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Viehweg's Conjecture for Families over $\mathbb{P}^n$

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ABSTRACT

The purpose of this note is to give a simple proof of a special case of a conjecture of Viehweg concerning base spaces of non-isotrivial families of smooth canonically polarized varieties.

Key Words: Varieties of general type; Families; Algebraic hyberbolicity.

1. INTRODUCTION

Let B be a smooth projective variety, Δ a normal crossing divisor on B and $B_0 = B \setminus \Delta$. Let $f_0 : X_0 \to B_0$ be a smooth family of canonically polarized varieties with Hilbert polynomial h, and assume that f_0 extends

#Dedicated to Professor Steven L. Kleiman on the occasion of his 60th birthday.
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to a family $f: X \to B$, semi-stable over the general points of the components of Δ. Our main focus in this note is the following conjecture.

Viehweg's Conjecture 1.1 (Viehweg, 2001). *If f_0 induces an étale map to $\mathfrak{M}_h$, the moduli stack of smooth canonically polarized varieties with Hilbert polynomial h, then $\omega_B(\Delta)$ is ample with respect to a non-empty open subset of B_0.*

Remark 1.1.1. In the past 8 years many related results have been obtained. The recent articles (Bedulev and Viehweg, 2000; Kovács, 1996, 1997a,b; Kovács, 2000; Kovács, 2002; Kovács, to appear; Migliorini, 1995; Oguiso and Viehweg, 2001; Viehweg and Zuo, 2001a,b; Viehweg and Zuo, 2002; Zhang, 1997) definitely should be mentioned. For more details on the relationship between these results and for further related references, see the introductions of Viehweg (2001), Kovács (2002) and Kovács (2001).

One should also note that there are other closely related conjectures made by Viehweg, but here we are only going to be concerned with 1.1.

The main result is the following theorem.

Theorem 1.2. *Let B be a smooth projective uniruled variety, Δ a normal crossing divisor on B and $B_0 = B \setminus \Delta$. Let $f_0: X_0 \to B_0$ be a smooth family of canonically polarized varieties with Hilbert polynomial h. If f_0 induces a generically finite map to $\mathfrak{M}_h$, the moduli stack of smooth canonically polarized varieties with Hilbert polynomial h, then $\omega_B(\Delta)^{-1}$ is not pseudo-effective. In particular, if in addition $\rho(B) = 1$, then $\omega_B(\Delta)$ is ample.*

The assumptions here are slightly more general than those of 1.1, because this statement does not require f to be semi-stable over the general points of the components of Δ or the induced map to moduli to be étale, only generically finite. On the other hand, the statement itself is not as strong as required by the above conjecture.

Perhaps the most interesting point here is that in this statement we do not need to assume that f_0 extends to a family over B. Unlike families over curves, families over higher-dimensional non-compact bases do not necessarily have a compactification, which is a *flat* family. Therefore this statement in fact applies to a larger class of families than required by 1.1.

Remark 1.2.1. Under the assumptions of 1.1, the statement of 1.2 also follows from the results of Viehweg and Zuo (2002). In fact, their results

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are much stronger in many aspects, but the proof presented here is considerably simpler.

Notice, however, that 1.2 is still strong enough to imply the following statement, which is simply Viehweg's conjecture for families over the projective space.

Corollary 1.3. *Let Δ be a normal crossing divisor on $\mathbb{P}^n$ and let $\phi : U \to \mathbb{P}^n \backslash \Delta$ be a smooth family of canonically polarized varieties. If the associated map of ϕ to the corresponding Hilbert scheme is generically finite, then*

$$\deg \Delta \geq n + 2.$$

The logic of the proof is very simple. The goal is to reduce the statement for the case of $B = \mathbb{P}^1$, in which case the statement is already known by the following theorem.

Theorem 1.4. (Kovács, 2000)**.** 1.3 *and hence Viehweg's conjecture hold for* $B = \mathbb{P}^1$.

Definitions and Notation. *Throughout the article the ground field will always be $\mathbb{C}$, the field of complex numbers.*

Let Δ_1, Δ_2 be divisors on a smooth projective variety. $\Delta_1 \cdot \Delta_2$ will denote the cycle-theoretic intersection of Δ_1 and Δ_2.

A divisor class is called *pseudo-effective* if its associated homology class lies on the boundary of the convex cone generated by the homology classes of effective divisors. A line bundle is *pseudo-effective* if so is its associated divisor class.

