THE INTUITIVE DEFINITION OF DU BOIS SINGULARITIES

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To Gerard van der Geer on the occasion of his 60th birthday

Gerard van der Geer-nek a 60-dik születésnapja alkalmából

Abstract. It is proved that for projective varieties having Du Bois singularities is equivalent to the condition that the coherent cohomology groups of the structure sheaf coincide with the appropriate Hodge components of the singular cohomology groups.

Kivonat. A cikk fő eredménye a következő: Egy projektív varietásnak pontosan akkor vannak Du Bois szingularitásai, ha a struktúra kéve koherens kohomológia csoportjai megegyeznek a szinguláris kohomológia csoportok megfelelő Hodge komponenseivel.

1. Introduction

If X is a smooth proper variety, then Hodge theory tells us that there is a strong link between toplogical (say singular) and analytic (say Dolbeault) cohomology. In particular, there is a surjective map

$$H^i(X, \mathbb{C}) \twoheadrightarrow H^i(X, \mathscr{O}_X). \tag{1.1}$$

This seemingly innocent fact has far reaching consequences: it plays an important role in the proof of the Kodaira vanishing theorem [Kol87] and has some nice consequences for deformations of smooth proper varieties.

Because of the usefulness of this map we are interested in finding out how this could be extended to (some) singular varieties. Let us first recall where this map comes from.

For a smooth proper variety, the Hodge-to-de-Rham (a.k.a. Frölicher) spectral sequence degenerates at E_1 hence the singular cohomology group $H^i(X, \mathbb{C})$ admits a Hodge filtration

$$H^i(X, \mathbb{C}) = F^0 H^i(X, \mathbb{C}) \supseteq F^1 H^i(X, \mathbb{C}) \supseteq \dots \tag{1.2}$$

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and in particular there exists a natural surjective map

$$H^i(X,\mathbb{C}) \twoheadrightarrow Gr^0_F H^i(X,\mathbb{C}) \tag{1.3}$$

where

$$Gr^0_F H^i(X,\mathbb{C}) \simeq H^i(X,\mathscr{O}_X). \tag{1.4}$$

Deligne's theory of (mixed) Hodge stuctures implies that even if X is singular (but still proper) there still exists a Hodge filtration and (1.3) remains true, but in general (1.4) fails.

However, there is something one can still say in general: Even if X is singular (but still proper) there exist natural maps between these groups; namely the map from (1.3) factors through $H^i(X,\mathscr{O}_X)$ (see (2.3) for a more precise statement):

$$H^i(X,\mathbb{C}) \xrightarrow[\beta]{} H^i(X,\mathscr{O}_X) \xrightarrow[\gamma]{} Gr^0_F H^i(X,\mathbb{C}), \quad \alpha = \gamma\circ\beta \ (\alpha: H^i(X,\mathbb{C}) \twoheadrightarrow Gr^0_F H^i(X,\mathbb{C})). \tag{1.5}$$

Du Bois singularities were introduced by Steenbrink to identify the class of singularities for which γ in the above diagram is an isomorphism, that is, those for which (1.4) remains true as well. However, naturally, one does not define a class of singularities by properties of proper varieties. Singularities should be defined by local properties and Du Bois singularities are indeed defined locally. For the definition see (2.4).

It is known that rational singularities are Du Bois (conjectured by Steenbrink and proved in [Kov99]) and so are log canonical singularities (conjectured by Kollár and proved in [KK10]). These properties make Du Bois singularities very important in higher dimensional geometry, especially in moduli theory (see [Kol11] for more details on applications).

Unfortunately the definition of Du Bois singularities is rather technical. The most important and useful fact about them is the consequence of (1.3) and (1.4) that if X is a proper variety over $\mathbb{C}$ with Du Bois singularities, then the natural map

$$H^i(X,\mathbb{C}) \twoheadrightarrow H^i(X,\mathscr{O}_X) \tag{1.6}$$

is surjective.

