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The cone of curves of a K3 surface

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Let X be a normal projective variety defined over the complex field. The homology class of a curve d on X will be denoted by $[d]$. Let

$$NE(X) = \left\{ \sum_{i=1}^{n} a_i[c_i] | c_i \text{ is an irreducible proper curve on } X, a_i \in \mathbb{R}, a_i \geqq 0 \right\}$$

$NE(X)$ is the *cone of curves of* X. The closed cone of curves of X – denoted by $\overline{NE}(X)$ – is the closure of $NE(X)$ in $H_2(X, \mathbb{R})$. $\overline{NE}(X)^\circ$ and $\partial\overline{NE}(X)$ denote the interior and the boundary of $\overline{NE}(X)$ in the vector space it spans in $H_2(X, \mathbb{R})$.

The aim of this article is to prove the following:

Theorem 1. *Let X be a smooth algebraic K3 surface with Picard number at least three. Then one of the following conditions is satisfied:*
(*) *X does not contain any curve of negative self-intersection.*
(**) $\overline{NE}(X) = \overline{\sum \mathbb{R}_+[\ell]}$ *where the sum runs over all smooth rational curves on X.*

In the next section after introducing the necessary notations, a complete classification of the possible cases will be given.

1 Notations and statement of the main results

Let X be a projective surface and h be an ample class on X. Let

$$\mathcal{Q}(X) = \{\xi \in H_2(X, \mathbb{R}) | (\xi \cdot \xi) = 0, (\xi \cdot h) \geqq 0\}.$$

A $\mathbb{Q}$-divisor on X is a $\mathbb{Q}$-linear combination of divisors on X. $NS(X)$ denotes the *Néron-Severi group*, that is, the image of $\mathrm{Pic}(X)$ in $H^2(X, \mathbb{Z})$. The *Picard* *number of* X, denoted by $\rho(X)$, is the rank of $NS(X)$, i.e. the dimension of $NS(X) \otimes \mathbb{Q}$.

A homology class that can be represented by an effective (resp. irreducible, ample) divisor is called *effective* (resp. *irreducible*, *ample*). An effective class is *indecomposable* if it is not the sum of two other effective classes. Note, that by definition an indecomposable class can be represented by an effective divisor. h will always mean an ample class.

A $K3$ *surface* is a two-dimensional compact complex Kähler manifold with trivial canonical class and such that its first Betti number is zero.

An *algebraic* $K3$ *surface* is a normal algebraic surface such that its minimal smooth resolution is a K3 surface.

Let X be a smooth algebraic K3 surface. The classes of the (-2)-curves – smooth rational curves of self-intersection -2 – are called the *nodal classes.* The set of (-2)-curves is denoted by $\mathcal{N}(X)$ and the set of irreducible rational curves of self-intersection zero is denoted by $\mathcal{E}(X)$.

Now with these definitions and notations the main result can be stated.

Theorem 2. *Let* X *be a smooth algebraic* $K3$ *surface. Then*

2.1 *One of the following statements holds*:

(i) $\rho(X) = 1$ *and* $\overline{NE}(X) = \mathbb{R}_+ h$, *where* h *is an ample class.*

(ii) $\rho(X) = 2$ *and* $\overline{NE}(X) = \mathbb{R}_+[e] + \mathbb{R}_+[\ell]$, *where* $e \in \mathcal{E}(X)$ *and* $\ell \in \mathcal{N}(X)$.

(iii) $2 \leqq \rho(X) \leqq 4$, $\overline{NE}(X) = Conv(\mathcal{Q}(X)))$ *and* $\partial\overline{NE}(X)$ *does not contain any effective class, i.e.* X *contains neither smooth rational nor smooth elliptic curves.*

(iv) $2 \leqq \rho(X) \leqq 11$, $\overline{NE}(X) = Conv(\mathcal{Q}(X))) = \overline{\sum_{e\in\mathcal{E}(X)} \mathbb{R}_+[e]}$ *in particular* X *does not contain a smooth rational curve.*

(v) $2 \leqq \rho(X) \leqq 20$ *and* $\overline{NE}(X) = \overline{\sum_{\ell\in\mathcal{N}(X)} \mathbb{R}_+[\ell]}$

2.2 *All of these cases do occur for every indicated value of* $\rho(X)$.

