array}{ccc} T & \xrightarrow{\tau} & B \\ {\scriptstyle\exists}\,\downarrow & \nearrow_{\iota} & \\ B' & & \end{array} .$$

PROOF. By replacing $\mathscr{L}$ by $\mathscr{L} \otimes \mathscr{K}^{-1}$ we may assume that $\mathscr{K} \simeq \mathscr{O}_X$. Observe that if $\mathscr{L}\big|_{X_b}$ is generated by a single section, then it gives an isomorphism

$$\mathscr{O}_{X_b} \xrightarrow{\simeq} \mathscr{L}\big|_{X_b}.$$

Consider

$$B''_{\text{red}} := \left\{ b \in B \mid h^0(X_b, \mathscr{L}\big|_{X_b}) \neq 0 \right\} = \operatorname{supp}(f_*\mathscr{L}).$$

This is closed by semi-continuity [**Har77**, III.12.8]. So far this is only a subset and we need to define a (natural) scheme structure. However, that is a local problem, so we may assume that B is affine. By cohomology-and-base-change [**Mum70**, §5] there exists a bounded complex of locally free sheaves

$$\mathscr{E}^0 \xrightarrow{\delta^0} \mathscr{E}^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{r-1}} \mathscr{E}^n$$

such that for any morphism $\tau : T \to B$,

$$R^i(f_T)_*\mathscr{L}_T \simeq H^i(\mathscr{E}_T^\bullet).$$

In particular,

$$(f_T)_*\mathscr{L}_T \simeq \ker[\delta_T^0 : \mathscr{E}_T^0 \to \mathscr{E}_T^1].$$

By definition, $B''_{\text{red}} = \operatorname{supp} \ker \delta^0$. Now define the ideal sheaf $\mathscr{I} \lhd \mathscr{O}_B$ as follows:

- If $B''_{\text{red}} = B_{\text{red}}$ in a neighbourhood of a point $b \in B$, then let $\mathscr{I} = 0$ near b.
- Otherwise write $\mathscr{E}^i = \oplus^{r_i} \mathscr{O}_B$ near $b \in B$. Since we are not in the previous case, we must have $r_1 \geq r_0$. Now let $\mathscr{I}$ be generated by the $r_0 \times r_0$ minors of

$$\delta^0 : \bigoplus^{r_0} \mathscr{O}_B \to \bigoplus^{r_1} \mathscr{O}_B.$$

Let the scheme structure on B''_{red} be defined by this ideal sheaf, i.e., let B'' the scheme with support B''_{red} and structure sheaf $\mathscr{O}_{B''} := \mathscr{O}_B \big/ \mathscr{I}$.

Now if $\tau : T \to B$ is such that $\mathscr{L}_T \sim_T \mathscr{O}_T$, then $(f_T)_* \mathscr{L}_T$ is a line bundle on T and if $\ker \delta^0_T$ contains a line bundle, then the image of $\tau^* \mathscr{I}$ in $\mathscr{O}_T$ has to be zero. In other words, τ factors through $B'' \hookrightarrow B$.

In the final step we construct B' as an open subscheme of B''. By our previous observation we may assume that $B'' = B$, in particular, $f_* \mathscr{L} \neq 0$ on a dense open set. Let

$$B''' := \left\{ b \in B \mid h^0(X_b, \mathscr{L}\big|_{X_b}) > 1 \right\}.$$

Again, by semi-continuity, B''' is closed. Next let $B^\circ = B \setminus B'''$, the largest open (possibly empty) subscheme of B with $f_* \mathscr{L}\big|_{B^\circ}$ invertible and let $Z \subseteq X$ be the support of $\operatorname{coker}[f^* f_* \mathscr{L} \to \mathscr{L}]$. Finally let

$$B' := (B \setminus f(Z)) \cap B^\circ \subseteq B.$$

It is easy to check that this B' satisfies the required condition. □

5.D. *Separatedness*

Boundedness and local closedness allows us to identify a subscheme of an appropriate Hilbert scheme consisting of the Hilbert points of the schemes in our moduli problem. This subscheme has a group action induced by the automorphism group of the ambient projective space. This already allows the construction of the moduli space as an algebraic space by taking the quotient by this group action. However, in order to effectively use this moduli space we hope that it will satisfy certain basic properties. Perhaps the most basic one is separatedness.

DEFINITION 5.9. A subfunctor $\mathcal{F} \subseteq \mathcal{MP}$ is *separated* if the following condition holds. Let R be a DVR and $T = \operatorname{Spec} R$ with general point $t_g \hookrightarrow T$ and $(X_i \to T, \mathscr{L}_i) \in \mathcal{F}(T)$ two families for $i = 1, 2$. Then any isomorphism $\alpha_g : ((X_1)_{t_g}, (\mathscr{L}_1)_{t_g}) \to ((X_2)_{t_g}, (\mathscr{L}_2)_{t_g})$ extends to an isomorphism $\alpha : X_1/T \to X_2/T$.

Separatedness of a moduli functor is a non-trivial property. Without further restrictions it will not hold as shown by the following examples.

EXAMPLE 5.10. Let $Z = \mathbb{P}^1 \times \mathbb{A}^1$ with coordinates $([x : y], t)$. Let the projections to the factors be $\pi_1 : Z \to \mathbb{P}^1$ and $\pi_2 : Z \to \mathbb{A}^1$. Further let $\mathscr{L} = \pi_1^* \mathscr{O}_{\mathbb{P}^1}(1)$, $R = k[t]_{(t)}$ (a DVR) and consider the base change to $T = \operatorname{Spec} R$. With the notation $f = (\pi_1)_T$, one has that $(f : Z_T \to T, \mathscr{L}_T) \in \mathcal{MP}(T)$. Now let $\alpha : Z_T \dashrightarrow Z_T$ be the map induced by $([x : y], t) \mapsto ([tx : y], t)$. This is an isomorphism over the general point of T, but is not even dominant over the special point.

REMARK 5.10.1. The main problem here comes from the fact that $\operatorname{Aut} \mathbb{P}^1$ is not discrete. The good news is that by a theorem of Matsusaka and Mumford [**MM64**] this problem can only occur if the fiber over the closed point is ruled.

EXAMPLE 5.11. Let Y be a smooth projective variety of dimension at least 2, $Z = Y \times \mathbb{A}^1$, $\pi : Z \to \mathbb{A}^1$ the projection to the second factor and $C_1, C_2 \subseteq Z$ two sections, i.e., curves in Z that are isomorphic to $\mathbb{A}^1$ via π and such that C_1 and C_2 intersect in a single point, P, transversally. Assume for simplicity that $\pi(P) = 0 \in \mathbb{A}^1$.

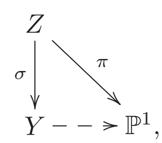
Let Z_1 be the variety obtained by first blowing up C_1 and then the proper transform of C_2. Similarly, let Z_2 be the variety obtained by first blowing up C_2 and then the proper transform of C_1.

Let $U = \{t \in \mathbb{A}^1 \mid t \neq 0\}$. Then $(Z_1)_U$ and $(Z_2)_U$ may be identified, but the isomorphism between $(Z_1)_U$ and $(Z_2)_U$ induced by this identification does not extend over $t = 0 \in \mathbb{A}^1$.

To make this example more interesting, assume that $\mathbb{A}^1$ admits an embedding into $\operatorname{Aut} Y$, i.e., Y admits a one-parameter group of automorphisms. Denote these automorphisms by α_t for $t \in \mathbb{A}^1$ and assume that $\alpha_0 = \mathrm{id}_Y$ and $(C_2)_t = \alpha_t((C_1)_t)$. In this case the automorphisms α_t induce an isomorphism between Z_1 and Z_2 including the fiber over $t = 0$. Observe that the restriction of this isomorphism is the identity on the fiber over $t = 0$, but different from the identity over any $t \in U$.

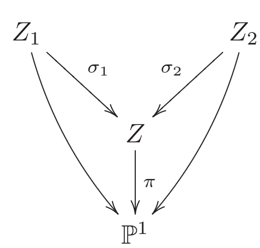
In this example Z_1 and Z_2 are isomorphic, but not all isomorphisms over U extend to an isomorphism over the entire $\mathbb{A}^1$.

EXAMPLE 5.12. This example is based on an example of Atiyah. Let $\iota : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^n$ be an arbitrary embedding and $Y \subseteq \mathbb{P}^{n+1}$ the projectivized cone over $\iota(\mathbb{P}^1 \times \mathbb{P}^1) \subseteq \mathbb{P}^n$ with vertex P. Let $L \subseteq \mathbb{P}^{n+1}$ be a general linear subspace of codimension 2. Notice that this implies that $P \notin L$. Consider the projection from L to a line, $\mathbb{P}^{n+1} \setminus L \to \mathbb{P}^1$. After blowing up L this extends to a morphism $\pi_L : Bl_L\mathbb{P}^{n+1} \to \mathbb{P}^1$. Let Z be the proper transform of Y on $Bl_L\mathbb{P}^{n+1}$ and $\pi = \pi_L\big|_Z$. Then one has the following diagram:



where π is flat projective with connected fibers and smooth general fiber and σ is the blowing up of $L \cap Y \subseteq Y$, hence birational and an isomorphism near $P \in Y$. Let $\widetilde{P} = \sigma^{-1}(P)$.