One could try to take this as a definition, but it would not lead to a good result for two reasons. As mentioned earlier, singularities should be defined locally and it is not at all likely that a global cohomological assumption would turn out to be a local property. Second, this particular condition could obviously hold "accidentally" and lead to the inclusion of singular spaces that should not be included, thereby further lowering the chances of having a local description of this class of singularities.

Therefore the reasonable approach is to keep Steenbrink's original definition, after all it has been proven to define a useful class. It does satisfy the first requirement above: it is defined locally. Once that is accepted, one might still wonder if proper varieties with Du Bois singularities could be characterized with a property that is close to requiring that (1.6) holds.

The main result of the present paper is exactly a characterization like that.

As we have already observed, simply requiring that (1.6) holds is likely to lead to a class of singularities that is too large. A more natural requirement is to ask that (1.4) holds, or in other words that γ is an isomorphism. Clearly, (1.4) implies (1.6) by (1.5), so our goal requirement is indeed satisfied.

The definition [Ste83, (3.5)] of Du Bois singularities easily implies that if X has Du Bois singularities and $H \subset X$ is a general member of a basepoint-free linear system, then H has Du Bois singularities as well. Therefore it is reasonable that in trying to give an intuitive definition of Du Bois singularities, one may assume that the defining condition holds for the intersection of general members of a fixed basepoint-free linear system.

I will prove here that this is actually enough to characterize Du Bois singularities (see (2.4) for their definition). This result is not geared for applications, it is mainly interesting from a philosophical point of view. It says that the local definition not only achieves the desired property for proper varieties, but does it in an economical way: it does not allow more than it has to.

At the same time, a benefit of this characterization is the fact that for the uninitiated reader this provides a relatively simple criterion without the use of derived categories or resolutions directly. In fact, one can make the condition numerical. This is a trivial translation of the "real" statement, but further emphasizes the simplicity of the criterion.

In order to do this we need to define some notation: Let X be a proper algebraic variety over $\mathbb{C}$ and consider Deligne's Hodge filtration $F^{\bullet}$ on $H^i(X,\mathbb{C})$ as in (1.2). Let

$$Gr_F^p H^i(X,\mathbb{C}) = F^p H^i(X,\mathbb{C}) \big/ F^{p+1} H^i(X,\mathbb{C})$$

and

$$\mathsf{f}^{\mathsf{p,i}}(X) = \dim_{\mathbb{C}} Gr_F^p H^i(X, \mathbb{C}).$$

I will also use the usual notation

$$\mathsf{h}^{\mathsf{i}}(X, \mathscr{O}_X) = \dim_{\mathbb{C}} H^i(X, \mathscr{O}_X).$$

Recall (cf. (1.5)) that the natural surjective map from $H^i(X, \mathbb{C})$ factors through $H^i(X, \mathscr{O}_X)$:

$$H^i(X, \mathbb{C}) \longrightarrow H^i(X, \mathscr{O}_X) \longrightarrow Gr_F^0 H^i(X, \mathbb{C}).$$

In particular, the natural morphism

$$H^i(X, \mathscr{O}_X) \twoheadrightarrow Gr_F^0 H^i(X, \mathbb{C}) \tag{1.7}$$

is also surjective and hence

$$\mathsf{h}^{\mathsf{i}}(X, \mathscr{O}_X) \geq \mathsf{f}^{\mathsf{0,i}}(X). \tag{1.8}$$

Now we are almost ready for the main theorem. It essentially says that if the opposite inequality of (1.8) holds for general complete intersections, then the ambient variety has Du Bois singularities.

The following definition will be used throughout the article:

Definition 1.9. Let X be a proper variety over $\mathbb{C}$. A *linear system* $\mathfrak{d}$ is the collection of effective (Cartier) divisors linearly equivalent to a fixed Cartier divisor. If X is not normal an effective Cartier divisor is defined as a subscheme defined by a single non-zero divisor at each point. If $\mathscr{L}$ is a line bundle, then global sections of $\mathscr{L}$ define effective Cartier divisors. A linear system is *basepoint-free* if for every $x \in X$ there exists a member of the linear system that does not contain x.