Corollary 1. *Let* X *be a smooth algebraic* $K3$ *surface. Then* $\overline{NE}(X)$ *is either circular or has no circular part at all.*

Corollary 2. *Let* X *be a smooth algebraic* $K3$ *surface such that* $\rho(X) \geqq 2$. *Then either* $\partial\overline{NE}(X)$ *does not contain any* $\mathbb{Q}$*-divisor class* (*i.e. there are no rational curves of self-intersection* 0 *or* -2) *or* $NE(X)$ *is generated by the homology classes of rational curves of self-intersection* 0 *or* -2.

Corollary 3. *Let* Y *be a singular algebraic* $K3$ *surface, then* $NE(Y)$ *is generated by rational curves.*

Throughout the article X will denote a smooth algebraic K3 surface.

X will be called *of type* (i)–(v) according to which case occurs for it in Theorem 2.

The rest of this section is devoted to fixing some notation and recalling an important result.

The *rank of* a cone is the dimension of the vector space it spans. A cone of rank 1 is called a *ray.* A vector of a convex cone $v \in C$ is *extremal* if

$u+w \in \mathbb{R}_+ v$ and $u, w \in C$ imply, that $u, w \in \mathbb{R}_+ v$. *Extremal ray* is a ray generated by an extremal vector. Let $C \subset \mathbb{R}^p$ be a closed convex cone. Let ∂C denote the boundary of C in $\mathbb{R}^p$ and let $v \in \partial C$. C is *locally finitely generated at* v if there exists a closed subcone B of C and finitely many vectors $v_1, \dots v_n \in C$ such that $v \notin B$ and C is generated by B and $\{v_1, \dots, v_n\}$. Let U be a nonempty open subset of ∂C. $\mathbb{R}_+ U$ is a *circular part of* C, if C is not locally finitely generated at any point of U. C is *circular* if ∂C is a circular part of C. By abuse of language, a cone will be said to be *contained in an open half space* if the cone minus the origin is contained in an open half space given by a hyperplane, that contains the origin. If A is a subset of a real vector space, *Conv*(A) will denote the convex hull of A.

A *lattice* $(\Lambda, <,>)$ is a free $\mathbb{Z}$-module of finite rank equipped with a $\mathbb{Z}$-valued symmetric bilinear form $<,>$. The *discriminant of* Λ, denoted discr(Λ) is the determinant of the matrix of its bilinear form. Λ is *non-degenerate* if discr$(\Lambda) \neq 0$. Λ is *unimodular* if discr$(\Lambda) = \pm 1$. Λ is *even* if for every $x \in \Lambda, \langle x, x\rangle \in 2\mathbb{Z}$ and it is *odd* otherwise. Let $(\Lambda, <,>)$ be a lattice. $-\Lambda$ shall mean the same module Λ, equipped with a bilinear form that is -1 times the one of Λ.

An embedding $\Sigma \hookrightarrow \Lambda$ of lattices is *primitive* if Λ/Σ is free. A sublattice is called *primitive* if it is the image of a primitive embedding.

Example 1.1 U denotes the *hyperbolic plane*, that is U is a free $\mathbb{Z}$-module of rank 2 whose bilinear form has matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Example 1.2. E_8 denotes the unique even unimodular positive definite lattice of rank 8. For the explicit description of its bilinear form see [BPV, I.2.7] or [Ser, V.1.4.2].

From standard results it is easy to see, that $H^2(X, \mathbb{Z})$ is torsion-free of rank 22 and, when equipped with the cupproduct pairing, isometric to $U^3 \oplus (-E_8)^2$, which is a unimodular lattice of signature (3, 19) and $\rho(X)$ ranges from 1 to 20.

The *transcendental lattice of* X, denoted by T_X, is the orthogonal complement of $NS(X)$ in $H^2(X, \mathbb{Z})$. Then the Hodge Index Theorem implies, that $NS(X)$ and T_X has signature $(1, \rho(X)-1)$ and $(2, 20-\rho(X))$ respectively.