Let B be a smooth projective variety. The rank of the Picard group of B is called the Picard number and denoted by $\rho(B)$.

By abuse of notation a divisor will sometimes also denote its own support, but this should not cause any confusion.

2. PROOF OF THEOREM 1.2

It turns out that the proof is rather simple; it follows very easily from 1.4 and a result of Keel and McKernan.

On the other hand, for families over projective spaces one can avoid using the above mentioned deep result of Keel and M^{c}Kernan and replace its use with an elementary argument.

This argument is in fact very simple for $B = \mathbb{P}^2$ and is included below.

2.1. Baby Case: Families over the Projective Plane

Let $\Delta \subseteq \mathbb{P}^2$ be a curve of degree d defined by the homogeneous polynomial $f \in k[x_0, x_1, x_2]$. Let $P \in \Delta$, and choose coordinates in such a way that $P = [1:0:0]$. Express f as a polynomial of x_0,

$$f = f_r x_0^{d-r} + f_{r+1} x_0^{d-r-1} + \cdots + f_{d-1} x_0 + f_d,$$

such that for each $i, f_i \in k[x_1, x_2]$ is a homogeneous polynomial of degree i.

Let $L \subseteq \mathbb{P}^2$ be a line such that $P \in L$. Then L is defined by a linear polynomial of the form $l = a_1 x_1 + a_2 x_2$. Choose L in such a way that it is tangent to the curve Δ at P, that is, l divides f_r. Suppose that $\omega_{\mathbb{P}^2}(\Delta)^{-1}$ is pseudo-effective, or in other words that deg $\Delta \leq 3$. By adding components if necessary, we may actually assume that deg $\Delta = 3$.

If $L \not\subseteq \Delta$, then L meets Δ in at most two points. If Δ is irreducible, this implies that, through a general point of $B = \mathbb{P}^2$, there exists a rational curve meeting Δ in at most two points. We want to prove that the same holds in all cases.

If Δ is reducible, then it is either the union of three lines, or the union of a line and an irreducible conic.

In the case of three lines, for a general point there exists a line L that goes through the given general point and (one of) the intersection point(s) of the lines. This line meets Δ in at most two points.

In the case of the union of a line and an irreducible conic, take a line that is tangent to the conic. Again, this will intersect Δ in at most two points.

This means that if deg $\Delta \leq 3$, then, through a general point of $B = \mathbb{P}^2$, there exists a rational curve meeting Δ in at most two points.

Consider $\phi : U \to \mathbb{P}^2 \backslash \Delta$ from 1.3 and restrict it to one of these curves. The resulting family is smooth over $\mathbb{P}^1 \backslash \{0, 1\}$, so by 1.4 it is isotrivial, that is, its induced map to moduli is constant. In other words, the map to moduli that ϕ induces, $\mathbb{P}^2 \backslash \Delta \to \mathfrak{M}_h$, has a positive-dimensional fibre through a general point and hence cannot be generically finite. This proves 1.3 for $B = \mathbb{P}^2$. □

A similar, elementary reduction argument should work for $B = \mathbb{P}^n$ for arbitrary n, but unfortunately the analyses of the renegade cases becomes more and more complicated, so a general proof does not really qualify for being elementary.

The first obstacle is that taking tangent hyperplane sections does not reduce the degree, so one cannot appeal to the lower-dimensional results directly.

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This, however, is not a serious problem, as the real key to make this work is choosing hyperplane cuts that reduce the difference between the degree of a component and the multiplicity of a fixed point on it. When this difference becomes zero, then one obtains a cone. In that case any general line through the vertex of the cone meets the cone in just that one point. More generally, the number of intersection points of this component and a general line through the fixed point is at most 1 higher than the difference between the degree of the component and the multiplicity of the fixed point.

Using this idea it is easy to give a proof of the case $B=\mathbb{P}^3$ that is similar to the above, but in higher dimensions some difficulties arise. The biggest problem appears when the intersection with a tangent hyperplane has irreducible components that do not contain the original point to which the tangent hyperplane belongs. This problem can most likely be treated through detailed analysis of the possibilities and by careful choice of the general points, but since we are going to give a short general proof anyway, it does not seem appropriate to include a cumbersome proof of a special case. Nevertheless, the idea of taking hyperplane sections with high intersection multiplicities should be noted as a way to reduce the dimension of the base and still carry on many of the assumptions.

One might also think that restricting to lines may limit our options. However, lines are the best candidates to intersect other varieties at the fewest possible number of points. Therefore it is quite the opposite; the reason this proof works is because of the abundance of lines on projective spaces.