Next let C_1 and C_2 be the images via ι of two general lines corresponding to the two different rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ and S_1 and S_2 their respective preimages on Y. Note that by construction C_1 and C_2 are disjoint from L. For the rest of this example anywhere i appears, it is meant to apply for both $i = 1, 2$. Let $\widetilde{S}_i = \sigma^{-1}S_i \subseteq Z$ and $\sigma_i : Z_i = Bl_{\widetilde{S}_i}Z \to Z$ the blow-up of Z along $\widetilde{S}_i$. Observe, that $\widetilde{S}_i \subseteq Z$ is a divisor and since Z is smooth away from $\widetilde{P}$, this implies that Z_i is isomorphic to Z away from $\widetilde{P}$, in particular $(Z_1)_{\mathbb{P}^1\setminus\{Q\}} \simeq (Z_2)_{\mathbb{P}^1\setminus\{Q\}}$ where $Q = \pi(\widetilde{P}) \in \mathbb{P}^1$. On the other hand, it is easy to check that $\sigma_i^{-1}(\widetilde{P}) \simeq \mathbb{P}^1$ is equal to the whole fiber of the blow-up $Bl_{S_i}\mathbb{P}^{n+1} \to \mathbb{P}^{n+1}$. (Since $P \notin L$, this computation can be done on Y).



Next, we wish to determine the fiber $Z_Q = \pi^{-1}(Q)$. Let $L_P = \langle L, P\rangle$, the linear span of L and P. Observe that $L_P \simeq \mathbb{P}^n$ is a general hyperplane through P in $\mathbb{P}^{n+1}$. Hence $L_P \cap Y$ is the cone over a general hyperplane section of a smooth projective surface, i.e., over a smooth projective curve. We conclude that Z_Q is the blow-up of this cone at its intersection with L which consists of finitely many points that are disjoint from P as well as from S_1 and S_2. Therefore, $(Z_i)_Q$ is a further blow-up along the proper transform of S_i.

Next suppose that ι is the standard quadratic embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$. In this case, $S_i \simeq \mathbb{P}^2$ are linear subspaces of $\mathbb{P}^4$ contained in Y, Z_Q is the blow-up at finitely many smooth points of a quadric cone and $(Z_i)_Q$ is the blow-up of Z_Q along one of the rays of the quadratic cone that miss the centers of the other blow-ups. Therefore $(Z_1)_Q \simeq (Z_2)_Q$, but this isomorphism does not extend to an isomorphism of Z_1 and Z_2.

This leads to a moduli space that is non-separated in a quite peculiar way: the point corresponding to the class of $(Z_1)_Q \simeq (Z_2)_Q$ completes the curve $\mathbb{P}^1 \setminus \{Q\}$ corresponding to the family $(Z_1)_{\mathbb{P}^1\setminus\{Q\}} \simeq (Z_2)_{\mathbb{P}^1\setminus\{Q\}}$ in two different way.

The result is the following curve. Let $Q \in \mathbb{P}^1$ a point. take two copies of this $\mathbb{P}^1$ and glue them together along $\mathbb{P}^1 \setminus \{Q\}$. Then glue the two copies of Q together but by a separate gluing. Therefore there are two separate ways to get to Q from the rest of the $\mathbb{P}^1$.

As we mentioned before, a result of Matsusaka and Mumford tells us that in our case these pathologies do not occur.

Theorem 5.13 [**MM64**, Theorem 1]. *Let R be a DVR and $T = \operatorname{Spec} R$ with closed point $t_s \in T$. Further let X_1/T be a proper T-scheme and X_2/T a reduced T-scheme of finite type such that $(X_2)_{t_s}$ is not ruled. Assume that X_1 and X_2 are birational. Then so are $(X_1)_{t_s}$ and $(X_2)_{t_s}$.*

We may use this result to prove separatedness of $\mathscr{M}_h^{\text{smooth}}$, but first we need an auxiliary theorem.

Theorem 5.14. *Let S be a scheme and $f_i : X_i \to S$ two proper S-schemes, $\mathscr{L}_i$ relatively ample line bundles on X_i/S and $j_i : U_i \hookrightarrow X_i$ open immersions with complement $Z_i = X_i \setminus U_i$ for $i = 1, 2$. Assume that*

(5.14.1) there exists an S-isomorphism $\alpha : U_1/S \xrightarrow{\simeq} U_2/S$ such that $\alpha^*\mathscr{L}_2 \simeq \mathscr{L}_1$, and

(5.14.2) $\operatorname{depth}_{Z_i} X_i \geq 2$ for $i = 1, 2$ (this is satisfied if for example X_i is normal and $\operatorname{codim}(Z_i, X_i) \geq 2$).

Then α extends to X_1 to give an isomorphism $X_1/S \simeq X_2/S$.

PROOF. Once α has an extension to X_1, it is unique, so the question is local on S and thus we may assume that it is affine. Let m be large enough that $\mathscr{L}_i^m$ is relatively very ample. First observe that (5.14.2) implies that $(j_i)_*\left(\mathscr{L}_i^m\big|_{U_i}\right) \simeq \mathscr{L}_i^m$ for $i = 1, 2$. Therefore

$$\left(f_i\big|_{U_i}\right)_*\left(\mathscr{L}_i^m\big|_{U_i}\right) \simeq (f_i)_*\,\mathscr{L}_i^m\big|_{f_i(U_i)}.$$

This implies that $\left(f_i\big|_{U_i}\right)_*\left(\mathscr{L}_i^m\big|_{U_i}\right)$ is coherent. Let $\mathscr{A}$ be an ample line bundle on S, then $\left(f_i\big|_{U_i}\right)_*\left(\mathscr{L}_i^m\big|_{U_i}\right) \otimes \mathscr{A}^r$ is generated by global sections for $r \gg 0$. As $\mathscr{L}_i^m$ is relatively very ample, this gives a surjection

$$f_i^*\left(\bigoplus^r \mathscr{A}^{-1}\right) \twoheadrightarrow \mathscr{L}_i^m,$$

which in turn induces an embedding $\phi_i : X_i \hookrightarrow \mathbb{P}^{r-1}_S$.

As the isomorphism α in (5.14.1) gives an isomorphism between the sheaves

$$\left(f_1\big|_{U_1}\right)_* \left(\mathscr{L}_1^m\big|_{U_1}\right) \simeq \left(f_2\big|_{U_2}\right)_* \left(\mathscr{L}_2^m\big|_{U_2}\right),$$

we may choose the generators defining the ϕ_i to be compatible with this isomorphism and conclude that $\phi_1\big|_{U_1} = \phi_2\big|_{U_2} \circ \alpha$.

Since U_i is dense in X_i, we obtain that $\phi_i(X_i)$ is the Zariski closure of $\phi_i(U_i)$ and hence we have

$$X_1 \xrightarrow[\phi_1]{\simeq} \phi_1(X_1) = \overline{\phi_1(U_1)} = \overline{\phi_2(U_2)} = \phi_2(X_2) \xleftarrow[\phi_2]{\simeq} X_2.$$

Clearly, this isomorphism restricted to U_1 coincides with α and so the statement is proved. □

Corollary 5.15. *$\mathcal{M}_h^{smooth}$ is separated.*

PROOF. Let R be a DVR and $T = \operatorname{Spec} R$ with general point $t_g \hookrightarrow T$. Further let $(X_i \to T, \mathscr{L}_i) \in \mathcal{M}_h^{\text{smooth}}(T)$ two families for $i = 1, 2$ and assume that there exists an isomorphism $\alpha_g : ((X_1)_{t_g}, (\mathscr{L}_1)_{t_g}) \to ((X_2)_{t_g}, (\mathscr{L}_2)_{t_g})$.

Let $U_i \subseteq X_i$ be the largest open sets for $i = 1, 2$ such that there exists an extension of α_g that gives an isomorphism $\alpha : U_1 \to U_2$.

Now observe that as α_g induces a birational equivalence between X_1 and X_2, by (5.13) it extends to a birational equivalence between $(X_1)_{t_s}$ and $(X_2)_{t_s}$ and hence these contain isomorphic open sets, which are then contained in U_1 and U_2 respectively. Therefore the conditions of (5.14) are satisfied and so α extends to an isomorphism $X_1/T \simeq X_2/T$. □

With this we have covered the most important properties of moduli functors, boundedness, local closedness, and separatedness. These properties, along with weak positivity and weak stability (see [**Vie95**, 7.16] for details), allows one to prove the following:

Theorem 5.16. [**Kol90**] [**Vie95**, 1.11] *There exists a quasi-projective coarse moduli scheme for $\mathcal{M}_h^{smooth}$.*

For more precise statements see [**Kol90**] and [**Vie95**, §1.2]. Other relevant sources are [**Kol85, KSB88, Vie89, Vie90a, Vie90b, Vie06**].