To prove the existence of certain $K3$ surfaces the following result of D. Morrison will be needed.

Theorem 1.3 [Mor, 1.9]. *Suppose S is a primitive sublattice of $L = U^3 \oplus (-E_8)^2$ of signature $(1, \rho-1)$. Then there exists an algebraic $K3$ surface X such that $NS(X) = S$.*

Corollary 1.4 [Mor, 2.9(i), 2.11]. *If $\rho \leqq 11$, then every even lattice S of signature $(1, \rho-1)$ occurs as the Néron-Severi group of some algebraic $K3$ surface.*

2 Convex cones and sets

Lemma 2.1. 2.1.1 *If C is a closed convex cone in $\mathbb{R}^p$, then every vector in C is the sum of vectors which are extremal in C.*

2.1.2 *If A is a cone in $\mathbb{R}^p$ such that $\overline{Conv(A)}$ is contained in an open half space, then $\overline{Conv(A)} = Conv(\bar{A})$*

2.1.3 *Every effective class can be written as a sum of indecomposable classes.*

2.1.4 *If A and B are subcones of $\overline{NE}(X), C = A + B$ and $\bar{A}, \bar{B} \subset C$, then C is closed.*

Proof. (i) and (ii) are easy exercises. For (iii) and (iv) one can use Kleiman's criterion for ampleness which guarantees the necessary compactness of a cross section of the cone. Q.E.D.

Lemma 2.2. 2.2.1 *Let $\ell_n \in \mathcal{N}(X), \lambda_n \in \mathbb{R}_+, \xi \in \overline{NE}(X)$ such that $\lambda_n \ell_n \to \xi$. Assume that $\{\ell_n | n \in \mathbb{N}\}$ is an infinite set, then*

$$(\xi \cdot \xi) = 0 .$$

2.2.2

$$\overline{\sum_{\ell \in \mathcal{N}(X)} \mathbb{R}_+[\ell]} \subset \sum_{\ell \in \mathcal{N}(X)} \mathbb{R}_+[\ell] + Conv(\mathcal{Q}(X))$$

Proof. Let ℓ be an arbitrary nodal class. For infinitely many $n, \ell_n \neq \ell$, so $(\ell_n \cdot \ell) \geqq 0$ and then $(\xi \cdot \ell) \geqq 0$. Apply this for every ℓ_n to see, that in fact $(\xi \cdot \xi) \geqq 0$, but it is also $\leqq 0$, since $(\lambda_n \ell_n \cdot \lambda_n \ell_n) < 0$. This proves 2.2.1 and that

$$\partial \left(\bigcup_{\ell \in \mathcal{N}(X)} \mathbb{R}_+[\ell] \right) \subset \bigcup_{\ell \in \mathcal{N}(X)} \mathbb{R}_+[\ell] \bigcup \mathcal{Q}(X) .$$

Lemma 2.1(ii) finishes the proof. Q.E.D.

Corollary 2.3. 2.3.1 $\overline{NE}(X) = \sum_{\ell \in \mathcal{N}(X)} \mathbb{R}_+[\ell] + Conv(\mathcal{Q}(X))$

2.3.2 $\overline{NE}(X) \cap \{\xi \in H_2(X, \mathbb{R}) | (\xi \cdot \xi) < 0\}$ *is locally finitely generated.*

2.3.3 *If ξ is extremal in $\overline{NE}(X)$, but it is not a multiple of a nodal class, then $(\xi \cdot \xi) = 0$.*

Proof. By [BPV, VIII.3.6] and [CKM, (4.4), (4.5)]

$$NE(X) \subset \sum_{\ell \in \mathcal{N}(X)} \mathbb{R}_+[\ell] + Conv(\mathcal{Q}(X)) .$$

Then Lemma 2.2 and Lemma 2.1(iv) implies 2.3.1.