Theorem 1.10. *Let X be a proper variety over $\mathbb{C}$ with a fixed basepoint-free linear system $\mathfrak{d}$. (For instance, X is projective with a fixed projective embedding). Then X has Du Bois singularities if and only if $\mathsf{h}^{\mathsf{i}}(L, \mathscr{O}_L) \leq \mathsf{f}^{\mathsf{0,i}}(L)$ for $i > 0$ for any $L \subseteq X$ which is the intersection of a set of general members of $\mathfrak{d}$. (X is included among these as the intersection of the empty set of general members of $\mathfrak{d}$).*

Corollary 1.11. *Let $X \subseteq \mathbb{P}^N$ be a projective variety over $\mathbb{C}$ with only isolated singularities. Then X has only Du Bois singularities if and only if $\mathsf{h}^{\mathsf{i}}(X, \mathscr{O}_X) \leq \mathsf{f}^{\mathsf{0,i}}(X)$ for $i > 0$.*

Proof. As X has only isolated singularities, a general hyperplane section is smooth and does not contain any of the singular points. Hence as soon as $\mathsf{h}^{\mathsf{i}}(X, \mathscr{O}_X) \leq \mathsf{f}^{\mathsf{0,i}}(X)$ one also has that $\mathsf{h}^{\mathsf{i}}(L, \mathscr{O}_L) \leq \mathsf{f}^{\mathsf{0,i}}(L)$ for any $L \subseteq X$ which is the intersection of general hyperplanes in $\mathbb{P}^N$. Therefore the statement follows from (1.10). □

These statements reiterate the fact that singularities impose restrictions on global cohomological conditions. In particular one has the following ad hoc consequence:

Corollary 1.12. *Let $X \subseteq \mathbb{P}^N$ be a projective variety over $\mathbb{C}$ with only isolated singularities. Assume that $\mathsf{h}^{\mathsf{i}}(X, \mathscr{O}_X) = 0$ for $i > 0$. Then X has only Du Bois singularities.*

Proof. As $\mathsf{f}^{0,\mathsf{i}}(X) \geq 0$, the statement follows from (1.11). □

Observe that (1.7) combined with the condition $\mathsf{h}^{\mathsf{i}}(L, \mathscr{O}_L) \leq \mathsf{f}^{0,\mathsf{i}}(L)$ implies that $H^i(L, \mathscr{O}_L) \to Gr^0_F H^i(L, \mathbb{C})$ is an isomorphism and hence (1.10) follows from the following.

Theorem 1.13. *Let X be a proper variety over $\mathbb{C}$ with a fixed basepoint-free linear system $\mathfrak{d}$. Then X has only Du Bois singularities if and only if for any $L \subseteq X$, which is the intersection of a (possibly empty) set of general members of $\mathfrak{d}$, the natural map,*

$$\nu_i = \nu_i(L) : H^i(L, \mathscr{O}_L) \to Gr^0_F H^i(L, \mathbb{C})$$

given by Deligne's theory[1] *is an isomorphism for all i.*

Remark 1.14. It is clear that if X has only Du Bois singularities then $\nu_i(L)$ is an isomorphism for all L. Therefore the interesting statement of the theorem is that the condition above implies that X has only Du Bois singularities.

Theorem 1.13 will be proven in two steps. A reduction step showing that it is enough to prove the statement in the case when the non-Du Bois locus is isolated (3.6) and the proof in that special case (3.8).

Definitions and Notation 1.15. Unless otherwise stated, all objects are assumed to be defined over $\mathbb{C}$, all schemes are assumed to be of finite type over $\mathbb{C}$ and a morphism means a morphism between schemes of finite type over $\mathbb{C}$.