At first sight it may seem that with the construction of this moduli scheme we have accomplished the plan laid down in (2.27). However, it is not entirely so. We should definitely consider this an answer if we only care about smooth canonically polarized varieties. After all, the moduli space does "classify" these objects. On the other hand, a canonical model produced by Plan 2.27 may not be smooth. So if we care about those cases, too, we have to work with singular varieties as well.

6. SINGULARITIES

In this section we will see that in order to accomplish our goal of classifying all canonical models (cf. (2.27)), we will have to allow our objects to have singularities.

There is another reason to do this. Even if we were only interested in smooth objects their degenerations provide important information. In other words, it is always useful to find complete moduli problems, i.e., extend our moduli functor so it would admit a complete (and preferably projective) coarse moduli space. This also leads to having to consider singular varieties.

However, we will have to be careful to limit the kind of singularities that we allow in order to be able to handle them. In other words, we have to revisit our definition of "nice" and we will change its definition according to our findings.

6.A. *Canonical singularities*

For an excellent introduction to this topic the reader is urged to take a thorough look at Miles Reid's original young person's guide [**Rei87**]. Here I will only touch on the subject.

Let X be a minimal surface of general type that contains a (-2)-curve (a smooth rational curve with self-intersection -2). For an example of such a surface consider the following.

EXAMPLE 6.1. $\widetilde{X} = (x^5 + y^5 + z^5 + w^5 = 0) \subseteq \mathbb{P}^3$ with the $\mathbb{Z}_2$-action that interchanges $x \leftrightarrow y$ and $z \leftrightarrow w$. This action has five fixed points, $[1 : 1 : -\varepsilon^i : -\varepsilon^i]$ for $i = 1, \dots, 5$ where ε is a primitive 5^{th} root of unity. Consequently the quotient $\widetilde{X}\big/\mathbb{Z}_2$ has five singular points, each a simple double point of type A_1. Let $X \to \widetilde{X}\big/\mathbb{Z}_2$ be the minimal resolution of singularities. Then X contains five (-2)-curves, the exceptional divisors over the singularities.

Let us return to the general case, that is, X is a minimal surface of general type that contains a (-2)-curve, $C \subseteq X$. As $C \simeq \mathbb{P}^1$, and X is smooth, the adjunction formula gives us that $K_X \cdot C = 0$. Therefore K_X is not ample.

On the other hand, since X is a minimal surface of general type, it follows that K_X is semi-ample, that is, some multiple of it is base-point free. In other words, there exists a morphism,

$$|mK_X| : X \to X_{\text{can}} \subseteq \mathbb{P}(H^0(X, \mathscr{O}_X(mK_X))).$$

This follows from various results, for example Bombieri's classification of pluri-canonical maps, but perhaps the simplest proof is provided by Miles Reid [**Rei97**, E.3].

It is then relatively easy to see that this morphism onto its image is independent of m. This constant image is called the *canonical model* of X, let us denote it by X_{can}.

The good news is that the canonical divisor of X_{can} is indeed ample, but the trouble with it is that it is singular. However, the singularity is not too bad, so we still have a good chance to do this. In fact, the singularities that can occur on the canonical model of a surface of general type belong to a much studied class. This class goes by several names; they are called *du Val singularities*, or *rational double points*, or *Gorenstein, canonical singularities*. For more on these singularities, refer to [**Dur79**], [**Rei87**].

6.B. *Normal crossings*

These singularities already appear in the construction of the moduli space of stable curves (or if the reader prefers, the construction of a compactificaton of the moduli space of smooth projective curves). As we want to understand degenerations of our preferred families, we have to allow normal crossings.

A *normal crossing* singularity is one that is locally analytically (or formally) isomorphic to the intersection of coordinate hyperplanes in a linear space. In other words, it is a singularity locally analytically defined as $(x_1 x_2 \cdots x_r = 0) \subseteq \mathbb{A}^n$ for some $r \leq n$. In particular, as opposed to the curve case, for surfaces it allows for triple intersections. However, triple intersections may be "resolved": Let $X = (xyz = 0) \subseteq \mathbb{A}^3$. Blow up the origin $O \in \mathbb{A}^3$, $\sigma : Bl_O\mathbb{A}^3 \to \mathbb{A}^3$ and consider the proper transform of X, $\sigma : \widetilde{X} \to X$. Observe that $\widetilde{X}$ has only double normal crossings.

Another important point to remember about normal crossings is that they are *not* normal. In particular they do not belong to the previous category. For some interesting and perhaps surprising examples of surfaces with normal crossings see [**Kol07**].

6.C. *Pinch points*

Another non-normal singularity that can occur as the limit of smooth varieties is the pinch point. It is locally analytically defined as $(x_1^2 = x_2 x_3^2) \subseteq \mathbb{A}^n$. This singularity is a double normal crossing away from the pinch point. Its normalization is smooth, but blowing up the pinch point does not make it any better. (Try it for yourself!)

6.D. *Cones*

Let $C \subseteq \mathbb{P}^2$ be a curve of degree d and $X \subseteq \mathbb{P}^3$ the projectivized cone over C. As X is a degree d hypersurface, it admits a smoothing.

EXAMPLE 6.2. Let $\Xi = (x^d + y^d + z^d + tw^d = 0) \subseteq \mathbb{P}^3_{x:y:z:w} \times \mathbb{A}^1_t$. The special fiber Ξ_0 is a cone over a smooth plane curve of degree d and the general fiber Ξ_t, for $t \neq 0$, is a smooth surface of degree d in $\mathbb{P}^3$.

This, again, suggests that we should deal with some singularities. The question is, whether we can limit the type of singularities we must deal with. More particularly to this case, can we limit the type of cones we need to deal with?

First we need an auxiliary computation.

EXAMPLE 6.3. Let W be a smooth variety and $X = X_1 \cup X_2 \subseteq W$ such that X_1 and X_2 are Cartier divisors in W. Then by the adjunction formula we have

$$K_X = (K_W + X)\big|_X$$
$$K_{X_1} = (K_W + X_1)\big|_{X_1}$$
$$K_{X_2} = (K_W + X_2)\big|_{X_2}$$

Therefore

$$K_X\big|_{X_i} = K_{X_i} + X_{3-i}\big|_{X_i} \tag{6.3.1}$$

for $i = 1, 2$, so we have that

$$K_X \text{ is ample} \quad \Leftrightarrow \quad K_X\big|_{X_i} = K_{X_i} + X_{3-i}\big|_{X_i} \text{ is ample for } i = 1, 2. \tag{6.3.2}$$

Next, let X be a normal projective surface with K_X ample and an isolated singular point $P \in \operatorname{Sing} X$. Assume that X is isomorphic to a cone $\Xi_0 \subseteq \mathbb{P}^3$ as in Example 6.2 locally analytically near P. Further assume that X is the special fiber of a smoothing family Ξ that itself is smooth. We would like to see whether we may resolve the singular point $P \in X$ and still stay within our moduli problem, i.e., that K would remain ample. For this purpose we may assume that P is the only singular point of X.

Let $\Upsilon \to \Xi$ be the blowing up of $P \in \Xi$ and let $\widetilde{X}$ denote the proper transform of X. Then $\Upsilon_0 = \widetilde{X} \cup E$ where $E \simeq \mathbb{P}^2$ is the exceptional divisor of the blow up. Clearly, $\sigma : \widetilde{X} \to X$ is the blow up of P on X, so it is a smooth surface and $\widetilde{X} \cap E$ is isomorphic to the degree d curve over which X is locally amalytically a cone.

We would like to determine the condition on d that ensures that the canonical divisor of Υ_0 is still ample. According to (6.3.2) this means that we need that $K_E + \widetilde{X}\big|_E$ and $K_{\widetilde{X}} + E\big|_{\widetilde{X}}$ be ample.

As $E \simeq \mathbb{P}^2$, $\omega_E \simeq \mathscr{O}_{\mathbb{P}^2}(-3)$, so $\mathscr{O}_E(K_E + \widetilde{X}\big|_E) \simeq \mathscr{O}_{\mathbb{P}^2}(d-3)$. This is ample if and only if $d > 3$.

As this computation is local near P the only relevant issue about the ampleness of $K_{\widetilde{X}} + E\big|_{\widetilde{X}}$ is whether it is ample in a neighbourhood of $E_X := E\big|_{\widetilde{X}}$. By the next claim this is equivalent to asking when $(K_{\widetilde{X}} + E_X) \cdot E_X$ is positive.

Claim. Let Z be a smooth projective surface with non-negative Kodaira dimension and $\Gamma \subset Z$ an effective divisor. If $(K_Z + \Gamma) \cdot C > 0$ for every proper curve $C \subset Z$, then $K_Z + \Gamma$ is ample.