Let h be a fixed ample class and $\varepsilon > 0$. By 2.2.1 there are only finitely many nodal classes that are not contained in $Conv(\mathcal{Q}_\varepsilon(X))$, where $\mathcal{Q}_\varepsilon(X) = \{\xi \in H_2(X, \mathbb{R}) | (\xi \cdot \xi) \geqq -\varepsilon(\xi \cdot h)^2, (\xi \cdot h) > 0\}$. Then by 2.3.1

$$\overline{NE}(X) = \sum_{i=1}^{n} \mathbb{R}_+[\ell_i] + \overline{NE}(X) \cap Conv(\mathcal{Q}_\varepsilon(X)). \qquad \text{Q.E.D.}$$

If an effective class is extremal, then it is clearly indecomposable, but not vice versa. However if it is indecomposable and of negative self-intersection, then it is a nodal class, so it is extremal, too. The next result shows that the same happens if it is of self-intersection zero.

Proposition 2.4. *Let e be an effective divisor class of self-intersection zero. Then e is indecomposable if and only if it is extremal.*

Proof. Let e be an indecomposable divisor class of self-intersection zero. $|e|$ defines an elliptic pencil that covers X, so if d is any other irreducible class, $(e \cdot d)$ will be positive, which means that e is numerically effective and an effective $\mathbb{Q}$-divisor class can have zero intersection product with e if and only if it is a multiple of it. Let $(e = 0) = \{\xi \in H_2(X, R) | (e \cdot \xi) = 0\}$. Now $(e = 0)$ is a supporting hyperplane that contains no other effective class than the multiples of e. Then $(e = 0) \cap \mathcal{Q}(X) = \mathbb{R}_+ e$.

Suppose now that $e = \sum_{i=1}^{n} \sigma_i, \sigma_i$ extremal. Since $(e = 0)$ is a supporting hyperplane, $\sigma_i \in (e = 0)$. $(e = 0)$ contains no other effective class than the multiples of e, so by Corollary 2.3 $\sigma_i \in \mathcal{Q}(X)$. Then $\sigma_i \in (e = 0) \cap \mathcal{Q}(X) = \mathbb{R}_+ e$ and e is extremal. Q.E.D.

Corollary 2.5. *Let $a \in \partial \overline{NE}(X)$ be indecomposable. Then a is extremal.*

The next lemma is true in some greater generality, but this is the form that will be needed later.

Lemma 2.6. *Let $Q \subset \mathbb{R}^p$ be a smooth compact quadratic hypersurface and $C \subset \mathbb{R}^p$ be a compact convex set. Assume, that $Q \not\subset C$, then there exists a U nonempty subset of Q such that $U \subset \partial Conv(Q \cup C)$.*

Proof. Let $q \in Q \backslash C$ and L be a hyperplane in $\mathbb{R}^p$, that separates q and C. There exists a $u \in Q \backslash C$ such that the tangent hyperplane of Q at u is parallel to L and then it is disjoint from C, so $u \in \partial Conv(Q \cup C)$. It is easy to see, that there is a neighborhood U of u such that the tangent hyperplane of any point of U is disjoint from C, so $U \subset \partial Conv(Q \cup C)$. Q.E.D.

3 Subcones of $\overline{NE}(X)$ of rank 2

The following lemma will play a very important role. Among other consequences it implies, that if the Picard number is two, then either none of the rays on the boundary or both of them can be generated by effective classes. It also implies that if $\overline{NE}(X)$ has a circular part and X contains a-2-curve, then it contains an effective divisor of self-intersection zero.

Lemma 3.1. *Let e, d be effective classes such that $e \in \partial\overline{NE}(X)$ and $(d \cdot d) > 0$. Let π be the plane generated by e and d in $H_2(X, \mathbb{R})$.*

3.1.1 *If $(e \cdot e) = 0$, then there exists an $f \in \pi \cap \overline{NE}(X)$, such that f is effective of self-intersection 0 and e and f are on opposite sides of d.*

3.1.2 *If $(e \cdot e) = -2$, then there exists an $f \in \pi \cap \overline{NE}(X)$, such that f is effective of self- intersection 0 or -2 and e and f are on opposite sides of d.*

Proof. Let a, b, c be $(d \cdot d), (d \cdot e), (e \cdot e)$ respectively and let $x = \xi d + \eta e$. Since $e \in \partial\overline{NE}(X)$, if $\eta < 0$, then $x \notin \overline{NE}(X)$ and

$$(x \cdot x) = a\xi^2 + 2b\xi\eta + c\eta^2 .$$

If $(e \cdot e) = 0$, then

$$(x \cdot x) = a\xi^2 + 2b\xi\eta = 0$$

has two rational solutions and since $(d \cdot d) > 0$, they must be different, on different sides of d.

