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Subvarieties of moduli stacks of canonically polarized varieties: generalizations of Shafarevich's conjecture

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CONTENTS

DISCLAIMER. In order to understand and follow this article the reader *does not* need to know what a stack is. In fact, the sole point of using the word "stack" is to make it easier to talk about subvarieties of moduli spaces that are induced by families that belong to the corresponding moduli problem. This is what I mean by "subvarieties of moduli stacks" and this is the only aspect of the theory of stacks that will be relevant.

1. INTRODUCTION

Moduli theory strives to understand how algebraic varieties deform and degenerate. Studying a moduli stack tells us a lot about these properties. A basic question one is interested in is whether a given moduli stack is proper, or if it is not, then how far it is from being proper. An even more simple question one may ask about the geometry of a given moduli stack is whether it contains any proper subvarieties. If it does, what kind can that proper subvariety be?

Naturally, the same questions may be asked about moduli spaces. The difference between the two is whether one is interested in any subvariety of the moduli space or only those that come from a family that belongs to the corresponding moduli problem. The latter

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ones provide subvarieties of the moduli stack and in this article we are mainly interested in those.

Consider M_g, the moduli space of smooth projective curves of genus $g \geq 3$. M_g admits a projective compactification, the Satake compactification, with a boundary of codimension two. Taking general hyperplane sections on this compactification one finds that M_g contains a proper curve through any general point. Unfortunately, this does not give explicit families of smooth projective curves that induce a non-constant map from a proper curve to M_g. On the other hand Kodaira constructed such families in [Kod67], cf. [Kas68], [BHPV04, V.14], [Zaa95], [GDH99]. However, the images of these curves in the corresponding moduli stack $\mathfrak{M}_g$ (or in the moduli space M_g) are confined to the special locus of curves that admit non-trivial automorphisms.

These results naturally imply the following question: Are there higher dimensional proper subvarieties contained in some $\mathfrak{M}_g$? The answer is positive. Kodaira's construction can be used to prove the following: For any $d \in \mathbb{N}$ there exists a $g = g(d) \in \mathbb{N}$ such that $\mathfrak{M}_g$ contains a proper subvariety of dimension d. For details on this construction see [Mil86], [FL99, pp.34-35], [Zaa99]. These examples are all based on the aforementioned construction of Kodaira and hence the proper subvarieties constructed this way all lie in the locus of curves that admit a morphism onto another positive genus curve.

One may argue that the really interesting question is whether there are higher dimensional proper subvarieties of $\mathfrak{M}_g$ that contain a general point of $\mathfrak{M}_g$. Unfortunately this is still an open question even for surfaces, i.e., it is not known whether there are proper surfaces through a general point of $\mathfrak{M}_g$, for any $g > 3$.

Naturally, since $\dim \mathfrak{M}_g = 3g - 4$, there is an obvious upper bound on the dimension of a proper subvariety of $\mathfrak{M}_g$ for a fixed g, but one may ask whether there is a better upper bound than $3g - 3$. Actually this is one of those questions when finding the answer for the moduli space, M_g, implies the same for the moduli stack, $\mathfrak{M}_g$, and not the other way around. The celebrated theorem of Diaz-Looijenga [Dia84, Dia87, Loo95] says that any proper subvariety of M_g has dimension at most $g - 2$. This estimate is trivially sharp for $g = 2, 3$, but it is not known to be sharp for any other values of g. The known examples are very far from this bound. Recently, Faber and van der Geer [FvdG04] pointed out that in $\operatorname{char} p$ there exists a natural subvariety of M_g of expected dimension $g - 2$, and hence seems a good candidate for a proper subvariety of maximal dimension. However, they also show that this subvariety has non-proper components and hence itself is not proper. On the other hand, Faber and van der Geer express hope that it might also have proper components. This would be enough to prove that the upper bound $g - 2$ is sharp.

Similar questions may be asked about other moduli spaces/stacks, for instance, replacing curves by abelian varieties. The reader interested in this question could start by consulting [Oor74], [KS03], and [VZ05c].

In this article we are interested in somewhat more sophisticated questions. On one hand, we are not only asking whether a given moduli stack contains proper subvarieties, but we would like to know what kind of proper subvarieties it contains. For instance, does it contain proper rational or elliptic curves? Furthermore, we are also interested in non-proper subvarieties. For instance, if it does not contain a proper rational curve, does it contain one that's isomorphic to the affine line?

Interestingly, already the question of containing proper rational curves differentiates between the moduli stack, $\mathfrak{M}_g$, and the moduli space, M_g: Parshin [Par68] proved that $\mathfrak{M}_g$ does not contain proper rational curves for any g, while Oort [Oor74] showed that there exists some g such that M_g does contain proper rational curves.

Our starting point in this article is Shafarevich's conjecture (2.1). This leads us to investigate related questions and eventually to a recent generalization, Viehweg's conjecture (5.6), which states that any subvariety of the moduli stack is of log general type.

The topic of this article has gone through an enormous transformation during the last decade and consequently it is impossible to cover all the new developments in as much detail as they deserve it. Hence the reader is encouraged to consult other surveys of related interest [Vie01], [Kov03a], [MVZ05].

The following notation will be preserved throughout the entire article:

NOTATION AND DEFINITIONS 1.1. Let k be an algebraically closed field of characteristic 0, B a smooth variety over k, and $\Delta \subseteq B$ a closed subset. Unless otherwise stated, all objects will be assumed to be defined over k.

A *family* over B is variety X together with a flat projective morphisms $f\colon X \to B$ with connected fibers.

For a morphism $Y \to S$ and another morphism $T \to S$, the symbol Y_T will denote $Y \times_S T$. In particular, for $Y = X$, $S = B$ and $b \in B$ we write $X_b = f^{-1}(b)$. In addition, if $T = \operatorname{Spec} F$, then Y_T will also be denoted by Y_F.