Let X be a complex scheme (i.e., a scheme of finite type over $\mathbb{C}$) of dimension n. Let $D_{\text{filt}}(X)$ denote the derived category of filtered complexes of $\mathscr{O}_X$-modules with differentials of order ≤ 1 and $D_{\text{filt,coh}}(X)$ the subcategory of $D_{\text{filt}}(X)$ of complexes K, such that for all i, the cohomology sheaves of $Gr^i_{\text{filt}} K^\bullet$ are coherent cf. [DB81], [GNPP88]. Let $D(X)$ and $D_{\text{coh}}(X)$ denote the derived categories with the same definition except that the complexes are assumed to have the trivial filtration. The superscripts $+, -, b$ carry the usual meaning (bounded below, bounded above, bounded). Isomorphism in these categories is denoted by $\simeq_{\text{qis}}$. A sheaf $\mathscr{F}$ is also considered as a complex $\mathscr{F}^\bullet$ with

[1]see [Del71, Del74, Ste83, GNPP88] (cf. (1.5), (2.3))

$\mathscr{F}^0 = \mathscr{F}$ and $\mathscr{F}^i = 0$ for $i \neq 0$. If $K^\bullet$ is a complex in any of the above categories, then $h^i(K^\bullet)$ denotes the i-th cohomology sheaf of $K^\bullet$.

The right derived functor of an additive functor F, if it exists, is denoted by $\mathcal{R}F$ and $\mathcal{R}^iF$ is short for $h^i \circ \mathcal{R}F$. Furthermore $\mathbb{H}^i$ will denote $\mathcal{R}^i\Gamma$, where Γ is the functor of global sections

2. Hyperresolutions and Du Bois' original definition

We will start with Du Bois's generalized De Rham complex, an object of $D_{\text{filt}}(X)$. The original construction of the Deligne-Du Bois's complex, $\underline{\Omega}^\bullet_X$, is based on simplicial resolutions. The reader interested in the details is referred to the original article [DB81]. Note also that a simplified construction was later obtained in [Car85] and [GNPP88] via the general theory of polyhedral and cubic resolutions. An easily accessible introduction can be found in [Ste85].

The word "hyperresolution" will refer to either simplicial, polyhedral, or cubic resolution. Formally, the construction of $\underline{\Omega}^\bullet_X$ is the same regardless the type of resolution used and no specific aspects of either types will be used.

Theorem 2.1 [DB81, 3.1, 3.2, 3.10, 4.5, 4.11]. *Let X be a complex scheme of finite type. Then there exists an object $\underline{\Omega}^\bullet_X \in \operatorname{Ob} D_{\text{filt}}(X)$, unique up to quasi-isomorphism, such that using the notation*

$$\underline{\Omega}^p_X := Gr^p_{\text{filt}}\, \underline{\Omega}^\bullet_X[p],$$

it satisfies the following properties

(2.1.1) $\underline{\Omega}^\bullet_X \simeq_{qis} \mathbb{C}_X$, *i.e., $\underline{\Omega}^\bullet_X$ is a resolution of the constant sheaf $\mathbb{C}$ on X.*

(2.1.2) $\underline{\Omega}^\bullet_{(\_\_)}$ *is functorial, i.e., if $\phi\colon Y \to X$ is a morphism of proper complex schemes of finite type, then there exists a natural map ϕ^* of filtered complexes*

$$\phi^*\colon \underline{\Omega}^\bullet_X \to R\phi_*\underline{\Omega}^\bullet_Y.$$

Furthermore, $\underline{\Omega}^\bullet_X \in \operatorname{Ob}\left(D^b_{filt,coh}(X)\right)$ and if ϕ is proper, then ϕ^* is a morphism in $D^b_{filt,coh}(X)$.