If $(e \cdot e) = -2$, then

$$(x \cdot x) = a\xi^2 + 2b\xi\eta - 2\eta^2 = -2\left(\left(b\frac{\xi}{2} - \eta\right)^2 - (2a + b^2)\left(\frac{\xi}{2}\right)^2\right) .$$

Now if $(2a + b^2)$ is not a square, then this equation has infinitely many integral solutions for $(x \cdot x) = -2$ [IR, Pell's equation 17.5.2]. So there exist positive integers u, v such that

$$u^2 - (2a + b^2)v^2 = 1 .$$

Since $a = (d \cdot d) > 0, u^2 > b^2v^2$, so $u > bv$. Set $\xi = 2v$ and $\eta = bv - u$ and let $f = \xi d + \eta e$. Then $(f \cdot f) = -2$ and by Riemann-Roch either f or $-f$ is effective. $e \in \partial\overline{NE}(X)$ and by construction $\xi > 0$ and $\eta < 0$, so $-f$ cannot be effective and e and f are on opposite sides of d.

If $(2a + b^2)$ is a square, then it has two rational solutions for $(x \cdot x) = 0$ and they are on different sides of d. Q.E.D.

Remark 3.2. 3.1.1 implies, that if $NS(X)$ represents zero, then the $\mathbb{Q}$-divisors in $\mathcal{Q}(X)$ form a dense subset of $\mathcal{Q}(X)$.

4 K3 surfaces with a circular cone

If X does not contain a smooth rational curve, then the cone will be generated by $\mathcal{Q}(X)$ and it will be circular. This section is devoted to the investigation of the circumstances under which this situation may take place.

The following result is an easy consequence of Nikulin's work on integral symmetric bilinear forms.

Lemma 4.1. *Let $\rho(X) \geqq 12$, then X contains a smooth rational curve.*

Proof. Let $t_{(+)} = 2, t_{(-)} = 20 - \rho(X) \leqq 8, l_{(+)} = 2$ and $l_{(-)} = 18$, then the conditions of [Nik 1, 1.12.4] are satisfied. By [Ser, V.2.2] $U^2 \oplus (-E_8)^2$ is the only unimodular lattice of signature (2, 18), so T_X admits an embedding ϕ_0 into $U^2 \oplus (-E_8)^2$.

Let $\phi = 0 \oplus \phi_0$ be the embedding of T_X into $L = U^3 \oplus (-E_8)^2$ that is the extension of ϕ_0 by zero to U. By [Ser, V.2.2] and [Nik 1, 1.14.4] the embedding of T_X into L is unique, so ϕ is isomorphic to the canonical embedding, in particular

$$U \hookrightarrow T_X^{\perp} = NS(X) = \mathrm{Pic}(X).$$

Thus there is a divisor class of self-intersection -2 and that implies the existence of a nodal class. Q.E.D.

Theorem 4.2. 4.2.1 *If $\rho(X) \geqq 5$ and X does not contain any (-2)-curve i.e. $\partial\overline{NE}(X) = \mathcal{Q}(X)$, then the $\mathbb{Q}$-divisor classes of self-intersection zero form a dense subset of $\partial\overline{NE}(X)$.*

4.2.2 *If $\partial\overline{NE}(X) = \mathcal{Q}(X)$, then $\rho(X) \leqq 11$*

4.2.3 *If $\rho \in \{2,3,4\}$, then there exists a K3 surface with $\rho(X) = \rho$ and such that $\partial\overline{NE}(X) = \mathcal{Q}(X)$ does not contain any $\mathbb{Q}$-divisor class.*

4.2.4 *If $\rho \in \{2,3,\ldots,11\}$, then there exists a K3 surface with $\rho(X) = \rho$ and such that $\partial\overline{NE}(X) = \mathcal{Q}(X)$ and the $\mathbb{Q}$-divisor classes of self-intersection zero form a dense subset of $\partial\overline{NE}(X)$.*

Proof. If $\rho(X) \geqq 5$, then the Hasse-Minkowski Theorem [Ser, IV.3.2] implies that there are effective classes of self-intersection zero and then Remark 3.2 proves 4.2.1.