Let $q \in \mathbb{N}$. Then $\mathfrak{M}_q$, respectively M_q, denotes the *moduli stack*, respectively the *coarse moduli space*, of smooth projective curves of genus q. Similarly $\overline{\mathfrak{M}}_q$, respectively $\overline{\mathsf{M}}_q$, denotes the *moduli stack*, respectively the *coarse moduli space*, of stable projective curves of genus q. Furthermore, $\mathfrak{M}_h$, respectively M_h, denotes the *moduli stack*, respectively the *coarse moduli space*, of smooth canonically polarized varietes with Hilbert polynomial h. We will say that M_h admits a *geometric compactification* if there exists a moduli stack $\overline{\mathfrak{M}}_h$ with a coarse moduli space $\overline{\mathsf{M}}_h$ such that $\overline{\mathsf{M}}_h$ is projective and contains M_h as an open subscheme.

A family $f\colon X \to B$ is *isotrivial* if $X_a \simeq X_b$ for any pair of general points $a, b \in B$. The family $f\colon X \to B$ will be called *admissible* (with respect to (B, Δ)) if it is not isotrivial and Δ contains the discriminant locus of f, i.e., the map $f\colon X\setminus f^{-1}(\Delta) \to B\setminus\Delta$ is smooth.

Let $\mathscr{L}$ be a line bundle on a scheme X. It is said to be *generated by global sections* if for every point $x \in X$ there exists a global section $\sigma_x \in H^0(X, \mathscr{L})$ such that the germ σ_x generates the stalk $\mathscr{L}_x$ as an $\mathscr{O}_X$-module. If $\mathscr{L}$ is generated by global sections, then the global sections define a morphism $\phi_{\mathscr{L}}\colon X \to \mathbb{P}^N = \mathbb{P}\left(H^0(X, \mathscr{L})\right)$. $\mathscr{L}$ is called *semi-ample* if $\mathscr{L}^m$ is generated by global sections for $m \gg 0$. $\mathscr{L}$ is called *ample* if it is semi-ample and $\phi_{\mathscr{L}^m}$ is an embedding for $m \gg 0$. A line bundle $\mathscr{L}$ on X is called *big* if the global sections of $\mathscr{L}^m$ define a rational map $\phi_{\mathscr{L}^m}\colon X \dashrightarrow \mathbb{P}^N$ such that X is birational to $\phi_{\mathscr{L}^m}(X)$ for $m \gg 0$. Note that in this case $\mathscr{L}^m$ is not necessarily generated by global sections, so $\phi_{\mathscr{L}^m}$ is not necessarily defined everywhere.

A smooth projective variety X is of *general type* if ω_X is big. It is easy to see that this condition is invariant under birational equivalence between smooth projective varieties. An arbitrary projective variety is of *general type* if so is a desingularization of it.

2. SHAFAREVICH'S CONJECTURE

Let B be a smooth projective curve of genus g and $\Delta \subset B$ a finite subset.

2.A. *The original conjecture*

Let us start with the aforementioned conjecture of Shafarevich [Sha63]:

2.1. SHAFAREVICH'S CONJECTURE. *Let (B, Δ) be fixed and $q \geq 2$ an integer. Then*

(2.1.1) *There exist only finitely many isomorphism classes of admissible families of curves of genus* q.

(2.1.2) *If* $2g - 2 + \#\Delta \leq 0$, *then there exist no such families.*

REMARK 2.2. The inequality in (2.1.2) can be satisfied only if B is either a rational or an elliptic curve:

$$2g - 2 + \#\Delta \leq 0 \quad \Leftrightarrow \quad \begin{cases} g = 0 & \text{and} \quad \#\Delta \leq 2, \\ g = 1 & \text{and} \quad \Delta = \emptyset. \end{cases}$$

Shafarevich showed a special case of (2.1.2): There exist no smooth families of curves of genus q over $\mathbb{P}^1$. (2.1.1) was confirmed by Parshin [Par68] for $\Delta = \emptyset$ and by Arakelov [Ara71] in general.

Our main goal is to generalize this statement to higher dimensional families. In order to do that we will have to reformulate the statement as Parshin and Arakelov did, but before doing so, we need a little bit of background on deformations and parameter spaces.

2.B. *Deformations and Parameter Spaces*

In general, deforming an object means to include that object in a family. There is a potentially confusing point here. Our main objects of study are families, that is, deformations of their members. However, we do not want to consider our families as deformations. We want to look at deformations *of* these families. This works just the same way as deformations of other objects. In addition, we want to fix the base of these families, so we are interested in deformations leaving the base fixed, which makes both the notation and the theory easier.

A *deformation* of a family $f \colon X \to B$ with the base fixed is a family $\mathscr{X} \to B \times T$, where T is connected and for some $t_0 \in T$ we have $(\mathscr{X}_{t_0} \to B \times \{t_0\}) \simeq (X \to B)$:

$$\begin{array}{ccc} X \simeq \mathscr{X}_{t_0} & \longrightarrow & \mathscr{X} \\ {\scriptstyle f}\downarrow & & \downarrow \\ B \simeq B \times \{t_0\} & \longrightarrow & B \times T. \end{array}$$

We say that two families $X_1 \to B$ and $X_2 \to B$ have the same *deformation type* if they can be deformed into each other, i.e., if there exists a connected T and a family $\mathscr{X} \to B \times T$ such that for some $t_1, t_2 \in T$, $(\mathscr{X}_{t_i} \to B \times \{t_i\}) \simeq (X_i \to B)$ for $i = 1, 2$.