Proof. By the assumption on the Kodaira dimension there exists an $m > 0$ such that mK_Z is effective, hence so is $m(K_Z + \Gamma)$. Then by the assumption on the intersection number, $(K_Z + \Gamma)^2 > 0$, so the statement follows by the Nakai-Moishezon criterium. $\square$

Now, observe that by the adjunction formula $(K_{\widetilde{X}} + E_X) \cdot E_X = \deg K_{E_X} = d(d-3)$ as E_X is isomorphic to a plane curve of degree d. Again, we obtain the same condition as above and thus conclude that K_{Υ_0} maybe ample only if $d > 3$.

For our moduli funtor this means that we have to allow cone singularities over curves of degree $d \leq 3$. The singularity we obtain for $d = 2$ is a rational double point, but the singularity for $d = 3$ is not even rational. This does not fit any of the earlier classes we discussed.

6.E. *Log canonical singularities*

Let us investigate the previous situation under more general assumptions.

COMPUTATION 6.4. Let $D = \sum_{i=0}^{r} \lambda_i D_i$ be a divisor with only normal crossing singularities (in some ambient variety) such that $\lambda_0 = 1$. Using the adjunction formula shows that in this situation (6.3.1) remains true even if the D_i are not hypersurfaces in $\mathbb{P}^n$:

$$K_D\big|_{D_0} = K_{D_0} + \sum_{i=1}^{r} \lambda_i D_i\big|_{D_0} \tag{6.4.1}$$

Let $f : \Xi \to B$ a projective family with $\dim B = 1$, Ξ smooth and K_{Ξ_b} ample for all $b \in B$. Further let $X = \Xi_{b_0}$ for some $b_0 \in B$ a singular fiber and let $\sigma : \Upsilon \to \Xi$ be an embedded resolution of $X \subseteq \Xi$. Finally let $Y = \sigma^* X = \widetilde{X} + \sum_{i=1}^{r} \lambda_i F_i$ where $\widetilde{X}$ is the proper transform of X and F_i are exceptional divisors for σ. We are interested in finding conditions that are necessary for K_Y to remain ample.

Let $E_i := F_i\big|_{\widetilde{X}}$ be the exceptional divisors for $\sigma : \widetilde{X} \to X$ and for the simplicity of computation, assume that the E_i are irreducible. For K_Y to be ample we need that $K_Y\big|_{\widetilde{X}}$ as well as $K_Y\big|_{F_i}$ for all i are all ample. Clearly, the important one of these for our purposes is $K_Y\big|_{\widetilde{X}}$ for which by (6.4.1) we have that

$$K_Y\big|_{\widetilde{X}} = K_{\widetilde{X}} + \sum_{i=1}^{r} \lambda_i E_i.$$

As usual, we may write $K_{\widetilde{X}} = \sigma^* K_X + \sum_{i=1}^{r} a_i E_i$, so we are looking for conditions to guarantee that $\sigma^* K_X + \sum (a_i + \lambda_i) E_i$ be ample. In particular, its restriction to any of the E_i has to be ample. To further simplify our computation let us assume that $\dim X = 2$. Then the condition that we want satisfied is that for all j,

$$\left(\sum_{i=1}^{r} (a_i + \lambda_i) E_i \right) \cdot E_j > 0. \tag{6.4.2}$$

Let

$$E_+ = \sum_{a_i+\lambda_i\geq 0} |a_i+\lambda_i|E_i, \quad \text{and}$$

$$E_- = \sum_{a_i+\lambda_i< 0} |a_i+\lambda_i|E_i, \quad \text{so}$$

$$\sum_{i=1}^{r}(a_i+\lambda_i)E_i = E_+ - E_-.$$

Choose a j such that $E_j \subseteq \operatorname{supp} E_+$. Then $E_- \cdot E_j \geq 0$ since $E_j \not\subseteq E_-$ and (6.4.2) implies that $(E_+ - E_-)\cdot E_j > 0$. These together imply that $E_+ \cdot E_j > 0$ and then that $E_+^2 > 0$. However, the E_i are exceptional divisors of a birational morphism, so their intersection matrix, $(E_i \cdot E_j)$ is negative definite.

The only way this can happen is if $E_+ = 0$. In other words, $a_i + \lambda_i < 0$ for all i. However, the λ_i are positive integers, so this implies that K_Y may remain ample only if $a_i < -1$ for all $i = 1, \dots, r$.

The definition of a *log canonical singularity* is the exact opposite of this condition. It requires that X be normal and admit a resolution of singularities, say $Y \to X$, such that all the $a_i \geq -1$. This means that the above argument shows that we may stand a fighting chance if we resolve singularities that are *worse* than log canonical, but have no hope to do so with log canonical singularities. In other words, this is another class of singularities that we have to allow. Actually, the class of singularities we obtained for the cones in the previous subsection belong to this class. In fact, all the normal singularities that we have considered so far belong to this class.

The good news is that by now we have covered pretty much all the ways that something can go wrong and found the class of singularities we must allow. Since we have already found that we have to deal with some non-normal singularities and in fact in this example we have not really needed that X be normal, we conclude that we will have to allow the non-normal cousins of log canonical singularities. These are called *semi-log canonical singularities* and the reader can find their definition in the next subsection.

6.F. *Semi-log canonical singularities*

As a warm-up, let us first define the normal and more traditional singularities that are relevant in the Minimal Model Program.

DEFINITION 6.5. A normal variety X is called $\mathbb{Q}$-*Gorenstein* if K_X is $\mathbb{Q}$-Cartier, i.e., some integer multiple of K_X is a Cartier divisor. Let X be a $\mathbb{Q}$-Gorenstein variety and $f : \widetilde{X} \to X$ a good resolution of singularities with exceptional divisor $E = \cup E_i$. Express the canonical divisor of $\widetilde{X}$ in terms of K_X and the exceptional divisors:

$$K_{\widetilde{X}} \equiv f^*K_X + \sum a_i E_i$$

where $a_i \in \mathbb{Q}$. Then

	terminal		$a_i > 0$.
X has	*canonical*	singularities if for all i,	$a_i \geq 0$.
	log terminal		$a_i > -1$.
	log canonical		$a_i \geq -1$.

The corresponding definitions for non-normal varieties are somewhat more cumbersome. I include them here for completeness, but the reader should feel free to skip them and assume that for instance "semi-log canonical" means something that can be reasonably considered a non-normal version of log canonical. These definitions will not be used in this article.

DEFINITION 6.6. Let X be a scheme of dimension n and $x \in X$ a closed point.

(6.6.1) $x \in X$ is a *double normal crossing* if it is locally analytically (or formally) isomorphic to the singularity

$$\{0 \in (x_0x_1 = 0)\} \subseteq \left\{0 \in \mathbb{A}^{n+1}\right\}.$$

(6.6.2) $x \in X$ is a *pinch point* if it is locally analytically (or formally) isomorphic to the singularity

$$\left\{0 \in (x_0^2 = x_1x_2^2)\right\} \subseteq \left\{0 \in \mathbb{A}^{n+1}\right\}.$$

(6.6.3) X is *semi-smooth* if all closed points of X are either smooth, or a double normal crossing, or a pinch point. In this case, unless X is smooth, $D_X := \operatorname{Sing} X \subseteq X$ is a smooth $(n-1)$-fold. If $\nu : \widetilde{X} \to X$ is its normalization, then $\widetilde{X}$ is smooth and $\widetilde{D}_X := \nu^{-1}(D_X) \to D_X$ is a double cover ramified along the pinch locus.

(6.6.4) A morphism, $f : Y \to X$ is a *semi-resolution* if

- f is proper,
- Y is semi-smooth,
- no component of D_Y is f-exceptional, and
- there exists a closed subset $Z \subseteq X$, with $\operatorname{codim}(Z, X) \geq 2$ such that

$$f\big|_{f^{-1}(X\setminus Z)} : f^{-1}(X \setminus Z) \xrightarrow{\simeq} X \setminus Z$$

  is an isomorphism.

Let E denote the exceptional divisor (i.e., the codimension 1 part of the exceptional set, not necessarily the whole exceptional set) of f. Then f is a *good semi-resolution* if $E \cup D_Y$ is a divisor with global normal crossings on Y.

(6.6.5) X has *semi-log canonical* (resp. *semi-log terminal*) singularities if

(a) X is reduced,

(b) X is S_2,

(c) K_X is $\mathbb{Q}$-Cartier, and

(d) there exist a good semi-resolution of singularities $f : \widetilde{X} \to X$ with exceptional divisor $E = \cup E_i$, and we write $K_{\widetilde{X}} \equiv f^*K_X + \sum a_iE_i$ with $a_i \in \mathbb{Q}$, then $a_i \geq -1$ (resp. $a_i > -1$) for all i.

REMARK 6.6.6. Note that a semi-smooth scheme has at worst hypersurface singularities, so in particular it is Gorenstein. This implies that a semi-log canonical variety is Gorenstein in codimension 1.

