The Cone of Curves of K3 Surfaces Revisited

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1 Introduction

The following theorem was proved in [4] over the complex numbers. It turns out that the proof given there works with very small adjustments in arbitrary characteristic. The main difference is that while in the original article we worked in a real homology group of the surface in question, here everything takes place in the group of $\mathbb{R}$-cycles modulo numerical equivalence. The arguments are essentially the same.

As already noted the purpose of this note is to verify the above statement, that is, to prove the following.

Theorem 1.1. *Let X be a K3 surface of Picard number at least three over an algebraically closed field of arbitrary characteristic. Then one of the following mutually exclusive conditions are satisfied:*

(1.1.1) *X does not contain any curve of negative self-intersection.*

(1.1.2) $\overline{NE}(X) = \overline{\sum \mathbb{R}_+ \ell}$ *where the sum runs over the classes of all smooth rational curves on X.*

Remark 1.2. To cover all cases one would also need to consider K3 surfaces with Picard number less than 3. The case of Picard number 1 is trivial and the case of Picard number 2 is handled in Corollary 3.2.

DEFINITIONS AND NOTATION 1.3. Let k be an algebraically closed field of arbitrary characteristic. Everything will be defined over k.

A K3 surface is a smooth projective surface X such that $\omega_X \simeq \mathscr{O}_X$ and $h^1(X, \mathscr{O}_X) = 0$.

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Let $Z_1(X)$ denote the free abelian group generated by the irreducible and reduced 1-dimensional subvarieties of X. Elements of this group are called 1-*cycles* on X. Two 1-cycles $C_1, C_2 \in Z_1(X)$ are called *numerically equivalent* if for any Cartier divisor D on X, the intersection numbers $D \cdot C_1$ and $D \cdot C_2$ agree. This relationship is denoted by $C_1 \equiv C_2$. All 1-cycles numerically equivalent to the 0-cycle form a subgroup of $Z_1(X)$, and the quotient is denoted by $N_1(X)_{\mathbb{Z}}$. By extension of scalars we define

$$N_1(X) := N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}.$$

The effective 1-cycles in $N_1(X)$ generate a subsemigroup denoted by $NE(X) \subseteq N_1(X)$. This subsemigroup is called the *cone of effective curves*. The *closed cone of effective curves* is the closure of this in $N_1(X)$:

$$\overline{NE}(X) := \overline{NE(X)} \subseteq N_1(X).$$

For more details about the construction and basic properties of this, one should consult [2, II.4].

From now on X is assumed to be a smooth projective surface defined over k. A class $\xi \in N_1(X)$ is called *integral* if it can be represented by a divisor on X. It is called *effective* (respectively *ample*) if it is integral and can be represented by an effective (respectively ample) divisor. A class $\xi \in \overline{NE}(X)$ is called *extremal* if it cannot be written as the sum of two incomparable classes in $\overline{NE}(X)$. The class of a smooth rational curve is called a *nodal* class. The set of all nodal classes is denoted by $\mathscr{N}(X)$.

Let h be an ample class and define

$$\mathscr{Q}(X) := \{\xi \in N_1(X) \mid \xi \cdot h > 0, \xi \cdot \xi = 0\}.$$

Note that by the positivity condition this is just half of a quadric cone.

The convex hull of a set will be denoted by Conv.

An open subset of the boundary of the cone is called *circular* if the cone is not locally finitely generated at any point in the open set.

2 Simple Facts

Let us start with an easy, well-known consequence of the Riemann-Roch theorem:

Lemma 2.1. *Let $a \in N_1(X)$ be an integral class. Then $a \cdot a$ is an even integer. If furthermore $a \cdot a \geq -2$, then either a or $-a$ is effective.*

Proof. As a is integral, there exists a line bundle $\mathscr{L}$ representing a. By Riemann-Roch

$$h^0(X, \mathscr{L}) - h^1(X, \mathscr{L}) + h^0(X, \mathscr{L}^\vee) = \frac{1}{2} a \cdot a + 2 \qquad (2.1.1)$$

and hence $a \cdot a$ has to be even.

If $a \cdot a \geq -2$, then the right hand side of Eq. (2.1.1) is positive, so either $h^0(X, \mathscr{L}) > 0$ or $h^0(X, \mathscr{L}^\vee) > 0$, so either a or $-a$ is effective. □

Corollary 2.2. *$\mathscr{Q}(X) \subset \overline{NE}(X)$ and for any $e \in \mathscr{Q}(X)$ the hyperplane $(e \cdot \_\_ = 0) = \{\xi \in N_1(X) \mid e \cdot \xi = 0\}$ is a supporting hyperplane of* $\mathrm{Conv}(\mathscr{Q}(X))$.