We will consider deformations of admissible families. It will be advantageous to restrict to deformations of the family over $B \setminus \Delta$. Doing so potentially allows more deformations than over the original base B: it can easily happen that a deformation over $B \setminus \Delta$, that is, a family $\mathscr{X} \to (B \setminus \Delta) \times T$, cannot be compactified to a (flat) family over $B \times T$, because the compactification could contain fibers of higher than expected dimension. This however, will not cause any problems because of the nature of our inquiry.

Given a family $f \colon X \to B$, we say that B *parametrizes* the members of the family. If all members of a class $\mathfrak{C}$ of varieties appear as fibers of f and all fibers are members of $\mathfrak{C}$, then we say that B is a *parameter space* for the class $\mathfrak{C}$. Note that we do not require that the members of $\mathfrak{C}$ appear only once in the family.

A very useful parameter space is the *Hilbert scheme,* a parameter space for subschemes of $\mathbb{P}^n$. The Hilbert scheme of $\mathbb{P}^n$, $\operatorname{Hilb}(\mathbb{P}^n)$, decomposes as the disjoint union of Hilbert schemes of subschemes with a given Hilbert polynomial h. The components of this union, $\operatorname{Hilb}_h(\mathbb{P}^n)$, are projective schemes (in particular of finite type). When one is hoping to parametrize the members of a class of varieties, then the most likely way to succeed is to try to find the parameter space as a subscheme of an appropriate Hilbert scheme. For more details on Hilbert schemes see [Kol96] and [Vie95]. For more on parameter spaces see [Harr95, Lectures 4, 21].

2.C. *The Parshin-Arakelov reformulation*

With regard to Shafarevich's conjecture, Parshin made the following observation. In order to prove that there are only finitely many admissible families, one can try to proceed the following way. Instead of aiming for the general statement, first try to prove that there are only finitely many deformation types. The next step then is to prove that admissible families are rigid, that is, they do not admit non-trivial deformations. Notice that if we prove these statements for families over $B \setminus \Delta$, then they also follow for families over B. Now since every deformation type contains only one family, and since there are only finitely many deformation types, the original statement follows.

The following is the reformulation of Shafarevich's conjecture that was used by Parshin and Arakelov:

2.3. SHAFAREVICH'S CONJECTURE (VERSION TWO). *Let (B, Δ) be fixed and $q \geq 2$ an integer. Then the following statements hold.*

- $(\mathbf{B})$ (BOUNDEDNESS) *There exist only finitely many deformation types of admissible families of curves of genus q with respect to $B \setminus \Delta$.*
- $(\mathbf{R})$ (RIGIDITY) *There exist no non-trivial deformations of admissible families of curves of genus q with respect to $B \setminus \Delta$.*
- $(\mathbf{H})$ (HYPERBOLICITY) *If $2g - 2 + \#\Delta \leq 0$, then no admissible families of curves of genus q exist with respect to $B \setminus \Delta$.*

REMARK 2.4. As we discussed above, $(\mathbf{B})$ and $(\mathbf{R})$ together imply (2.1.1) and $(\mathbf{H})$ is clearly equivalent to (2.1.2).

2.D. *Shafarevich's conjecture for number fields*

Shafarevich's conjecture has an analogue for number fields. The number field version played a prominent role in Faltings' proof of the Mordell conjecture. This section is a brief detour to this very exciting area, but it is disconnected from the rest of the article. The reader should feel free to skip this section and continue with the next one.

DEFINITION 2.5. Let $(R, \mathfrak{m})$ be a DVR, $F = \operatorname{Frac}(R)$, and C a smooth projective curve over F. C is said to have *good reduction over* R if there exists a scheme Z, smooth and projective over $\operatorname{Spec} R$, such that $C \simeq Z_F$,

$$\begin{array}{ccccc} C & \xrightarrow{\simeq} & Z_F & \longrightarrow & Z \\ & & \downarrow & & \downarrow \\ & & \operatorname{Spec} F & \longrightarrow & \operatorname{Spec} R \end{array}$$

DEFINITION 2.6. Let R be a Dedekind ring, $F = \operatorname{Frac}(R)$, and C a smooth projective curve over F. C has *good reduction at the closed point* $\mathfrak{m} \in \operatorname{Spec} R$ if it has good reduction over $R_{\mathfrak{m}}$.

2.7. SHAFAREVICH'S CONJECTURE (NUMBER FIELD CASE). *Let $q \geq 2$ be an integer.*

(2.7.1) *Let F be a number field, $R \subset F$ the ring of integers of F, and $\Delta \subset \operatorname{Spec} R$ a finite set. Then there exists only finitely many smooth projective curves over F of genus q that have good reduction outside Δ.*

(2.7.2) *There are no smooth projective curves of genus* q *over* $\operatorname{Spec}\mathbb{Z}$.

REMARK 2.8. Shafarevich's Conjecture in the number field case has been confirmed: (2.7.1) by Faltings [Fal83b, Fal84] and (2.7.2) by Fontaine [Fon85].

One can reformulate (2.1.1) to resemble the above statement:

2.9. SHAFAREVICH'S CONJECTURE (FUNCTION FIELD CASE, VERSION THREE). *Let* $q \geq 2$ *be an integer and* $F = K(B)$ *the function field of* B*. Let* $\Delta \subset B$ *a finite subset such that* $B \setminus \Delta = \operatorname{Spec} R$ *for a (Dedekind) ring* R*. Then there exist only finitely many smooth projective non-isotrivial curves of genus* q *over* F *having good reduction over all closed points of* $\operatorname{Spec} R$.

DEFINITION 2.10. If C is a smooth projective curve over F (an arbitrary field), then there exists a morphism $C \to \operatorname{Spec} F$. Sections, $\operatorname{Spec} F \to C$, of this morphism correspond in a one-to-one manner to F*-rational points* of C, points that are defined over the field F. F-rational points of C will be denoted by $C(F)$.