(2.1.3) *Let $U \subseteq X$ be an open subscheme of X. Then*

$$\underline{\Omega}^\bullet_X\big|_U \simeq_{qis} \underline{\Omega}^\bullet_U.$$

(2.1.4) *If X is proper, there exists a spectral sequence degenerating at E_1 and abutting to the singular cohomology of X such that the resulting filtration coincides with Deligne's Hodge filtration:*

$$E_1^{pq} = \mathbb{H}^q\left(X, \underline{\Omega}^p_X\right) \Rightarrow H^{p+q}(X, \mathbb{C}).$$

In particular,

$$Gr_F^p H^{p+q}(X,\mathbb{C}) \simeq \mathbb{H}^q\left(X, \underline{\Omega}^p_X\right).$$

(2.1.5) *If $\varepsilon_\bullet : X_\bullet \to X$ is a hyperresolution, then*

$$\underline{\Omega}^\bullet_X \simeq_{qis} \mathcal{R}\varepsilon_{\bullet\,*}\Omega^\bullet_{X_\bullet}.$$

In particular, $h^i\left(\underline{\Omega}^p_X\right) = 0$ for $i < 0$.

(2.1.6) *There exists a natural map, $\mathscr{O}_X \to \underline{\Omega}^0_X$, compatible with (2.1.2).*

(2.1.7) *If X is smooth, then*

$$\underline{\Omega}^\bullet_X \simeq_{qis} \Omega^\bullet_X.$$

In particular,

$$\underline{\Omega}^p_X \simeq_{qis} \Omega^p_X.$$

(2.1.8) *If $\phi\colon Y \to X$ is a resolution of singularities, then*

$$\underline{\Omega}^{\dim X}_X \simeq_{qis} \mathcal{R}\phi_*\omega_Y.$$

(2.1.9) *If $\pi : \widetilde{Y} \to Y$ is a projective morphism, $X \subset Y$ is a reduced closed subscheme such that π is an isomorphism outside of X, E is the reduced subscheme of $\widetilde{Y}$ with support equal to $\pi^{-1}(X)$, and $\pi' : E \to X$ is the induced map, then for each p one has an exact triangle in the derived category,*

$$\underline{\Omega}^p_Y \longrightarrow \underline{\Omega}^p_X \oplus \mathcal{R}\pi_*\underline{\Omega}^p_{\widetilde{Y}} \xrightarrow{\ -\ } \mathcal{R}\pi'_*\underline{\Omega}^p_E \xrightarrow{+1}.$$

Corollary 2.2. *Let X be a complex scheme of finite type and $H \subset X$ a general member of a basepoint-free linear system. Then $\underline{\Omega}^\bullet_H \simeq_{qis} \underline{\Omega}^\bullet_X \otimes_L \mathscr{O}_H$ and hence in particular $\underline{\Omega}^0_H \simeq_{qis} \underline{\Omega}^0_X \otimes_L \mathscr{O}_H$.*

Proof. Let $\varepsilon_\bullet : X_\bullet \to X$ be a hyperresolution. Since H is general, it is a reduced effective Cartier divisor and the fiber product $X_\bullet \times_X H \to H$ provides a hyperresolution of H. Then the statement follows from (2.1.5) applied to both X and H. □

It turns out that the Deligne-Du Bois complex behaves very much like the de Rham complex for smooth varieties. Observe that (2.1.4) says that the Hodge-to-de Rham spectral sequence works for singular varieties if one uses the Deligne-Du Bois complex in place of the de Rham complex. This has far reaching consequences and if the associated graded pieces, $\underline{\Omega}^p_X$ turn out to be computable, then this single property leads to many applications.