2.2. The Case of a Smooth Hypersurface of Maximal Degree

Another special case that can be treated via elementary methods is when $B=\mathbb{P}^n$ and $\Delta\subseteq\mathbb{P}^n$ is a smooth hypersurface of degree $d=n+1$. Let Δ be defined by the homogeneous polynomial $f\in k[x_0,\dots,x_n]$, and let $P\in\Delta$ be a general point. Choose coordinates in such a way that $P=[1:0:\cdots:0]$, and express f as a polynomial of x_0,

$$f=f_1x_0^{d-1}+f_2x_0^{d-2}+\cdots+f_{d-1}x_0+f_d,$$

such that for each j, $f_j\in k[x_1,\dots,x_n]$ is a homogeneous polynomial of degree j.

We are going to define a series of pairs B_i, Δ_i, where B_i is a codimension i linear subspace of $\mathbb{P}^n$ and $\Delta_i=\Delta|_{B_i}$ is a divisor on B_i.

Let $B_0=B$, $\Delta_0=\Delta$ and $r_0=1$. Assuming that B_i and Δ_i are already defined, by abuse of notation, let f still denote the restriction of f to the homogeneous coordinate ring of B_i, which is isomorphic to $k[x_0, \dots, x_{n-i}]$. Again, choose coordinates on B_i, such that $P = [1:0:\cdots:0]$, and express f as a polynomial of x_0,

$$f = f_{r_i} x_0^{d-r_i} + f_{r_i+1} x_0^{d-r_i-1} + \cdots + f_{d-1} x_0 + f_d,$$

such that for each j, $f_j \in k[x_1, \dots, x_{n-i}]$ is a homogeneous polynomial of degree j. Let $B_{i+1} \subseteq B_i$ be a hyperplane defined by a linear polynomial l_i, such that $P \in B_{i+1}$. Choose B_{i+1} in such a way that it is tangent to Δ_i at P, that is, l_i divides f_{r_i}. Finally, let $\Delta_{i+1} = \Delta_i|_{B_{i+1}} = \Delta|_{B_{i+1}}$.

Notice that since Δ is smooth of degree $n+1$, it is not uniruled and hence does not contain a rational curve, in particular a line or a linear subspace, through a general point. Therefore, $B_{i+1} \not\subseteq \Delta$, and hence Δ_{i+1} is indeed a divisor on B_{i+1}. Further notice that by the choice of B_{i+1}, the multiplicity of P on Δ_{i+1} is higher than on Δ_i, i.e., $r_{i+1} > r_i$, and hence $r_i \geq i+1$.

Next consider $B_{n-1} \simeq \mathbb{P}^1$. By the argument above, the multiplicity of P on Δ_{n-1} is at least n. However, the degree of Δ_{n-1} is $n+1$, and hence the support of Δ_{n-1} contains at most one point other than P.

It is clear that as P runs through the general points of Δ, the line B_{n-1} sweeps a dense subset of $B = \mathbb{P}^n$.

Therefore, as in the previous case, we established that through a general point of B there exists a rational curve meeting Δ in at most two points.

Now we can run the same argument as before. Consider $\phi : U \to B \setminus \Delta$ from 1.3 and restrict it to one of these curves. The resulting family is smooth over $\mathbb{P}^1 \setminus \{0, 1\}$, so by 1.4 its induced map to moduli is constant. Therefore, the map to moduli that ϕ induces, $B \setminus \Delta \to \mathfrak{M}_h$, has a positive-dimensional fibre through a general point and hence cannot be generically finite. This finishes the proof of this case. □

2.3. Proof of 1.2: General Case

Finally, here is the aforementioned, almost embarrassingly short proof. Assume that $\omega_B(\Delta)^{-1}$ is pseudo-effective. Then B_0 is covered by images of $\mathbb{P}^1 \setminus \{0, 1\}$ by Keel and McKernan (1995, 5.4).

In other words, through a general point of B there exists a rational curve meeting Δ in at most two points. Let $C \subseteq B$ be a rational curve meeting Δ in at most two points and consider the map to moduli, $\eta : B_0 \to \mathfrak{M}_h$, induced by f_0. By 1.4, η maps C to a point, in other words,

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η has a positive-dimensional fibre through any general point and hence cannot be generically finite. This finishes the proof of 1.2. □

One should note that 1.1 is sharp in a certain way: Beauville (1981) gave an example of a smooth family of curves over $\mathbb{P}^1 \backslash \{0, 1, \infty\}$, whose map to the appropriate moduli space is an embedding. This implies that for $\dim B = 1$ one cannot expect anything better.