EXAMPLE 2.11.

- The $\mathbb{R}$-rational points of the curve $x^2 + y^2 - z^2 = 0$ form a circle, its $\mathbb{C}$-rational points form a sphere.
- The curve $x^2 + y^2 + z^2 = 0$ has no $\mathbb{R}$-rational points.
- Let C_n be the curve defined by the equation $x^n + y^n - z^n = 0$. If $n \geq 3$, then according to Wiles' Theorem (Fermat's Last Theorem),

$$C_n(\mathbb{Q}) = \begin{cases} \{[1:0:1],[0:1:1],[1:-1:0]\}, & \text{if } n \text{ is odd,} \\ \{[1:0:1],[0:1:1],[1:0:-1],[0:1:-1]\} & \text{if } n \text{ is even.} \end{cases}$$

As mentioned earlier, Faltings used (2.7) to prove:

2.12. FALTINGS' THEOREM (MORDELL'S CONJECTURE) [Fal83b, Fal84]. *Let* F *be a number field and* C *a smooth projective curve of genus* $q \geq 2$ *defined over* F*. Then* $C(F)$ *is finite.*

The function field version of this conjecture was proved earlier by Manin:

2.13. MANIN'S THEOREM (MORDELL CONJECTURE FOR FUNCTION FIELDS) [Man63]. *Let* F *be a function field (i.e., the function field of a variety over* k*, where* k *is an algebraically closed field of characteristic* 0*) and let* C *be a smooth projective non-isotrivial curve over* F *of genus* $q \geq 2$*. Then* $C(F)$ *is finite.*

REMARK 2.14. The essential case to settle is when $\operatorname{tr.deg}_k F = 1$, i.e., $F = K(B)$, where B is a smooth projective curve over k.

2.E. *From Shafarevich to Mordell: Parshin's trick*

Shafarevich's conjecture implies Mordell's in both the function field and the number field case by an argument due to Parshin. The first step is a clever way to associate different (families of) curves to different sections:

2.15. PARSHIN'S COVERING TRICK. *For every* F*-rational point,* $P \in C(F)$*, or equivalently, for every section* $X \overset{\sigma_P}{\underset{\curvearrowleft}{\to}} B$*, there exists a finite cover of* X*,* $W_P \xrightarrow{\pi_P} X$ *such that*

- *the degree of* π_P *is bounded in terms of* q*,*
- *the projection* $W_P \to B$ *is smooth over* $B \setminus \Delta$*,*
- *the map* π_P *is ramified exactly over the image of* σ_P*,*
- *the genus of the fibers of* $W_P \to B$ *is bounded in terms of* q*.*

For details on the construction of the covers, $W_P \xrightarrow{\pi_P} X$, see [Lan97, IV.2.1] and [Cap02, §4]. The second step is an old result:

2.16. DE FRANCHIS'S THEOREM [dF13, dF91]. *Let C and D be smooth projective curves of genus at least two. Then there exist only finitely many dominant rational maps $D \to C$.*

Shafarevich's Conjecture implies that there are only finitely many different W_P's. Viewing W_P and X as curves over F, de Franchis's theorem implies that a fixed W_P can admit only finitely many different maps to X.

Since those maps are ramified exactly over the image of the corresponding σ_P, this means that there are only finitely many σ_P's, i.e., $C(F)$ is finite, and therefore Mordell's conjecture follows from that of Shafarevich.

We end our little excursion to the number field case here. In the rest of the article we work in the function field case and use the notation and assumptions of (1.1).

3. HYPERBOLICITY AND BOUNDEDNESS

3.A. *Hyperbolicity*

DEFINITION 3.1. [Bro78] A complex analytic space X is called *Brody hyperbolic* if every holomorphic map $\mathbb{C} \to X$ is constant.

REMARK 3.2. Another important, related notion is *Kobayashi hyperbolicity*. For its definition and relation to Brody hyperbolicity the reader is referred to [Kob70] and [Lan87].

REMARK 3.3. Let T be a complex torus. If X is Brody hyperbolic, then since every holomorphic map $\mathbb{C}^* \to X$ is constant, it follows that every holomorphic map $T \to X$ is also constant.

We would like to define the algebraic analogue of hyperbolicity motivated by this observation. Algebraic maps are more restrictive than holomorphic ones. For instance the universal covering map, $\mathbb{C} \to E$, of an elliptic curve, E, is not algebraic. In particular, excluding algebraic maps from $\mathbb{C}$ to X does not exclude maps from E to X. The same argument goes for abelian varieties. Since there exist simple abelian varieties (i.e., such that do not contain other abelian varieties) of arbitrary dimension, we have to take into consideration arbitrary dimensional abelian varieties.

The following definition of *algebraic hyperbolicity* is an algebraic version of Brody hyperbolicity and perhaps it should be called "algebraic Brody hyperbolicity" to emphasize that fact. However, this is not the established terminology.

Complicating the matter is that there are some related, but different definitions that are used with the same name [Dem97], [Che04]. As usual in similar cases, these different variants were introduced around the same time and hence it is hard to go back and change the terminology. In the next section we will introduce the notion of *weak boundedness*, which is closer in spirit to Demailly's notion of hyperbolicity. As the reader will see, the known results of hyperbolicity (as used in this article) follow from weak boundedness (3.8), hence the statements actually remain true even if one uses Demailly's definition of algebraic hyperbolicity.

One major advantage of the definition used here is that it extends naturally to stacks, which is exactly the context we would like to use it.

DEFINITION 3.4. An algebraic stack $\mathfrak{X}$ is called *algebraically hyperbolic* if

– every morphism $\mathbb{A}^1 \setminus \{0\} \to \mathfrak{X}$ is constant, and
– every morphism $A \to \mathfrak{X}$ is constant for an abelian variety, A.