Proof. The first statement follows directly from Lemma 2.1 and since for any $\xi \in \mathrm{Conv}(\mathscr{Q}(X))$ (effective) irreducible class, $\xi \cdot e \geq 0$, this also implies the second statement. □

Corollary 2.3. *Let $e, d \in \overline{NE}(X)$ such that $e \cdot e = 0$ and $d \cdot d > 0$. Then d is in the interior of $\overline{NE}(X)$ and $e \cdot d > 0$.*

Proof. Since d is in the interior of $\mathscr{Q}(X)$ and $\mathscr{Q}(X) \subset \overline{NE}(X)$, it follows that d is in the interior of $\overline{NE}(X)$. Then since $(e \cdot \_\_ = 0)$ is a supporting hyperplane, d cannot be contained in it and hence $e \cdot d > 0$. □

Next we establish that nodal rays can only accumulate along the cone generated by $\mathscr{Q}(X)$.

Lemma 2.4. *Let $\{\ell_n\} \subset \mathscr{N}(X)$ be an infinite sequence of nodal classes such that*

$$\mathbb{R}_+ \ell_n \xrightarrow[n \to +\infty]{} \mathbb{R}_+ \xi \quad \text{for some } \xi \in \overline{NE}(X). \text{ Then } \xi \cdot \xi = 0.$$

Proof. Let $\ell \in \mathscr{N}(X)$. Then for infinitely many $n \in \mathbb{N}$, $\ell_n \neq \ell$, so $\ell_n \cdot \ell \geq 0$ and hence $\xi \cdot \ell \geq 0$. Applying this with $\ell = \ell_n$ yields that $\xi \cdot \xi \geq 0$. On the other hand since $\ell_n \cdot \ell_n < 0$, it follows that $\xi \cdot \xi \leq 0$ as well and so we must have $\xi \cdot \xi = 0$. □

Corollary 2.5. *Let $h \in \overline{NE}(X)$ be an ample class and $\varepsilon > 0$ a real number. Define*

$$\mathscr{Q}_\varepsilon(X) := \{\xi \in N_1(X) \mid \xi \cdot h = 1, \xi \cdot \xi \geq -\varepsilon\}.$$

Then the number of nodal classes not contained in $\mathrm{Conv}(\mathscr{Q}_\varepsilon(X))$ *is finite.*

Proof. The set $\{\xi \in \overline{NE}(X) \mid \xi \cdot h = 1\}$ is compact by Kleiman's criterion [3, 1.18] and hence any infinite set contained in it has an accumulation point. By Lemma 2.4 all accumulation points have to be contained in $\mathscr{Q}_\varepsilon(X)$, which implies the desired statement. □

Corollary 2.6. *Let $\xi \in \overline{NE}(X)$ be an extremal class which is not a multiple of a nodal class. Then $\xi \cdot \xi = 0$.*

Proof. It follows from Riemann-Roch and [2, II.4.14] that

$$\overline{NE}(X) = \sum_{\ell \in \mathscr{N}(X)} \mathbb{R}_+ \ell + \mathrm{Conv}(\mathscr{Q}(X)), \tag{2.6.1}$$

in particular $\xi \cdot \xi \leq 0$, and then for any $\varepsilon > 0$,

$$\overline{NE}(X) = \sum_{\substack{\ell \in \mathscr{N}(X) \\ \ell \notin \operatorname{Conv}(\mathscr{Q}_\varepsilon(X))}} \mathbb{R}_+ \ell + (\overline{NE}(X) \cap \operatorname{Conv}(\mathscr{Q}_\varepsilon(X))), \qquad (2.6.2)$$

where the above sum is finite by Corollary 2.5.

Suppose that $\xi \cdot \xi < 0$. Without loss of generality we may assume that $\xi \cdot h = 1$ and choose an $\varepsilon > 0$ such that $\xi \notin \operatorname{Conv}(\mathscr{Q}_\varepsilon(X))$. Then ξ is a multiple of a nodal class by Eq. (2.6.2). □

3 Subcones Generated by Two Elements

The following is a simple, but important computation.

Lemma 3.1. *Let e, d be effective classes such that e is indecomposable and $d \cdot d > 0$. Let L be the 2-dimensional linear subspace generated by e and d in $N_1(X)$ and $\mathscr{C} = L \cap \overline{NE}(X)$. Then:*

(3.1.1) If $e \cdot e = 0$, then there exists an $f \in \mathscr{C}$ such that f is effective, $e \cdot f > 0$, e and f are on opposite sides of d, and $f \cdot f = 0$.