REMARK 6.6.7. In the definition of a semi-resolution, one could choose to require that the exceptional set be a divisor. This leads to slightly different notions and at the time of the writing of this article it has not been settled whether either of the definitions and the notions of singularities they lead to are unnecassary. For more on singularities related to semi-resolutions see [**KSB88**] and [**Kol92**].

We are now ready to update our definition of "nice" to its final form cf. (2.10).

DEFINITION 6.7. Let X be *nice* if X is semi-log canonical, projective and ω_X is an ample $\mathbb{Q}$-line bundle.

7. FAMILIES AND MODULI FUNCTORS

A very important issue in considering higher dimensional moduli problems is that, as opposed to the case of curves, when studying families of higher dimensional varieties one must put conditions on the admissible families that restrict the kind of families and not only the kind of fibers that are allowed. This is perhaps better understood through an example of bad behaviour.

7.A. *An important example*

SETUP:

- Let $R \subseteq \mathbb{P}^4$ be a quartic rational normal curve, i.e., the image of the embedding of $\mathbb{P}^1$ into $\mathbb{P}^4$ by the global sections of $\mathscr{O}_{\mathbb{P}^1}(4)$. For example take
$$R = \{[u^4 : u^3v : u^2v^2 : uv^3 : v^4] \in \mathbb{P}^4 \mid [u : v] \in \mathbb{P}^1\}.$$
- Let $T \subseteq \mathbb{P}^5$ be a quartic rational scroll, i.e., the image of the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^5$ by the global sections of $\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1, 2)$. Let f_1 and f_2 denote the divisor classes of the two rulings on T. For example take
$$T = \{[xz^2 : xzt : xt^2 : yz^2 : yzt : yt^2] \in \mathbb{P}^5 \mid ([x : y], [z : t]) \in \mathbb{P}^1 \times \mathbb{P}^1\}.$$
- Let $C_R \subseteq \mathbb{P}^5$ be the projectivized cone over R in $\mathbb{P}^5$ and $C_T \subseteq \mathbb{P}^6$ the projectivized cone over T in $\mathbb{P}^6$. For the above choices, these are represented by
$$C_R = \{[u^4 : u^3v : u^2v^2 : uv^3 : v^4 : w^4] \in \mathbb{P}^5 \mid [u : v : w] \in \mathbb{P}^2\}, \text{and}$$
$$C_T = \{[xz^2 : xzt : xt^2 : yz^2 : yzt : yt^2 : pq^2] \in \mathbb{P}^6 \mid ([x : y : p], [z : t : q]) \in \mathbb{P}^2 \times \mathbb{P}^2\}.$$
- Let $V \subseteq \mathbb{P}^5$ be a Veronese surface, i.e., the image of the Veronese embedding; the embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$ by the global sections of $\mathscr{O}_{\mathbb{P}^2}(2)$. For example take
$$V = \{[u^2 : vw : uv : uw : v^2 : w^2] \mid [u : v : w] \in \mathbb{P}^2\}.$$
Another possible parametrization is obtained when the Veronese embedding is combined with the 4-to-1 endomorphism of $\mathbb{P}^2$, $[u : v : w] \mapsto [u^2 : v^2 : w^2]$:
$$V = \{[u^4 : v^2w^2 : u^2v^2 : u^2w^2 : v^4 : w^4] \mid [u : v : w] \in \mathbb{P}^2\}.$$
- Let $W \subseteq \mathbb{P}^5 \times \mathbb{P}^1$ be the following quasi-projective threefold:
$$W = \big\{\big([u^4 : u^3v + \lambda(v^2w^2 - u^3v) : u^2v^2 : uv^3 + \lambda(u^2w^2 - uv^3) : v^4 : w^4], \lambda\big)\big| \\ \big|[u : v : w] \in \mathbb{P}^2, \lambda \in \mathbb{A}^1\big\} \subseteq \mathbb{P}^5 \times \mathbb{A}^1.$$

OBSERVATIONS:

- *V is a smoothing of C_R.* Indeed, the second projection of $\mathbb{P}^5 \times \mathbb{P}^1$ exhibits W as a family of surfaces $W \to \mathbb{P}^1$. Both C_R and V appear as members of this family. For $\lambda = 0, 1 \in \mathbb{A}^1$; $W_0 \simeq C_R$ and $W_1 \simeq V$.
- *R is a hyperplane section of T.* Indeed let $H \subseteq \mathbb{P}^5$ be a general hyperplane. Then $C := H \cap T$ is a smooth curve such that $C \sim_T \mathsf{f}_1 + 2\mathsf{f}_2$. Then by the adjunction formula $2g(C) - 2 = (-2\mathsf{f}_1 - 2\mathsf{f}_2 + C) \cdot C = -2$, hence $C \simeq \mathbb{P}^1$. Furthermore, then $C^2 = 4$, so $\mathscr{O}_T(1, 2)\big|_C \simeq \mathscr{O}_C(4)$. Therefore C is a quartic rational curve in $H \simeq \mathbb{P}^4$, and thus it may be identified with R.
- *T is also a smoothing of C_R.* Indeed, both T and C_R are hyperplane sections of C_T. The latter statement follows from the previous observation.

ANALYSIS:

- It is relatively easy, and thus left to the reader, to compute that C_R has log terminal singularities. In particular, this type of singularity is among those we have to be able to handle.

- The problem this example points to is that if we allow arbitrary families, then we may get unwanted results. For example, using the families derived from C_T and W would mean that $T \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $V \simeq \mathbb{P}^2$ should be considered to have the same deformation type. However, there are obviously no smooth families that they both belong to, they are topologically very different. For instance, $K_T^2 = 8$ while $K_V^2 = 9$.
- The crux of the matter is that C_T is *not* $\mathbb{Q}$-Gorenstein and consequently the family obtained from it is not a $\mathbb{Q}$-Gorenstein *family*. This is actually an important point: the members of the family are $\mathbb{Q}$-Gorenstein surfaces, but the relative canonical bundle of the family is not $\mathbb{Q}$-Cartier. In particular, the canonical divisors of the members of the family are not consistent.
- The family obtained from W *is* $\mathbb{Q}$-Gorenstein and consequently ensures that the canonical divisors of the members of the family are similar to some extent. Among other things this implies that $K_{C_R}^2 = 9$. One may also use the parametrization of C_R given above to verify this fact independently. It is interesting to note that K_{C_R} is $\mathbb{Q}$-Cartier, but not Cartier even though its self-intersection number is an integer.

7.B. $\mathbb{Q}$-*Gorenstein families*

We have seen that we have to extend the definition of the moduli functor (see (3.3)) to allow (some) singular varieties.

WARNING. Here we are entering a somewhat uncharted territory. Some of the notions and conditions are still evolving. It has not crystallized yet what are the "right" or optimal conditions to assume. Accordingly, on occasion, we may assume too much or too little. This section is intended to give a peak into the forefront of the research that is conducted in this area.

The previous example shows that it is not enough to restrict the kind of members of the families we allow but we have to restrict the kind of families we allow as well.

DEFINITION 7.1. Let k be an algebraically closed field of characteristic 0 and Sch_k the category of k-schemes. We define $\mathcal{M}^{\text{wst}} : \mathsf{Sch}_k \to \mathsf{Sets}$, the *moduli functor of weakly stable canonically polarized* $\mathbb{Q}$*-Gorenstein varieties*, the following way.

(7.1.1) A morphism $f : X \to B$ is called a *weakly stable family* if the following hold:
  (a) f is flat and projective with connected fibers,
  (b) $\omega_{X/B}$ is a relatively ample $\mathbb{Q}$-line bundle, and
  (c) for all $b \in B$, X_b has only semi-log canonical singularities.

(7.1.2) For an object $B \in \operatorname{Ob} \mathsf{Sch}_k$,

$$\mathcal{M}^{\text{wst}}(B) := \{f : X \to B \mid f \text{ is a weakly stable family}\} \big/ \simeq$$

where "$\simeq$" is defined as in 3.3.

(7.1.3) For a morphism $\alpha \in \operatorname{Hom}_{\mathsf{Sch}_k}(A, B)$,

$$\mathcal{M}^{\text{wst}}(\alpha) := (\underline{\quad}) \times_B \alpha.$$

REMARK 7.1.1. Note that it is not obvious from the definition that this is indeed a functor. However, this functor (if it is a functor) is actually not yet the one we are interested in. We will use this to define others. The fact that those others are indeed functors follows from Lemma 7.3.

As mentioned above, this functor is not yet the right one. There are two additional conditions to which we have to pay attention. The first is to keep track of the Hilbert polynomials of the polarizations. This is straightforward, although somewhat different from the smooth case in that now we have to also keep track of what power of the $\mathbb{Q}$-line bundle we consider giving the polarization. This is done as follows.