REMARK 3.5. The first row in the following diagram is the statement of condition $(\mathbf{H})$. The last row shows equivalent conditions for both the assumption and the conclusion. Recall that $\mathfrak{M}_q$ stands for the moduli stack (of curves of genus q), so maps of the form $B \setminus \Delta \to \mathfrak{M}_q$ are exactly the ones that are induced by families over $B \setminus \Delta$.

$$\begin{array}{ccc} 2g-2+\#\Delta \leq 0 & \Longrightarrow & \nexists\, f\colon X \to B \text{ admissible} \\ \Updownarrow & & \Updownarrow \\ \begin{array}{c} B \setminus \Delta \supseteq \mathbb{A}^1 \setminus \{0\} \text{ or,} \\ \Delta = \emptyset \text{ and } B \text{ is an elliptic curve.} \end{array} & & \begin{array}{c} \nexists\, B \setminus \Delta \to \mathfrak{M}_q \\ \text{non-constant} \end{array} \end{array}$$

This implies that proving $(\mathbf{H})$ is equivalent to proving that there does not exist a non-constant morphism of the form $\mathbb{A}^1 \setminus \{0\} \to \mathfrak{M}_q$ or $E \to \mathfrak{M}_q$, where E is an arbitrary elliptic curve.

Corollary 3.6. *If $\mathfrak{M}_q$ is algebraically hyperbolic, then $(\mathbf{H})$ holds.*

3.B. *Weak Boundedness*

In addition to properties $(\mathbf{B})$, $(\mathbf{R})$, and $(\mathbf{H})$, there is another important property to study. Its importance lies in the fact that it implies $(\mathbf{H})$ and if some additional conditions hold it also implies $(\mathbf{B})$.

$(\mathbf{WB})$ (WEAK BOUNDEDNESS) *For an admissible family $f\colon X \to B$, the degree of $f_*\omega^m_{X/B}$ is bounded above in terms of $g(B), \#\Delta, g(X_{\rm gen}), m$. In particular, the bound is independent of f.*

The traditional proof of hyperbolicity for curves proceeds via some form of weak boundedness. The key point is that the upper bound obtained on $\deg f_*\omega^m_{X/B}$ has the form of

$$(2g(B)-2+\#\Delta)\cdot c(g(B), \#\Delta, g(X_{\rm gen}), m),$$

where $c(g(B), \#\Delta, g(X_{\rm gen}), m) > 0$. This proves hyperbolicity. Since $\det f_*\omega^m_{X/B}$ is ample, its degree is positive, so any upper bound of it is positive as well.

In higher dimensions, the bounds obtained are not always in this form. However, perhaps somewhat surprisingly, hyperbolicity follows already from the fact of weak boundedness, not only from the explicit bound.

Theorem 3.7. $(\mathbf{WB}) \Rightarrow (\mathbf{H})$

A more precise and somewhat more general formulation is the following:

Theorem 3.8 ([Kov02, 0.9], cf. [Par68]). *Let $\mathfrak{F}$ be a collection of smooth varieties of general type, B a smooth projective curve and $\Delta \subset B$ a finite subset of B. Let*

$$\operatorname{Fam}(B,\Delta,\mathfrak{F}) = \{f\colon X \to B \mid X \text{ is smooth, } f \text{ is flat and } f^{-1}(t) \in \mathfrak{F} \text{ for all } t \in B\setminus\Delta\}.$$

Assume that $\operatorname{Fam}(B,\Delta,\mathfrak{F})$ contains non-isotrivial families and that there exist $M, m \in \mathbb{N}$ such that for all $(f\colon X \to B) \in \operatorname{Fam}(B,\Delta,\mathfrak{F})$,

$$\deg\left(f_*\omega^m_{X/B}\right) \leq M.$$

Then $2g(B) - 2 + \#\Delta > 0$.

3.C. *From Weak Boundedness to Boundedness*

By [Par68, Theorem 1] there exists a scheme V that parametrizes admissible families of curves of genus q. Hence $(\mathbf{B})$ is equivalent to the statement that V has finitely many components. Therefore, in order to prove $(\mathbf{B})$, it is enough to prove that V is a scheme of finite type.

V is naturally embedded into $\operatorname{Hom}((B, B \setminus \Delta), (\overline{\mathsf{M}}_q, \mathsf{M}_q))$. For a family $f\colon X \to B$, let $\mu_f\colon B \to \overline{\mathsf{M}}_q$ be the moduli map. If for a fixed ample line bundle $\mathscr{L}$ on $\overline{\mathsf{M}}_q$, one can establish that $\deg \mu_f^* \mathscr{L}$ is bounded on B, the bound perhaps depending on B, Δ and q, but not on f, then one may conclude that the image of V in $\operatorname{Hom}((B, B \setminus \Delta), (\overline{\mathsf{M}}_q, \mathsf{M}_q))$ is contained in finitely many components and hence is of finite type.

The final piece of the puzzle is provided by the construction of $\overline{\mathsf{M}}_q$. For p sufficiently large and divisible there exist line bundles $\lambda_m^{(p)}$ on $\overline{\mathsf{M}}_q$ such that for a family of stable curves $f\colon X \to B$, if $\bar{\mu}_f\colon B \to \overline{\mathsf{M}}_q$ is the induced moduli map, then

$$\left(\det\left(f_* \omega_{X/B}^m\right)\right)^p = \bar{\mu}_f^* \lambda_m^{(p)}.$$

Hence $(\mathbf{WB})$ gives exactly the above required boundedness result and so we obtain the following statement.

Theorem 3.9. *For families of curves* $(\mathbf{WB})$ *implies* $(\mathbf{B})$.