(3.1.2) If $e \cdot e = -2$, then there exists an $f \in \mathscr{C}$ such that f is effective, $e \cdot f > 0$, e and f are on opposite sides of d, and $f \cdot f = 0$ or $f \cdot f = -2$.

Proof. Let $A = d \cdot d$, $B = e \cdot d$, and $C = e \cdot e$. Set $f = \alpha d - \beta e$. Then

$$f \cdot f = A\alpha^2 - 2B\alpha\beta + C\beta^2.$$

If $C = e \cdot e = 0$, then $B = e \cdot d > 0$ by Corollary 2.3 and hence the equation

$$0 = A\alpha^2 - 2B\alpha\beta = \alpha(A\alpha - 2B\beta)$$

has a positive integer solution, $\alpha = 2B$, $\beta = A$ such that the class $f = \alpha d - \beta e$ has $f \cdot f = 0$. Since $\alpha d = f + \beta e$, f and e lie on opposite sides of d and it also follows that f is effective by Lemma 2.1 and since $e \cdot f > 0$.

If $C = e \cdot e = -2$, then set $x = B\alpha/2 + \beta$, $y = \alpha/2$, and $N = 2A + B^2$. Then

$$f \cdot f = A\alpha^2 - 2B\alpha\beta - 2\beta^2 = -2\left((B\alpha/2 + \beta)^2 - (2A + B^2)(\alpha/2)^2\right) = -2(x^2 - Ny^2).$$

Now if N is a square, then as above there are two effective solutions for $f \cdot f = 0$ and they are on opposite sides of d and hence one of them is on the side of d opposite to e.

If N is not a square then finding an f with $f \cdot f = -2$ is equivalent to solving Pell's equation $x^2 - Ny^2 = 1$ [1, 17.5.2]. One may choose a solution with both $x, y > 0$ which again ensures that e and f are on opposite sides of d and that $e \cdot f > 0$. This completes the proof. □

Corollary 3.2. *If $\rho(X) = 2$, let $\overline{NE}(X) = \mathbb{R}_+\xi + \mathbb{R}_+\eta$. Then one of the following mutually exclusive cases hold:*

(3.2.1) Neither $\mathbb{R}_+\xi$ nor $\mathbb{R}_+\eta$ contain any effective classes.
(3.2.2) Both $\mathbb{R}_+\xi$ and $\mathbb{R}_+\eta$ contain an effective class of 0 or -2 self-intersection.

Proof. If $\mathscr{N}(X) = \emptyset$, then the decomposition in Eq. (2.6.1) implies that $\overline{NE}(X) = \mathrm{Conv}(\mathscr{Q}(X))$. If there exists an integral (equivalently, effective) class in $\mathscr{Q}(X)$, then by (3.1.1) we are in case (3.2.2). If there are no integral classes in $\mathscr{Q}(X)$ then we are in case (3.2.1).

If $\mathscr{N}(X) \neq \emptyset$, then by the decomposition in Eq. (2.6.1) and (3.1.2) we are in case (3.2.2). □

4 K3 Surfaces Containing a Smooth Rational Curve

Theorem 4.1. *Let X be a K3 surface and $\xi \in \overline{NE}(X)$ an extremal vector. Assume that $\rho(X) \geq 3$ and X contains a smooth rational curve. Then:*

(4.1.1) $\overline{NE}(X)$ has no circular part.

(4.1.2) There exists a sequence $\ell_n \in \mathscr{N}(X)$ such that $\mathbb{R}_+\ell_n \xrightarrow[n\to+\infty]{} \mathbb{R}_+\xi$.

(4.1.3) $\overline{NE}(X) = \overline{\sum_{\ell \in \mathscr{N}(X)} \mathbb{R}_+\ell}$.

Proof. Let ℓ denote the class of a smooth rational curve, guaranteed by the assumption, and let $\sigma : X \to X'$ be the morphism contracting ℓ to a point. Recall that X' is still projective and let h_1 and h_2 be two linearly independent ample classes on X'. Let $d_i = \sigma^* h_i$ for $i = 1, 2$ and observe that d_1 and d_2 are effective classes on X such that $\ell \cdot d_1 = \ell \cdot d_2 = 0$ and ℓ, d_1, d_2 are linearly independent.