DEFINITION 7.2. Let k be an algebraically closed field of characteristic 0, Sch_k the category of k-schemes and $N \in \mathbb{N}$. We define $\mathcal{M}^{\text{wst},[N]} : \mathsf{Sch}_k \to \mathsf{Sets}$, the *moduli functor of weakly stable canonically polarized* $\mathbb{Q}$*-Gorenstein varieties of index* N, as the subfunctor of $\mathcal{M}^{\text{wst}}$ with the additional condition that $\omega_{X/B}^{[N]}$ is a line bundle:

$$\mathcal{M}^{\text{wst},[N]}(B) := \left\{ (f : X \to B) \in \mathcal{M}^{\text{wst}}(B) \mid \omega_{X/B}^{[N]} \text{ is a line bundle.} \right\}.$$

Now let $h \in \mathbb{Q}[t]$. Then

$$\mathcal{M}_h^{\text{wst},[N]}(B) := \left\{ (f : X \to B) \in \mathcal{M}^{\text{wst},[N]}(B) \mid \chi(X_b, \omega_{X_b}^{[mN]}) = h(m) \right\}.$$

In order to use the polarization given by the appropriate reflexive power of the canonical sheaves of the fibers we need to know that the powers of the relative canonical sheaf commute with base change. The following shows that for objects in $\mathcal{M}^{\text{wst},[N]}(B)$, this holds for multiples of the index.

Lemma 7.3. [**HK04**, 2.6] *Given a weakly stable family of canonically polarized* $\mathbb{Q}$*-Gorenstein varieties of index* N*,* $f : X \to B$*, and a morphism* $\alpha : T \to B$*, we have*

$$\alpha_X^* \omega_{X/B}^{[N]} \simeq \omega_{X_T/T}^{[N]}.$$

PROOF. Let $U \subset X$ be the largest open subset U of X such that ω_{U_b} is a line bundle for all $b \in B$ or equivalently the largest open subset U of X such that $\omega_{X/B}\big|_U \simeq \omega_{U/B}$ is a line bundle. Then $\omega_{X/B}^{[N]}\big|_U \simeq \omega_{U/B}^N$ and hence

$$\alpha_X^* \omega_{X/B}^{[N]}\big|_{\alpha_X^{-1}U} \simeq \alpha_X^* \omega_{U/B}^N \simeq \omega_{\alpha_X^{-1}U/T}^N \simeq \omega_{X_T/T}^{[N]}\big|_{\alpha_X^{-1}U}.$$

Now $\operatorname{codim}(U_b, X_b) \geq 2$ for all $b \in B$ (cf. (6.6.6)), so $\operatorname{codim}((\alpha_X^{-1}U)_t, (X_T)_t) \geq 2$ for all $t \in T$ and hence $\operatorname{codim}(\alpha_X^{-1}U, X_T) \geq 2$. Finally $\alpha_X^* \omega_{X/B}^{[N]}$ and $\omega_{X_T/T}^{[N]}$ are reflexive, so since they are isomorphic on $\alpha_X^{-1}U$, they are isomorphic on X_T. □

However, this may not be enough to encode the main topological properties of the fibers. As a solution, Kollár suggests to require more.

DEFINITION 7.4 : KOLLÁR'S CONDITION. We say that *Kollár's condition* holds for a family $(f : X \to B) \in \mathcal{M}^{\text{wst}}(B)$, if for all $\ell \in \mathbb{Z}$ and for all $b \in B$,

$$\omega_{X/B}^{[\ell]}\big|_{X_b} \simeq \omega_{X_b}^{[\ell]}.$$

The important difference between this condition and the situation in the previous lemma is that this condition requires that the restriction of *all* reflexive powers commute with base change, not only those that are line bundles.

It is relatively easy to see using the same argument as in the proof of (7.3) that this condition is equivalent to the requirement that the restriction of all reflexive powers to the fibers be reflexive themselves.

Now we are ready to define the "right" moduli functor.

DEFINITION 7.5. Let k be an algebraically closed field of characteristic 0 and Sch_k the category of k-schemes. We define $\mathcal{M} = \mathcal{M}^{\text{st}} : \mathsf{Sch}_k \to \mathsf{Sets}$, the *moduli functor of* *stable canonically polarized* $\mathbb{Q}$*-Gorenstein varieties*, as the subfunctor of $\mathcal{M}^{\text{wst}}$ with the additional condition that a family $(f : X \to B) \in \mathcal{M}^{\text{wst}}(B)$ satisfy Kollár's condition:

$$\mathcal{M}(B) := \left\{ (f : X \to B) \in \mathcal{M}^{\text{wst}}(B) \;\middle|\; \forall \ell \in \mathbb{Z}, b \in B, \omega_{X/B}^{[\ell]}\big|_{X_b} \simeq \omega_{X_b}^{[\ell]} \right\}.$$

Finally, let $h \in \mathbb{Q}[t]$ and $N \in \mathbb{N}$. Then we define $\mathcal{M}_h^{[N]}$ as the subfunctor of $\mathcal{M}^{[N]}$ with the additional condition that for a family $(f : X \to B) \in \mathcal{M}^{[N]}(B)$, the Hilbert polynomial of the fibers agree with h:

$$\mathcal{M}_h^{[N]}(B) := \left\{ (f : X \to B) \in \mathcal{M}^{[N]}(B) \;\middle|\; \forall b \in B, \chi(X_b, \omega_{X_b}^{[mN]}) = h(m) \right\}.$$

The difference between the moduli functors $\mathcal{M}_h^{\text{wst},[N]}$ and $\mathcal{M}_h^{[N]}$ is very subtle. They parametrize the same objects and as long as one restricts to Gorenstein varieties, they allow the same families. This means that if one is only interested in the compactification of the coarse moduli space of $\mathcal{M}_h^{\text{smooth}}$, then the difference between these two moduli functors does not matter as they lead to the same reduced scheme. The difference may only show up in their scheme structure. However, the usefulness of a moduli space is closely related to its "right" scheme structure, so it is important to find that.

A somewhat troubling point is that we do not actually know for a fact that these two moduli functors are really different in characteristic 0. In other words, we do not know an example of a family that belongs to $\mathcal{M}_h^{\text{wst},[N]}$, but not to $\mathcal{M}_h^{[N]}$. The following example of Kollár shows that these functors *are* different in characteristic $p > 0$, but there is no similar example known in characteristic 0.

EXAMPLE 7.6 : KOLLÁR'S EXAMPLE (UNPUBLISHED). Note that the first part of the discussion (7.6.1) works in arbitrary characteristic. It shows that a family with the required properties belongs to $\mathcal{M}_h^{\text{wst},[N]}$, but not to $\mathcal{M}_h^{[N]}$. In the second part (7.6.2) it is shown that in characteristic $p > 0$ a family satisfying another set of properties also has the ones required in (7.6.1). Finally, it is easy to see that the example in §§7.A admits these later properties, so we do indeed have an explicit example for this behaviour.

(7.6.1) Suppose that $g : Y \to B$ is a family of canonically polarized $\mathbb{Q}$-Gorenstein varieties (with only semi-log canonical singularities) and assume that $B = \operatorname{Spec} R$ with $R = (R, \mathfrak{m})$ a DVR. Let $B_n = \operatorname{Spec} R_n$ where $R_n := R/\mathfrak{m}^n$ and consider the restriction of the family g over B_n, $g_n : Y_n = Y \times_B B_n \to B_n$. Finally assume that ω_{Y_n/B_n} is $\mathbb{Q}$-Cartier of index r_n for all n but $r_n \to \infty$ as $n \to \infty$ (recall that the index means the smallest integer m such that the m^{th} reflexive power is a line bundle). Note that by Lemma 7.3 this implies that $\omega_{Y/B}$ cannot be $\mathbb{Q}$-Cartier.

We claim that g_n is a weakly stable family of canonically polarized $\mathbb{Q}$-Gorenstein varieties of index r_n (Definition 7.1), but it does not satisfy Kollár's condition (Definition 7.4) for all but possibly a finite number of n.

The first part of the claim is obvious from the assumptions. For the second part consider the following argument. If g_n satisfied Kollár's condition, then for any $m < n$ the restriction of $\omega_{Y_n/B_n}^{[r_m]}$ to Y_m (hence to Y_1) would be a line bundle implying, via Nakayama's lemma, that $\omega_{Y_n/B_n}^{[r_m]}$ itself is a line bundle. That however would further imply that $r_n \leq r_m$, but since $r_n \to \infty$ as $n \to \infty$, this can only happen for a finite number of n's.

(7.6.2) Next we will show (following Kollár) that a family such as in (7.6.1) does exist in characteristic $p > 0$. It is currently not known whether such an example exists in characteristic 0.

As above, let $g : Y \to B$ be a family of canonically polarized $\mathbb{Q}$-Gorenstein varieties with only log canonical singularities, such that $B = \operatorname{Spec} R$ with $R = (R, \mathfrak{m})$ a DVR. Assume that g, Y, and B are defined above a field k of characteristic $p > 0$. Let $B_n = \operatorname{Spec} R_n$ where $R_n := R/\mathfrak{m}^n$ and consider the restriction of the family g over B_n, $g_n : Y_n = Y \times_B B_n \to B_n$. For a concrete example one may consider the smoothing of C_R to T via C_T (reduced over $k[x]_{(x)}$) from the example in §§7.A.