4. Higher dimensional fibers

Next we turn to higher dimensional generalizations. As a first step, we will keep the base of the family be a curve and allow higher dimensional fibers. Independently, or simultaneously, one can study families over higher dimensional bases. Furthermore, generalizing the conditions on the fibers naturally leads to the study of families of singular varieties. We will discuss all of these directions.

In order to generalize Shafarevich's conjecture to the case of families of higher dimensional varieties the first task is to generalize both the statement and the conditions. The condition that a curve has genus at least 2, i.e., our assumption that $g(X_{\mathrm{gen}}) \geq 2$, is equivalent to the condition that $\omega_{X_{\mathrm{gen}}}$ is ample. In higher dimensions, the role of the genus is played by the Hilbert polynomial, so fixing $g(X_{\mathrm{gen}})$ will be replaced by fixing $h_{\omega_{X_{\mathrm{gen}}}}$, the Hilbert polynomial of $\omega_{X_{\mathrm{gen}}}$. Therefore we have the following starting data:

- a fixed smooth curve B of genus g,
- a fixed finite subset $\Delta \subset B$, and
- a fixed polynomial h.

DEFINITION 4.1. An *admissible family* with respect to B, Δ and h is a non-isotrivial family $f\colon X \to B$, such that X is a smooth projective variety and for all $b \in B \setminus \Delta$, the variety X_b is smooth and projective with ω_{X_b} ample and $h_{\omega_{X_b}} = h$. Two such families are *equivalent* if they are isomorphic over $B \setminus \Delta$.

Having made this definition, the various parts of Shafarevich's conjecture make sense in any dimension.

4.2. HIGHER DIMENSIONAL SHAFAREVICH CONJECTURE. *Fix* B, Δ *and* h. *Then*

- $(\mathbf{B})$ (BOUNDEDNESS) *there exist only finitely many deformation types of admissible families of canonically polarized varieties with respect to* B, Δ *and* h,
- $(\mathbf{R})$ (RIGIDITY) *there exist no non-trivial deformations of admissible families of canonically polarized varieties with respect to* B, Δ *and* h,

($\mathbf{H}$) (HYPERBOLICITY) *if* $2g(B) - 2 + \#\Delta \leq 0$*, then no admissible families of canonically polarized varieties with respect to* B, Δ *and* h *exist, and*

($\mathbf{WB}$) (WEAK BOUNDEDNESS) *for an admissible family* $f\colon X \to B$*, the degree of* $f_*\omega^m_{X/B}$ *is bounded in terms of* $g(B), \#\Delta, h$ *and* m*.*

Next we will discuss the state of affairs with regard to these conjectures and the many results obtained during the past decade. Because of the interdependency of the various results it makes more sense to follow a different order than they are listed in the conjecture.

4.A. *Rigidity*

Let $Y \to B$ be an arbitrary non-isotrivial family of curves of genus ≥ 2, and C a smooth projective curve also of genus ≥ 2. Then $f\colon X = Y \times C \to B$ is an admissible family, and a deformation of C gives a deformation of f. Therefore ($\mathbf{R}$) fails as stated.

This leads naturally to the following question.

QUESTION 4.3. Under what additional conditions does ($\mathbf{R}$) hold?

A possible answer to this question will be given in Section 8.B.

4.B. *Hyperbolicity*

Migliorini [Mig95] showed that for families of minimal surfaces a somewhat weakened hyperbolicity statement holds, namely that $\Delta \neq \emptyset$ if $g \leq 1$. The same conclusion was shown in [Kov96] for families of minimal varieties of arbitrary dimension. Later ($\mathbf{H}$) for families of minimal surfaces was proved in [Kov97b], and then in general for families of canonically polarized varieties in [Kov00a].

Theorem 4.4 [Kov00a]. *Let* $X \to B$ *be an admissible families of canonically polarized varieties with respect to* B, Δ *and* h*. Then* $2g(B) - 2 + \#\Delta > 0$*.*

Finally, Viehweg and Zuo [VZ03b] proved the analytic version of ($\mathbf{H}$):

Theorem 4.5 [VZ03b]. $\mathfrak{M}_h$ *is Brody hyperbolic.*

4.C. *Weak Boundedness*

Bedulev and Viehweg [BV00] proved the following:

Theorem 4.6 [BV00]. *Let* $f\colon X \to B$ *be an admissible family with* B, Δ, h *fixed. Let* $\delta = \#\Delta$*,* $g = g(B)$*, and* $n = \dim X_{\text{gen}} = \dim X - 1$*. If* $f_*\omega^m_{X/B} \neq 0$*, then there exists a positive integer* $e = e(m, h)$ *such that*

$$\deg f_*\omega^m_{X/B} \leq m \cdot e \cdot \operatorname{rk} f_*\omega^m_{X/B} \cdot (n(2g - 2 + \delta) + \delta).$$

This clearly implies ($\mathbf{WB}$) and as a byproduct of the explicit bound it also implies ($\mathbf{H}$).

Viehweg and Zuo [VZ01] extended ($\mathbf{WB}$) to families of varieties of general type and of varieties admitting a good minimal model. In [Kov02] similar results were obtained with different methods allowing the fibers to have rational Gorenstein singularities, but restricting to the case of families of minimal varietes of general type.

The proof of (3.8) still works in this generality, so ($\mathbf{WB}$) implies ($\mathbf{H}$) in all dimensions [Kov02, 0.9].

4.D. *Boundedness*

Using the existence of moduli spaces of canonically polarized varieties and the description of ample line bundles on them, Bedulev and Viehweg [BV00] also proved a boundedness-type statement:

Theorem 4.7 [BV00]. *Let B, Δ and h be fixed and assume that M_h admits a geometric compactification $\overline{\mathsf{M}}_h$. Then there exists a subscheme of $\operatorname{Hom}((B, B \setminus \Delta), (\overline{\mathsf{M}}_h, \mathsf{M}_h))$ of finite type that contains the classes of all morphisms $B \to \overline{\mathsf{M}}_h$ induced by admissible families.*

Unfortunately this statement does not imply $(\mathbf{B})$. However, a recent result of Kovács and Lieblich does.