Observation 2.3. Notice that (2.1.6) gives a natural map $\mathscr{O}_X \to \underline{\Omega}^0_X$. This implies that the natural map $H^i(X,\mathbb{C}) \to \mathbb{H}^i(X,\underline{\Omega}^0_X)$, which is surjective when X is proper because of the degeneration at E_1 of the spectral sequence in (2.1.4), factors as

$$H^i(X,\mathbb{C}) \longrightarrow H^i(X,\mathscr{O}_X) \longrightarrow \mathbb{H}^i(X,\underline{\Omega}^0_X) = Gr^0_F H^i(X,\mathbb{C}).$$

The induced map $H^i(X,\mathscr{O}_X) \to Gr^0_F H^i(X,\mathbb{C})$ is the one that appears in (1.13).

Definition 2.4. A scheme X is said to have *Du Bois singularities* (or *DB singularities* for short) if the natural map $\mathscr{O}_X \to \underline{\Omega}^0_X$ from (2.1.6) is a quasi-isomorphism.

Remark 2.5. If $\varepsilon_{\bullet} : X_{\bullet} \to X$ is a hyperresolution of X then X has Du Bois singularities if and only if the natural map $\mathscr{O}_X \to \mathscr{R}\varepsilon_{\bullet\,*}\mathscr{O}_{X_{\bullet}}$ is a quasi-isomorphism.

Example 2.6. It is easy to see that smooth points are Du Bois and Deligne proved that normal crossing singularities are Du Bois as well cf. [DJ74, Lemme 2(b)].

3. The proof of (1.13)

As observed in (1.14), we only need to prove that if for every $i > 0$ and for every $L \subseteq X$ which is the intersection of general members of $\mathfrak{d}$, the natural map

$$\nu_i : H^i(L,\mathscr{O}_L) \to Gr^0_F H^i(L,\mathbb{C}) \tag{3.1}$$

is an isomorphism, then X has Du Bois singularities.

Observation 3.2. Note that it follows that ν_i is an isomorphism for all $i \in \mathbb{Z}$. Indeed, both sides are zero for $i < 0$ and have the same dimension for $i = 0$. Since ν_i is surjective this implies the claim.

Definition 3.3. Let X be a complex scheme of finite type and let $\Sigma_X \subseteq X$ denote the locus of points where X does not have Du Bois singularities, i.e., Σ_X is the smallest closed subset of X such that $X \setminus \Sigma_X$ has Du Bois singularities. Using this notation X has Du Bois singularities if and only if $\Sigma_X = \emptyset$.

Definition 3.4. [Kov11, 2.9] The *DB defect* of X, denoted by $\underline{\Omega}^{\times}_X$, is defined as the mapping cone of the natural morphism $\mathscr{O}_X \to \underline{\Omega}^0_X$. In other words, by definition there exists an exact triangle,

$$\mathscr{O}_X \longrightarrow \underline{\Omega}^0_X \longrightarrow \underline{\Omega}^{\times}_X \xrightarrow{+1} . \tag{3.4.1}$$

Observe that $\Sigma_X = \operatorname{supp}(\underline{\Omega}^{\times}_X) = \cup_i \operatorname{supp} h^i(\underline{\Omega}^{\times}_X)$ and X has Du Bois singularities if and only if $\underline{\Omega}^{\times}_X \simeq_{\text{qis}} 0$.

Lemma 3.5. *Let X be a complex scheme of finite type and $H \subset X$ a reduced effective Cartier divisor such that $\underline{\Omega}^{\times}_H \simeq_{qis} \underline{\Omega}^{\times}_X \otimes_L \mathscr{O}_H$ holds. Then $\Sigma_H = \Sigma_X \cap H$.*

Proof. If $\underline{\Omega}^{\times}_X \simeq_{\text{qis}} 0$, then so is $\underline{\Omega}^{\times}_H \simeq_{\text{qis}} \underline{\Omega}^{\times}_X \otimes_L \mathscr{O}_H$ and hence $\Sigma_H \subseteq \Sigma_X \cap H$.