Claim. ω_{Y_n/B_n} is $\mathbb{Q}$-Cartier.

Proof. The question is local on Y, so we may assume that Y_n is a local scheme. In particular, we will assume that all line bundles on Y_1 are trivial. Let

$$\iota_n : U_n = (Y_n \setminus \operatorname{Sing} Y_n) \hookrightarrow Y_n.$$

By assumption, Y_n is normal (R_1 and S_2) for all n, so

$$\omega_{Y_n/B_n}^{[m]} \simeq (\iota_n)_* \omega_{U_n/B_n}^{\otimes m}$$

for all m.

Next, consider the restriction maps to the special fiber of the family from all the infinitesimal thickenings:

$$\varrho_n : \operatorname{Pic} U_n \to \operatorname{Pic} U_1.$$

The key observation is the following: the kernel of this map is a (p-power) torsion group (cf. [**Har77**, Ex.III.4.6]). In other words, any line bundle on U_n whose restriction to U_1 is trivial extends to a $\mathbb{Q}$-Cartier divisor on Y_n.

Recall that by assumption, $\omega_{Y_1/B_1} = \omega_{Y_1}$ is $\mathbb{Q}$-Cartier (of index r_1), in particular $\omega_{Y_1/B_1}^{[r_1]}$ is trivial. Therefore,

$$\varrho_n(\omega_{U_n/B_n}^{\otimes r_1}) = \omega_{U_n/B_n}^{\otimes r_1}\big|_{U_1} \simeq \omega_{U_1/B_1}^{\otimes r_1}$$

is also trivial. Consequently,

$$\omega_{U_n/B_n}^{\otimes r_1} \in \ker \varrho_n.$$

Recall that this a torsion group, so there exists an $m_n \in \mathbb{N}$ such that $\left(\omega_{U_n/B_n}^{\otimes r_1}\right)^{\otimes m_n}$ is trivial. That however, implies that then so is

$$\omega_{Y_n/B_n}^{[r_1 \cdot m_n]} \simeq (\iota_n)_* \omega_{U_n/B_n}^{\otimes (r_1 \cdot m_n)} \simeq (\iota_n)_* \mathscr{O}_{U_n} \simeq \mathscr{O}_{Y_n}.$$

We conclude that ω_{Y_n/B_n} is indeed $\mathbb{Q}$-Cartier. □

It is left for the reader to prove that if $\omega_{Y/B}$ is not $\mathbb{Q}$-Cartier, then the index of ω_{Y_n/B_n} has to tend to infinity. It is easy to check that this happens in the case of C_T considered as a non-$\mathbb{Q}$-Gorenstein smoothing of C_R as above.

REMARK 7.7. The previous example also shows an important aspect of why Kollár's condition is useful. Let Z be a canonically polarized $\mathbb{Q}$-Gorenstein variety of index m with only semi-log canonical singularities. If we want to find a moduli space where Z appears, we may choose the moduli functor $\mathscr{M}_h^{\mathrm{wst},[a \cdot m]}$ for any $a \in \mathbb{N}$ (where h is the Hilbert polynomial of $\omega_Z^{[a \cdot m]}$). The previous example shows that the scheme structure of the corresponding moduli space will depend on which a we choose. As a grows, the moduli scheme gets thicker. Consequently, there is not a unique moduli scheme where Z would naturally belong to. This does not happen for the functor $\mathscr{M}_h^{[a \cdot m]}$ because Kollár's condition makes sure that the choice of a makes no difference.

7.C. *Projective moduli schemes*

With the definition of $\mathcal{M}_h^{[N]}$ we have reached the moduli functor that should be the right one. This functor accounts for all canonical models, even a little bit more, as well as all degenerations of smooth canonical models.

The natural next step would be to state the equivalent of Theorem 5.16 for $\mathcal{M}_h^{[N]}$. However, we can't quite do that exactly.

Boundedness was proven for moduli of surfaces (i.e., $\deg h = 2$) in [**Ale94**] (cf. [**AM04**]) A more general result was obtained in [**Kar00**] assuming that certain conjectures from the Minimal Model Program were true. Fortunately these conjectures have been recently proven in [**HM07, BCHM06**], so this piece of the puzzle is in place.

Separatedness follows from [**KSB88**] and [**Kaw05**].

Projectivity follows from [**Kol90**].

Local closedness for $\mathcal{M}_h^{\mathrm{wst},[N]}$ was proven in [**HK04**]. Hacking obtained partial results toward the local closedness of $\mathcal{M}_h^{[N]}$ in [**Hac04**]. Local closedness of $\mathcal{M}_h^{[N]}$ in general has been proved by Abramovich and Hassett, but this result has not appeared in any form yet at the time of this writing. Even more recently a general flattening result that implies the local closedness of $\mathcal{M}_h^{[N]}$ has been proved by Kollár [**Kol08**]. Kollár's result essentially closes the question of local closedness for good.

So, the conclusion is that all the pieces are in place, even though the statement of the existence of a projective coarse moduli scheme for $\mathcal{M}_h^{[N]}$ has not yet appeared in print and thus I will not formulate it as a theorem here.

7.D. *Moduli of pairs and other generalizations*

As it has become clear in higher dimensional geometry in recent years, the "right" formulation of (higher dimensional) problems deals with pairs, or log varieties (cf. [**Kol97b**]). Accordingly, one would like to have a moduli theory of log varieties. In fact, one would like to go through this entire article and replace all objects with log varieties, canonical models with log canonical models, etc.

However, this is not as straightforward as it may appear at the first sight and the formulation of the moduli functor itself is not entirely obvious. Nonetheless, work is being done in this area and perhaps by the time these words appear in print, there will be concrete results to speak of about log varieties.

There are many related results I did not have the chance to mention in detail. Here is a somewhat random sample of those results: Valery Alexeev has been particularly prolific and the interested reader should take a look at his results, a good chunk of which is joint work with Michel Brion: [**Ale96, Ale02, Ale01, AB04a, AB04b, AB05**]. Paul Hacking solved the long standing problem of compactifying the moduli space of plane curves in a geometrically meaningful way [**Hac04**]. Hacking jointly with Keel and Tevelev has done the same for the moduli space of hyperplane arrangements [**HKT06**] and Del Pezzo surfaces [**HKT07**].

REFERENCES

[Ale94] V. ALEXEEV: *Boundedness and K^2 for log surfaces*, Internat. J. Math. **5** (1994), no. 6, 779–810. MR1298994 (95k:14048)

[Ale96] V. ALEXEEV: *Moduli spaces $M_{g,n}(W)$ for surfaces*, Higher-dimensional complex varieties (Trento, 1994), de Gruyter, Berlin, 1996, pp. 1–22. MR1463171 (99b:14010)

[Ale01] V. ALEXEEV: *On extra components in the functorial compactification of A_g*, Moduli of abelian varieties (Texel Island, 1999), Progr. Math., vol. 195, Birkhäuser, Basel, 2001, pp. 1–9. MR1827015 (2002d:14070)

[Ale02] V. ALEXEEV: *Complete moduli in the presence of semiabelian group action*, Ann. of Math. (2) **155** (2002), no. 3, 611–708. MR1923963 (2003g:14059)

[AB04a] V. ALEXEEV AND M. BRION: *Stable reductive varieties. I. Affine varieties*, Invent. Math. **157** (2004), no. 2, 227–274. MR2076923

[AB04b] V. ALEXEEV AND M. BRION: *Stable reductive varieties. II. Projective case*, Adv. Math. **184** (2004), no. 2, 380–408. MR2054021

[AB05] V. ALEXEEV AND M. BRION: *Moduli of affine schemes with reductive group action*, J. Algebraic Geom. **14** (2005), no. 1, 83–117. MR2092127

[AM04] V. ALEXEEV AND S. MORI: *Bounding singular surfaces of general type*, Algebra, arithmetic and geometry with applications (West Lafayette, IN, 2000), Springer, Berlin, 2004, pp. 143–174. MR2037085

[BCHM06] C. BIRKAR, P. CASCINI, C. D. HACON, AND J. MCKERNAN: *Existence of minimal models for varieties of log general type*, preprint, 2006. arXiv:math.AG/0610203

[BLR90] S. BOSCH, W. LÜTKEBOHMERT, AND M. RAYNAUD: *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990. MR1045822 (91i:14034)

[Dur79] A. H. DURFEE: *Fifteen characterizations of rational double points and simple critical points*, Enseign. Math. (2) **25** (1979), no. 1-2, 131–163. MR543555 (80m:14003)

[Gro62a] A. GROTHENDIECK: *Fondements de la géométrie algébrique. [Extraits du Séminaire Bourbaki, 1957–1962.]*, Secrétariat mathématique, Paris, 1962. MR0146040 (26 #3566)

[Gro62b] A. GROTHENDIECK: *Fondements de la géométrie algébrique. [Extraits du Séminaire Bourbaki, 1957–1962.]*, Secrétariat mathématique, Paris, 1962. MR0146040 (26 #3566)