4.2.2 follows directly from Lemma 4.1.

Let $\rho \in \{2,3,4\}$. It is easy to see – looking at it modulo 8 – that $q(x_1,\ldots,x_\rho) = 7x_1^2 - \sum_{i=2}^{\rho} x_i^2$ does not represent zero, so $4q$ is an even quadratic form of rank ρ and of signature $(1, \rho - 1)$, that represents neither 0 nor -2. By Corollary 1.4 this implies 4.2.3.

To prove 4.2.4, take $4(x_1^2 - \sum_{i=2}^{\rho} x_i^2)$ and use Corollary 1.4, Corollary 2.3 and Remark 3.2. Q.E.D.

5 K3 surfaces with Picard number two

Theorem 5.1. *Let $\rho(X) = 2$ and let $\overline{NE}(X) = \mathbb{R}_+\xi + \mathbb{R}_+\eta$. Then one of the following statements holds:*

(a) *Neither $\mathbb{R}_+\xi$ nor $\mathbb{R}_+\eta$ contains any effective classes.*

(b) *Both $\mathbb{R}_+\xi$ and $\mathbb{R}_+\eta$ contains an effective class of 0 or -2 self-intersection.*

Furthermore, all of these cases do occur.

Proof. By Lemma 3.1 it is easy to see, that there is no other possibility. By Theorem 4.2 there are K3 surfaces of type (a) and type (b) of both self-intersection being zero. Now take the following two quadratic forms: $-2x_1^2 + 8x_1x_2 - 2x_2^2$ and $-2x_1^2 + 6x_1x_2$. By Corollary 1.4 they give rise to K3 surfaces of type (b) of both self-intersection being -2 and of self-intersection 0 and -2 respectively. Q.E.D.

The two quadratic forms and corresponding cones, mentioned in the proof can be realized easily. For the first one take a general degree four surface in $\mathbb{P}^3$ containing a smooth conic. The second one is the cone of a general degree four surface in $\mathbb{P}^3$ containing a line.

6 K3 surfaces containing a -2 curve

In this section $\rho(X)$ will be assumed to be at least three and $\mathcal{N}(X)$ to be not empty i.e. X contains a smooth rational curve ℓ. This will imply the existence of several other smooth rational curves.

Theorem 6.1. *Let $\rho(X) \geqq 3$, suppose X contains a smooth rational curve and let ξ be extremal in $\overline{NE}(X)$. Then*

6.1.1 $\overline{NE}(X)$ has no circular part.

6.1.2 There exists a sequence $\{\ell_n\} \subset \mathcal{N}(X)$ such that $\mathbb{R}_+[\ell_n] \to \mathbb{R}_+\xi$.

6.1.3 $\overline{NE}(X) = \sum_{\ell \in \mathcal{N}(X)} \mathbb{R}_+[\ell]$.

Proof. Let ℓ denote the class of the smooth rational curve, that is guaranteed by the assumption. Let $\sigma : X \to X_0$ be the map that contracts ℓ to a point P. Since $\rho(X) \geqq 3, \rho(X_0) \geqq 2$. Let h be ample on X_0 and d be a divisor on X_0, independent from h. Replacing d with $d + nh$ d may be assumed to be base point free. So h and d can be represented in $X_0 \backslash \{P\} \simeq X \backslash \ell$. Then they give two divisors d_1, d_2 on X such that $(\ell \cdot d_1) = (\ell \cdot d_2) = 0$ and d_1, d_2, ℓ are independent in Pic(X).

Suppose U is a nonempty open subset of $\partial\overline{NE}(X)$ such that $\mathbb{R}_+U$ is a circular part of $\overline{NE}(X)$. By Corollary 2.3 $U \subset \mathcal{Q}(X)$. Then in a neighborhood of any point of $\mathbb{R}_+U$ every effective class is of nonnegative self-intersection.