Suppose there exists $U \subset \partial\overline{NE}(X)$ a non-empty open subset of $\partial\overline{NE}(X)$ such that $\mathbb{R}_+U$ is a circular part (i.e., nowhere locally finitely generated) of $\overline{NE}(X)$. By Lemma 2.4 it follows that $U \subseteq \mathscr{Q}(X)$ and hence in a neighbourhood of $\mathbb{R}_+U$ every effective class has non-negative self-intersection.

Let h be an arbitrary ample class and observe that Lemma 3.1 implies that in the 2-dimensional linear subspace generated by h and ℓ there is an effective class f with either $f \cdot f = 0$ or $f \cdot f = -2$ and such that f is on the side of h opposite to ℓ. We may repeat the same procedure with ℓ replaced by f and h replaced by another ample class and find that these classes are all over near the boundary of $\overline{NE}(X)$. In particular, we can find an ample class $h \in \overline{NE}(X)$ and an effective class f with either $f \cdot f = 0$ or $f \cdot f = -2$ such that the 2-dimensional linear subspace generated by h and f intersects U non-trivially. Then applying Lemma 3.1 again and combining it by the observation above we obtain that there exists an effective class $e \in U$ such that $e \cdot e = 0$.

Next let d be one of d_1 and d_2 such that e, d, ℓ are linearly independent. Let $A = d \cdot d$, $B = e \cdot d$, and $C = e \cdot \ell$. Recall that by the choice of d we have $d \cdot \ell = 0$.

Claim 4.1.4. $2B^2 \neq AC^2$

Proof. Suppose $2B^2 = AC^2$ and let $f = ACe - BCd + B^2\ell$. Then $f \cdot f = B^2(AC^2 - 2B^2) = 0$ and $f \cdot e = 0$. Then by Lemma 2.1 f or $-f$ is effective. However, since $e \in U$ which is a circular part of $\partial\overline{NE}(X)$, the only effective classes contained in the hyperplane $(e \cdot \_\_ = 0)$ are multiples of e. This implies that $BC = B^2 = 0$, so $B = 0$. Applying the same argument for d, it would follow that d is a multiple of e which is impossible by the choice of d and e. Therefore we reached a contradiction and hence the claim is proven. □

Continuation of the proof of Theorem 4.1. Next let $n \in \mathbb{N}$ and for $n \geq 3$ define

$$\pm d_n = (2(2B^2 - AC^2)Cn^2 - 4Bn)e + (2C^2n)d + (1 - 2BCn)\ell.$$

Then $d_n \cdot d_n = -2$ and by Lemma 2.1 either d_n or $-d_n$ is effective. Choose d_n to be effective. By Claim 4.1.4 $2B^2 - AC^2 \neq 0$ and hence $\mathbb{R}_+ d_n \to \mathbb{R}_+ e$, but this contradicts the observation that in a neighbourhood of U every effective class has non-negative self-intersection. Therefore Theorem 4.1.1 is proven.

Now let $\xi \in \overline{NE}(X)$ extremal. If no multiple of ξ is in $\mathscr{N}(X)$, then $\xi \in \mathscr{Q}(X)$ by Corollary 2.6. If ξ were not contained in the closure of the convex cone generated by $\mathscr{N}(X)$, then by [4, 2.6] $\overline{NE}(X)$ would have a circular part, so (4.1.2) follows from (4.1.1).

Finally, since every class in $\overline{NE}(X)$ may be written as a sum of finitely many extremal classes, (4.1.3) follows from (4.1.2). □

Corollary 4.2. *Let X be a K3 surface of Picard number at least three over an algebraically closed field of arbitrary characteristic. Then one of the following mutually exclusive conditions are satisfied:*

$$\overline{NE}(X) = \operatorname{Conv}(\mathscr{Q}(X)) \tag{4.2.1}$$

$$\overline{NE}(X) = \overline{\sum_{\ell \in \mathscr{N}(X)} \mathbb{R}_+ \ell}. \tag{4.2.2}$$

Proof. If X does not contain any curve of negative self-intersection, then (4.2.1) follows from Eq. (2.6.1). Otherwise (4.2.2) follows from Theorem 4.1. □

Remark 4.3. Clearly Corollary 4.2 is equivalent to Theorem 1.1.

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References

1. K. IRELAND AND M. ROSEN, *A classical introduction to modern number theory*, second ed., Graduate Texts in Mathematics, vol. 84, Springer-Verlag, New York, 1990.
2. J. KOLLÁR, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 32, Springer-Verlag, Berlin, 1996.
3. J. KOLLÁR AND S. MORI, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.
4. S. J. KOVÁCS, *The cone of curves of a $K3$ surface*, Math. Ann. **300**, no. 4, 681–691, (1994).