[Gro95] A. GROTHENDIECK: *Fondements de la géométrie algébrique. Commentaires [MR0146040 (26 #3566)]*, Séminaire Bourbaki, Vol. 7, Soc. Math. France, Paris, 1995, pp. 297–307. MR1611235

[Hac04] P. HACKING: *Compact moduli of plane curves*, Duke Math. J. **124** (2004), no. 2, 213–257. MR2078368 (2005f:14056)

[HKT06] P. HACKING, S. KEEL, AND J. TEVELEV: *Compactification of the moduli space of hyperplane arrangements*, J. Algebraic Geom. **15** (2006), no. 4, 657–680. MR2237265 (2007j:14016)

[HKT07] P. HACKING, S. KEEL, AND J. TEVELEV: *Stable pair, tropical, and log canonical compact moduli of del pezzo surfaces*, 2007. arXiv:math/0702505

[HM06] C. D. HACON AND J. MCKERNAN: *Boundedness of pluricanonical maps of varieties of general type*, Invent. Math. **166** (2006), no. 1, 1–25. MR2242631 (2007e:14022)

[HM07] C. D. HACON AND J. MCKERNAN: *Extension theorems and the existence of flips*, Flips for 3-folds and 4-folds, Oxford Lecture Ser. Math. Appl., vol. 35, Oxford Univ. Press, Oxford, 2007, pp. 76–110. MR2359343

[HMo98] J. HARRIS AND I. MORRISON: *Moduli of curves*, Graduate Texts in Mathematics, vol. 187, Springer-Verlag, New York, 1998. MR1631825 (99g:14031)

[Har77] R. HARTSHORNE: *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR0463157 (57 #3116)

[HK04] B. HASSETT AND S. J. KOVÁCS: *Reflexive pull-backs and base extension*, J. Algebraic Geom. **13** (2004), no. 2, 233–247. MR2047697 (2005b:14028)

[Hir64] H. HIRONAKA: *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) 79 (1964), 109–203; ibid. (2) **79** (1964), 205–326. MR0199184 (33 #7333)

[Iit82] S. IITAKA: *Algebraic geometry*, Graduate Texts in Mathematics, vol. 76, Springer-Verlag, New York, 1982, An introduction to birational geometry of algebraic varieties, North-Holland Mathematical Library, 24. 84j:14001

[Kar00] K. KARU: *Minimal models and boundedness of stable varieties*, J. Algebraic Geom. **9** (2000), no. 1, 93–109. MR1713521 (2001g:14059)

[Kaw05] M. KAWAKITA: *Inversion of adjunction on log canonicity*, 2005. arXiv:math.AG/0511254

[Kaw84] Y. KAWAMATA: *The cone of curves of algebraic varieties*, Ann. of Math. (2) **119** (1984), no. 3, 603–633. MR744865 (86c:14013b)

[KeM97] S. KEEL AND S. MORI: *Quotients by groupoids*, Ann. of Math. (2) **145** (1997), no. 1, 193–213. MR1432041 (97m:14014)

[Kol84] J. KOLLÁR: *The cone theorem. Note to a paper: "The cone of curves of algebraic varieties"* [Ann. of Math. (2) **119** (1984), no. 3, 603–633; MR0744865 (86c:14013b)] *by Y. Kawamata*, Ann. of Math. (2) **120** (1984), no. 1, 1–5. MR750714 (86c:14013c)
[Kol85] J. KOLLÁR: *Toward moduli of singular varieties*, Compositio Math. **56** (1985), no. 3, 369–398. MR814554 (87e:14009)
[Kol87] J. KOLLÁR: *The structure of algebraic threefolds: an introduction to Mori's program*, Bull. Amer. Math. Soc. (N.S.) **17** (1987), no. 2, 211–273. MR903730 (88i:14030)
[Kol90] J. KOLLÁR: *Projectivity of complete moduli*, J. Differential Geom. **32** (1990), no. 1, 235–268. MR1064874 (92e:14008)
[Kol94] J. KOLLÁR: *Moduli of polarized schemes*, unpublished manuscript, 1994.
[Kol96] J. KOLLÁR: *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 32, Springer-Verlag, Berlin, 1996. MR1440180 (98c:14001)
[Kol97a] J. KOLLÁR: *Quotient spaces modulo algebraic groups*, Ann. of Math. (2) **145** (1997), no. 1, 33–79. MR1432036 (97m:14013)
[Kol97b] J. KOLLÁR: *Singularities of pairs*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 221–287. MR1492525 (99m:14033)
[Kol07] J. KOLLÁR: *Two examples of surfaces with normal crossing singularities*, 2007. arXiv:0705.0926v2
[Kol08] J. KOLLÁR: *Hulls and husks*, 2008. arXiv:0805.0576v2 [math.AG]
[KM98] J. KOLLÁR AND S. MORI: *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR1658959 (2000b:14018)
[KSB88] J. KOLLÁR AND N. I. SHEPHERD-BARRON: *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988), no. 2, 299–338. MR922803 (88m:14022)
[Kol92] J. KOLLÁR ET. AL: *Flips and abundance for algebraic threefolds*, Société Mathématique de France, Paris, 1992, Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992). MR1225842 (94f:14013)
[Mat72] T. MATSUSAKA: *Polarized varieties with a given Hilbert polynomial*, Amer. J. Math. **94** (1972), 1027–1077. MR0337960 (49 #2729)
[MM64] T. MATSUSAKA AND D. MUMFORD: *Two fundamental theorems on deformations of polarized varieties*, Amer. J. Math. **86** (1964), 668–684. MR0171778 (30 #2005)
[Mor82] S. MORI: *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math. (2) **116** (1982), no. 1, 133–176. MR662120 (84e:14032)
[Mor87] S. MORI: *Classification of higher-dimensional varieties*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 269–331. MR927961 (89a:14040)
[Mor88] S. MORI: *Flip theorem and the existence of minimal models for* 3*-folds*, J. Amer. Math. Soc. **1** (1988), no. 1, 117–253. MR924704 (89a:14048)
[MFK94] D. MUMFORD, J. FOGARTY, AND F. KIRWAN: *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR1304906 (95m:14012)
[Mum70] D. MUMFORD: *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Published for the Tata Institute of Fundamental Research, Bombay, 1970. MR0282985 (44 #219)
[Nag62] M. NAGATA: *Imbedding of an abstract variety in a complete variety*, J. Math. Kyoto Univ. **2** (1962), 1–10. MR0142549 (26 #118)
[Rei83] M. REID: *Projective morphisms according to Kawamata*, unpublished manuscript, 1983.
[Rei87] M. REID: *Young person's guide to canonical singularities*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 345–414. MR927963 (89b:14016)
[Rei97] M. REID: *Chapters on algebraic surfaces*, Complex algebraic geometry (Park City, UT, 1993), IAS/Park City Math. Ser., vol. 3, Amer. Math. Soc., Providence, RI, 1997, pp. 3–159. MR1442522 (98d:14049)
[Sho85] V. V. SHOKUROV: *A nonvanishing theorem*, Izv. Akad. Nauk SSSR Ser. Mat. **49** (1985), no. 3, 635–651. MR794958 (87j:14016)
[Sho03] V. V. SHOKUROV: *Prelimiting flips*, Tr. Mat. Inst. Steklova **240** (2003), no. Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 82–219. MR1993750 (2004k:14024)

[Sho04] V. V. SHOKUROV: *Letters of a bi-rationalist. V. Minimal log discrepancies and termination of log flips*, Tr. Mat. Inst. Steklova **246** (2004), no. Algebr. Geom. Metody, Svyazi i Prilozh., 328–351. MR2101303 (2006b:14019)

[Vie89] E. VIEHWEG: *Weak positivity and the stability of certain Hilbert points*, Invent. Math. **96** (1989), no. 3, 639–667. MR996558 (90i:14037)

[Vie90a] E. VIEHWEG: *Weak positivity and the stability of certain Hilbert points. II*, Invent. Math. **101** (1990), no. 1, 191–223. MR1055715 (91f:14032)

[Vie90b] E. VIEHWEG: *Weak positivity and the stability of certain Hilbert points. III*, Invent. Math. **101** (1990), no. 3, 521–543. MR1062794 (92f:32033)

[Vie91] E. VIEHWEG: *Quasi-projective quotients by compact equivalence relations*, Math. Ann. **289** (1991), no. 2, 297–314. MR1092177 (92d:14028)

[Vie95] E. VIEHWEG: *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 30, Springer-Verlag, Berlin, 1995. MR1368632 (97j:14001)

[Vie06] E. VIEHWEG: *Compactifications of smooth families and of moduli spaces of polarized manifolds*, 2006. arXiv:math.AG/0605093

UNIVERSITY OF WASHINGTON, DEPARTMENT OF MATHEMATICS, 354350, SEATTLE, WA 98195, USA
E-mail address: kovacs@math.washington.edu
URL: http://www.math.washington.edu/∼kovacs