Using Beauville's families and Hurwitz's theorem one can construct a family of canonically polarized varieties of dimension n over the base

$$B_0 = \prod_{i=1}^{n} (\mathbb{P}^1 \backslash \{0, 1, \infty\}),$$

whose associated map to the appropriate moduli space is an embedding. Let

$$B = \prod_{i=1}^{n} \mathbb{P}^1 \quad \text{and} \quad \Delta = B \backslash B_0.$$

Then $\omega_B(\Delta)$ is ample, and in a rather strong sense it is minimal with respect to this property. Indeed, any line bundle $\mathscr{L}$ on B is a product of line bundles $\mathscr{L}_i$, pulled back from the different components, and $\mathscr{L}$ is ample if and only if $\mathscr{L}_i$ is the pull back of an ample line bundle for all i. This, however, means that any ample line bundle on B actually contains $\omega_B(\Delta)$. Incidentally, changing Δ such that $\omega_B(\Delta)$ is no longer ample means that the degree of Δ on one of the components is lowered to be at most two. Then any family will have to be constant in this direction by 1.4, in which case the associated map to moduli is not generically finite.

This shows that for example over $B = \prod_{i=1}^{n} \mathbb{P}^1$ one cannot expect a stronger statement than 1.1. It would, however, be interesting to see whether there is a similar example over a base with Picard number one, in particular over $\mathbb{P}^n$. It would also be interesting to see an example that is not constructed via products of families of curves.

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REFERENCES

Beauville, A. (1981). Le nombre minimum de fibres singulières d'une courbe stable sur $\mathbb{P}^1$. *Astérisque* 86:97–108.

Bedulev, E., Viehweg, E. (2000). On the Shafarevich conjecture for surfaces of general type over function fields. *Inv. Math.* 139:603–615.

Keel, S., McKernan, J. (1995). Rational curves on quasi-projective varieties. *Memoirs of the AMS* 669, viii+153 pp.

Kovács, S. J. (1996). Smooth families over rational and elliptic curves. *J. Alg. Geom.* 5:369–385.

Kovács, S. J. (1997a). Families over a base with a birationally nef tangent bundle. *Math. Ann.* 308:347–359.

Kovács, S. J. (1997b). On the minimal number of singular fibres in a family of surfaces of general type. *J. Reine Angew. Math.* 487: 171–177.

Kovács, S. J. (2000). Algebraic hyperbolicity of fine moduli spaces. *J. Alg. Geom.* 9:165–174.

Kovács, S. J. (2001). Families of varieties of general type: the Shafarevich conjecture and related problems. Lecture notes, "Workshop on higher dimensional varieties and rational points" Rényi Institute.

Kovács, S. J. (2002). Logarithmic vanishing theorems and Arakelov-Parshin boundedness for singular varieties. *Comp. Math.* 131(3): 291–317.

Kovács, S. J. Vanishing theorems, boundedness and hyperbolicity over higher dimensional bases. *Proc. AMS*, to appear.

Migliorini, L. (1995). A smooth family of minimal surfaces of general type over a curve of genus at most one is trivial. *J. Alg. Geom.* 4: 353–361.

Oguiso, K., Viehweg, E. (2001). On the isotriviality of families of elliptic surfaces. *J. Alg. Geom.* 10:569–598.

Viehweg, E. (2001). Positivity of direct image sheaves and applications to families of higher dimensional manifolds. Lecture notes, "School on Vanishing Theorems and Effective Results in Algebraic Geometry", ICTP Lecture Notes Series **VI** 249–284.

Viehweg, E., Zuo, K. (2001a). On the isotriviality of families of projective manifolds over curves. *J. Alg. Geom.* 10:781–799.

Viehweg, E., Zuo, K. (2001b). On the Brody hyperbolicity of moduli spaces for canonically polarized manifolds. Preprint, 36 pp.

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Viehweg, E., Zuo, K. (2002). Base spaces of non-isotrivial families of smooth minimal models. In: *Complex Geometry, Collection of Papers dedicated to Hans Grauert*. Springer, pp. 279–328.
Zhang, Q. (1997). Global holomorphic one-forms on projective manifolds with ample canonical bundles. *J. Alg. Geom.* 6:777–787.

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