By Lemma 3.1 find an effective divisor $e \in \mathbb{R}_+U \cap \mathcal{Q}(X)$. e is extremal and as in the proof of Proposition 2.4, $(e = 0)$ contains no other effective classes than the multiples of e, in particular $(\ell \cdot e) \neq 0$.

Let d be d_1 or d_2 such that e, d, ℓ be independent. Let $\delta = (d \cdot d), \alpha = (d \cdot e)$ and $\beta = (\ell \cdot e)$, then $\alpha \neq 0, \beta \neq 0$ and $(d \cdot \ell) = 0$.

Suppose $2\alpha^2 - \delta\beta^2 = 0$. Let $f = \delta\beta e - \alpha\beta d + \alpha^2\ell$. Then $(f \cdot f) = \alpha^2(\delta\beta^2 - 2\alpha^2) = 0$ and $(f \cdot e) = 0$. By Riemann-Roch f or $-f$ is effective and then since $(e = 0)$ contains no other effective classes than the multiples of $e, \alpha\beta = \alpha^2 = 0$ so $(d \cdot e) = \alpha = 0$ which is impossible. So $2\alpha^2 - \delta\beta^2 \neq 0$.

Let $n \in \mathbb{N}$ and let

$$\pm d_n = (2(2\alpha^2 - \partial\beta^2)\beta n^2 - 4\alpha n)e + (2\beta^2 n)d + (1 - 2\alpha\beta n)\ell \ .$$

Easy computation shows that $(d_n \cdot d_n) = -2$, so by Riemann-Roch either d_n or $-d_n$ is effective. Let d_n be effective. Since $(2\alpha^2 - \delta\beta^2)\beta \neq 0, \mathbb{R}_+ d_n \to \mathbb{R}_+ e$.

Since in a neighborhood of $\mathbb{R}_+ e$ every effective class is of nonnegative self-intersection, this is a contradiction and 6.1.1 is proven.

Let $\xi \in \mathcal{Q}(X)$ be extremal and $H = \{\eta \in H_2(X, \mathbb{R}) | (\eta \cdot h) = (\xi \cdot h)\}$, where h is an ample class. $Q = H \cap \mathcal{Q}(X)$ is a smooth compact quadratic hypersurface and $\xi \in Q$. Let $N = \{v \in H | \exists \ell \in \mathcal{N}(X), v \in \mathbb{R}_+[\ell]\}$ and $C = Conv(\bar{N}) = \overline{Conv(N)}$.

It is easy to see, that if there were a $U \subset Q$ nonempty open subset such that $U \subset \partial Conv(Q \cup C)$, then $\mathbb{R}_+ U$ would be a circular part of $\overline{NE}(X)$, so by 6.1.1 and Lemma 2.6 Q is a subset of C. Then $\xi \in C = Conv(\bar{N})$ and since ξ is extremal, $\xi \in \bar{N}$. This implies 6.1.2.

If $\eta \in \overline{NE}(X)$ arbitrary, let $\eta = \sum_{i=1}^{n} \eta_i, \eta_i$ extremal. Then

$$\eta \in \overline{\sum_{\ell \in \mathcal{N}(X)} \mathbb{R}_+[\ell]} \,. \qquad \text{Q.E.D.}$$

Corollary 6.2. *Theorem 1 holds true.*

7 Proof of Theorem 2

Lemma 7.1. *Let $e \in \mathcal{Q}(X)$ be an indecomposable class. Then there is an irreducible rational curve on X that represents e.*

Proof. e defines an elliptic fibration of $X, \vartheta : X \to \mathbb{P}^1$. If this fibration had only non-singular fibres then $e(X)$ would be zero by the formula [BPV, III.11.4]. Since it is 24 ([BPV, VIII.3.1]), there must be a singular fibre, too. Its arithmetic genus is one and it is singular, so it must be rational and since e is indecomposable, it must be irreducible. Q.E.D.