Next let $P \in X \setminus \Sigma_H$. Then $\underline{\Omega}^{\times}_H \simeq_{\text{qis}} 0$ in a neighborhood of P, so in the same neighborhood we have that $\underline{\Omega}^{\times}_X \otimes_L \mathscr{O}_X(-H) \simeq_{\text{qis}} \underline{\Omega}^{\times}_X$. Since $\mathscr{O}_X(-H)$ is locally free this implies that for all i,

$$h^i(\underline{\Omega}^{\times}_X) \simeq h^i(\underline{\Omega}^{\times}_X) \otimes \mathscr{O}_X(-H),$$

which in turn implies that H is disjoint from $\operatorname{supp} h^i(\underline{\Omega}^{\times}_X)$ for all i in this neighborhood of P and hence $P \notin \Sigma_X \cap H$ and so the statement follows. □

Corollary 3.6. *Let X be a complex scheme of finite type and $H \subset X$ a general member of a basepoint-free linear system. Then $\Sigma_H = \Sigma_X \cap H$.*

Proof. By (2.2) $\underline{\Omega}^0_H \simeq_{\text{qis}} \underline{\Omega}^0_X \otimes_L \mathscr{O}_H$, so by (3.4.1) $\underline{\Omega}^{\times}_H \simeq_{\text{qis}} \underline{\Omega}^{\times}_X \otimes_L \mathscr{O}_H$ and hence the statement follows from (3.5). □

Remark 3.7. The last statement fails if H is not general since there exist non-Du Bois hypersurfaces [Kov99, 3.6]. However, the implication $\Sigma_H \supseteq \Sigma_X \cap H$ holds for arbitrary Cartier divisors by [KS11, 4.1].

As our goal is to prove that $\Sigma_X = \emptyset$, using (3.5) we may replace X with an intersection of general members of $\mathfrak{d}$ and assume that Σ_X is finite. In other words, (1.13) follows from the following special case:

Theorem 3.8. *Let X be a proper variety over $\mathbb{C}$ and assume that there exists a finite set $\Sigma \subseteq X$ such that $X \setminus \Sigma$ has Du Bois singularities. Then X has Du Bois singularities if and only if*

$$\nu_i : H^i(X, \mathscr{O}_X) \to Gr^0_F H^i(X, \mathbb{C})$$

is an isomorphism for all i.

Proof. By (3.2) and (2.1.4), $H^i(X, \mathscr{O}_X) \xrightarrow{\simeq} \mathbb{H}^i(X, \underline{\Omega}^0_X)$ is an isomorphism for all $i \in \mathbb{Z}$, and hence

$$\mathbb{H}^i(X, \underline{\Omega}^{\times}_X) = 0 \tag{3.8.1}$$

for all $i \in \mathbb{Z}$. On the other hand there exists a spectral sequence computing $\mathbb{H}^i(X, \underline{\Omega}^{\times}_X)$:

$$H^p(X, h^q(\underline{\Omega}^{\times}_X)) \Rightarrow \mathbb{H}^{p+q}(X, \underline{\Omega}^{\times}_X).$$

Observe that $\operatorname{supp} h^q(\underline{\Omega}^{\times}_X) \subseteq \Sigma_X \subseteq \Sigma$ and hence it is 0-dimensional. Consequently

$$H^p(X, h^q(\underline{\Omega}^{\times}_X)) = 0$$

for $p > 0$, and therefore

$$\mathbb{H}^i(X, \underline{\Omega}^{\times}_X) = H^0(X, h^i(\underline{\Omega}^{\times}_X)) = h^i(\underline{\Omega}^{\times}_X)$$

for all $i \in \mathbb{Z}$. Comparing with (3.8.1) we obtain that $h^i(\underline{\Omega}^{\times}_X) = 0$ for all $i \in \mathbb{Z}$ and hence $\underline{\Omega}^{\times}_X \simeq_{\text{qis}} 0$. By the definition of the DB defect this implies (cf. (3.4.1)) that X has Du Bois singularities. □

This proves (1.13) by the argument preceding (3.8). By (1.14) that implies (1.10), so all desired statements are now proven.

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