Theorem 4.8 [KL06]. *Let B, Δ and h be fixed. Then there exist only finitely many deformation types of admissible families of canonically polarized varieties with respect to B, Δ and h.*

4.E. *Shafarevich's conjecture for other types of varieties*

One may ask whether the Shafarevich problem holds for families of other types of varieties. There are some known results in this setting as well.

Faltings [Fal83a] studied the Shafarevich problem for families of abelian varieties and proved that $(\mathbf{B})$ holds, while $(\mathbf{R})$ fails in general. He also gave an equivalent condition for $(\mathbf{R})$ to hold in this case.

Oguiso and Viehweg [OV01] considered $(\mathbf{H})$ for families of non-general type surfaces. Their work combined with the previous results show that $(\mathbf{H})$ holds for families of minimal surfaces of non-negative Kodaira dimension.

Recent results have been obtained by Jorgensen and Todorov [JT02], Liu, Todorov, Yau and Zuo [LTYZ05] and Viehweg and Zuo [VZ05b] for families of Calabi-Yau varieties.

5. HIGHER DIMENSIONAL BASES

The next natural generalization is to allow B to have arbitrary dimension. Let B be a smooth projective variety, $\Delta \subset B$ a divisor with normal crossings and h a polynomial. The definition of an admissible family is formally the same as in (4.1). As before, for an admissible family, $f \colon X \to B$, the moduli map $b \mapsto [X_b]$ is denoted by $\mu_f \colon B \setminus \Delta \to \mathfrak{M}_h$.

Since B is now allowed to be higher dimensional, the notion of isotriviality is no longer the best one to consider. Observe that f is isotrivial if and only if μ_f is constant. Saying that f is not isotrivial would allow the family to be isotrivial in certain directions. What we want to assume is that the family "truly" changes in any direction on B. To express this we define the family's variation in moduli.

DEFINITION 5.1 cf. [Vie83a], [Vie83b], [Kol87a]. $\operatorname{Var} f := \dim(\mu_f(B)) \;\; (\leq \dim B)$.

We are interested in the case $\operatorname{Var} f = \dim B$. In (3.6), we observed that hyperbolicity follows if we know that the stack $\mathfrak{M}_h$ is algebraically hyperbolic. In fact, for hyperbolicity over a 1-dimensional base, we only needed the corresponding property of $\mathfrak{M}_h$ for curves. However, we would also like to know that every morphism $A \to \mathfrak{M}_h$ induced by a family is constant, where A is an arbitrary abelian variety. This is the extra content of the next theorem.

Theorem 5.2. [Kov97a], [Kov00a] *$\mathfrak{M}_h$ is algebraically hyperbolic.*

REMARK 5.3. This statement also follows from boundedness by an argument similar to the one used in the proof of (3.8). It also follows from (4.5).

As before, this implies that if $f \colon X \to \mathbb{P}^1$ is an admissible family, then $\#\Delta > 2$. More generally, for an admissible family $f \colon X \to \mathbb{P}^m$ with $\operatorname{Var} f = m$, this implies that

$\deg \Delta > 2$. However, we expect that in this case $\deg \Delta$ should be larger than $m+1$. This is indeed the case.

Theorem 5.4 [VZ02],[Kov03c]. *Let $f\colon X \to \mathbb{P}^m$ be an admissible family. Then $\omega_{\mathbb{P}^m}(\Delta)$ is ample, or equivalently $\deg \Delta > m+1$.*

REMARK 5.5. Viehweg and Zuo actually prove a lot more than this in [VZ02]. Please see the article for details.

It is now natural to suspect that a more general statement should hold. The following statement to this effect is part of a more general conjecture of Viehweg [Vie01].

5.6. VIEHWEG'S CONJECTURE. *If $f\colon X \to B$ is an admissible family, then $\omega_B(\Delta)$ is big.*

For $\dim B = 1$, this is simply $(\mathbf{H})$. For $\dim B > 1$, it is known to be true for families of curves by [Vie01, 2.6] and for $B = \mathbb{P}^n$ and various other special cases by [VZ02], [Kov03b] and [Kov03c]. It was recently confirmed for $\dim B = 2$ by Kebekus and Kovács in [KK05]. However, this question is far from being completely settled. The reader is encouraged to read Viehweg's discussion of this and other related open questions in [Vie01].

6. UNIFORM AND EFFECTIVE BOUNDS

6.A. *Families of curves*

A finiteness result such as (2.1.1) naturally leads to the question whether the obtained bound dependends on the actual curve, or only on its genus. In other words, is it possible to give a *uniform bound* that works for all base curves B of genus g?

This question is actually more subtle than it might seem at first. Consider the argument before (3.9). That proves that $(\mathbf{WB})$ implies $(\mathbf{B})$, but it does not shed any light on the obtained bound. Even if the bound appearing in $(\mathbf{WB})$ depends only on the genus, it might happen that the number of deformation types still depends on the actual curve. The argument uses the fact that a subscheme of a scheme of finite type itself is of finite type. That means that the subscheme has finitely many components, which is what is needed for $(\mathbf{B})$, but it says nothing about how big that finite numner is. The number of components of a subscheme has nothing to do with the number of components of the ambient scheme.