Proof of Theorem 2. 2.1 If $\rho(X) = 1$ then X is of type (i). If $\rho(X) = 2$, then by Theorem 5.1 X is of type (ii), (iii), (iv) or (v). If $\rho(X) \geqq 3$ and X contains a smooth rational curve, then by Theorem 6.1 it is of type (v). If $\rho(X) \geqq 3$ and X does not contain a smooth rational curve, then by Theorem 4.2 and Lemma 7.1 it is of type (iii) or (iv).

2.2 By the Noether-Lefschetz Theorem a general degree four surface in $\mathbb{P}^3$ is of type (i). By Theorem 4.2 and Theorem 5.1 types (ii), (iii), (iv) and (v) with $\rho(X) = 2$ do occur.

If $3 \leqq \rho \leqq 11$ let M be any primitive sublattice of $(-E_8)^2$ of rank $\rho - 2$. Then $U \oplus M$ and Lemma 4.1 shows that for any $3 \leqq \rho \leqq 20$ there exists a primitive sublattice of L of signature $(1, \rho - 1)$ containing a vector x with $(x \cdot x) = -2$. Then by Corollary 1.4 there exists a K3 surface with $\rho(X) = \rho$ containing a divisor of self-intersection -2 and then X contains a smooth rational curve. Therefore by the first part of the Theorem it is of type (v).

Q.E.D.

Proof of Corollary 1. Assume, that $\rho(X) \geqq 3$. If X is of type (iii) or (iv), then $\overline{NE}(X)$ is circular. If X is of type (v), then it follows from 6.1.1. Q.E.D.

Proof of Corollary 2. If X is of type (ii) or (iii), then the statement is true. If X is of type (iv) or (v), then

$$\overline{NE}(X) = \overline{\sum_{f \in \mathcal{N}(X) \cup \mathcal{E}(X)} \mathbb{R}_+[f]}\,.$$

Now let d be an effective class. It can be written as a sum of indecomposable classes and by Riemann-Roch they are of self-intersection at least -2. So $d = d_1 + d_2$ such that $(d_1 \cdot d_1) > 0$ and

$$d_2 \in \sum_{f \in \mathcal{N}(X) \cup \mathcal{E}(X)} \mathbb{R}_+[f]\,.$$

Then by [CKM, (4.4), (4.5)] $d_1 \in \overline{NE}(X)^\circ$.

$$\overline{\sum_{f \in \mathcal{N}(X) \cup \mathcal{E}(X)} \mathbb{R}_+[f]} \Big\backslash \sum_{f \in \mathcal{N}(X) \cup \mathcal{E}(X)} \mathbb{R}_+[f] \subset \partial \left(\sum_{f \in \mathcal{N}(X) \cup \mathcal{E}(X)} \mathbb{R}_+[f] \right),$$

so $d_1 \in \sum_{f \in \mathcal{N}(X) \cup \mathcal{E}(X)} \mathbb{R}_+[f]$ since if C is a convex set, then $\partial C = \partial \bar{C}$.
Q.E.D.

Proof of Corollary 3. Let X be the minimal smooth resolution of Y. Then it contains a smooth rational curve so $NE(X)$ is generated by rational curves. Q.E.D.

Remark 7.2. It is well known, that for $\rho(X) \geqq 3$, $\overline{NE}(X)$ is polyhedral if and only if Aut(X) is finite ([PS-S, Ste]). Nikulin has classified the lattices that occur as the Picard group of a K3 surface that has finite automorphism group ([Nik 2, Nik 3, Nik 4]). Shioda and Inose proved that a K3 surface of Picard number 20 has an infinite automorphism group ([Sh-In]).

Using these facts one can see easily, that there exists a K3 surface X such that $\rho(X) = \rho$ and X contains finitely many -2-curves i.e. $NE(X)$ is a closed polyhedral cone if and only if $1 \leqq \rho \leqq 19$ and for every $\rho, 3 \leqq \rho \leqq 20$ there exists a K3 surface X such that $\rho(X) = \rho$ and X contains infinitely many -2-curves i.e. $\overline{NE}(X)$ is non-polyhedral of type (v). In particular every Kummer surface gives such an example.

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