Despite these difficulties, uniform boundedness is known. The first such result was obtained by Caporaso:

Theorem 6.1 [Cap02, Cap03] cf. [Cap04]. *There exists a constant $c(q,d,\delta)$ such that for any smooth irreducible variety $B \subseteq \mathbb{P}^r$ of* degree d *and for any closed subscheme $\Delta \subset B$ of degree δ, the number of admissible families of curves of genus q with respect to (B,Δ) is at most $c(q,d,\delta)$.*

REMARK 6.2. If B is one dimensional, then one may write $c(q,d,\delta) = c'(q,g,\delta)$ using $g = g(B)$ the genus of B instead of d.

The next question is whether the constant $c(q,d,\delta)$ (or in the case of a base curve $c'(q,g,\delta)$) is computable. In other words, is it possible to give an *effective* uniform bound? For families over a base curve this was achieved by Heier:

Theorem 6.3 [Hei04]. *Let B be a smooth projective curve of genus g and $\Delta \subset B$ a finite subset. Then $c'(q,g,\delta)$ can be expressed as an explicit function of q, g and δ.*

REMARK 6.4. The expression itself is rather complicated and can be found in the original article.

6.B. *Higher dimensional families*

For higher dimensional families rigidity fails and so we cannot expect a similar finiteness statement as above. However, one may still ask whether *uniform boundedness* holds and if so, whether there is an effective bound on the number of deformation types over a base with a fixed Hilbert polynomial.

Uniform boundedness was recently proved by Kovács and Lieblich.

Theorem 6.5 [KL06]. *Let h be a fixed polynomial. Then the set of deformation types of admissible families of canonically polarized varieties with Hilbert polynomial h is finite and uniformly bounded in any quasi-compact family of base varieties B°.*

On the other hand, no effective (uniform) bound is known at this time.

7. TECHNIQUES

7.A. *Positivity of direct images*

One of the most important ingredients in the proofs of the known results is an appropriate variant of a fundamental positivity result due to the work of Fujita, Kawamata, Kollár and Viehweg. In this section we will assume, for simplicity, that $\dim B = 1$.

DEFINITION 7.1. A locally free sheaf, $\mathscr{E}$, is *ample* if $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ on $\mathbb{P}(\mathscr{E})$ is ample.

Theorem 7.2 [Fuj78], [Kaw82b], [Kol87a], [Kol90], [Vie83a], [Vie83b]. *Let $f\colon X \to B$ be an admissible family and $m > 1$. If $f_*\omega^m_{X/B} \neq 0$, then $f_*\omega^m_{X/B}$ is ample on B.*

Corollary 7.3. *Let $f\colon X \to B$ be an admissible family and $m > 1$. If $f_*\omega^m_{X/B} \neq 0$, then* $\deg f_*\omega^m_{X/B} > 0$.

The methods used to prove (7.2) give a more precise estimate of the positivity of these push-forwards as shown by Esnault and Viehweg:

Theorem 7.4 [EV90, 2.4]. *Let $f\colon X \to B$ be an admissible family, and $\mathscr{M}$ a line bundle on B. Assume that there exists an integer $m > 1$ such that $\deg \mathscr{M} < \deg f_*\omega^m_{X/B}$. Let r denote the rank of $f_*\omega^m_{X/B}$. Then there exists a positive integer $e = e(m,h)$, such that $(f_*\omega^m_{X/B})^{\otimes e\cdot r} \otimes \mathscr{M}^{-1}$ is ample on B.*

Corollary 7.5 [Kov96, 2.15], [Kov00a, 2.1], [Kov02, 7.6]. (for $\Delta = \emptyset$) *Let $\mathscr{N}$ be a line bundle on B such that $\deg \mathscr{N}^{m\cdot e\cdot r} < \deg f_*\omega^m_{X/B}$. Then $\omega_{X/B} \otimes f^*\mathscr{N}^{-1}$ is ample on X.*

PROOF (SKETCH). As $\Big(f_*(\omega^m_{X/B} \otimes f^*\mathscr{N}^{-m})\Big)^{\otimes e\cdot r} \simeq (f_*\omega^m_{X/B})^{\otimes e\cdot r} \otimes \mathscr{N}^{-m\cdot e\cdot r}$, we obtain that (7.4) implies that $f_*(\omega^m_{X/B} \otimes f^*\mathscr{N}^{-m})$ is ample on B. Furthermore, by assumption one has that $\omega^m_{X/B} \otimes f^*\mathscr{N}^{-m}|_{X_{\mathrm{gen}}} \simeq \omega^m_{X_{\mathrm{gen}}}$ is ample on X_{gen}. Hence $\omega_{X/B} \otimes f^*\mathscr{N}^{-1}$ is ample both "horizontally" and "vertically", so it is ample. For details about the last step see [Kov02, 7.6]. □

This allows us to reduce the proof of $(\mathbf{WB})$ to finding an appropriate line bundle on B according to the following plan.

PLAN 7.6. First, find a line bundle $\mathscr{N}$ on B, depending only on B and Δ, such that $\omega_{X/B} \otimes f^*\mathscr{N}^{-1}$ is not ample on X. Then one has $\deg \mathscr{N}^{m\cdot e\cdot r} \not< \deg f_*\omega^m_{X/B}$ by (7.5). In other words one has that

$$\deg f_*\omega^m_{X/B} \leq m \cdot e \cdot r \cdot \deg \mathscr{N}.$$

We find such an $\mathscr{N}$ using vanishing theorems. The main idea is the following: we want to find a line bundle such that twisting with the relative dualizing sheaf does not yield an ample line bundle. Ample line bundles appear in many vanishing theorems, so one way to prove that a given line bundle is not ample is to prove that a cohomology group does not vanish that would if the line bundle were ample. Next we are going to look at the needed vanishing theorems.

7.B. *Vanishing theorems*

Vanishing theorems have played a central role in algebraic geometry for the last couple of decades, especially in classification theory. Kollár [Kol87b] gives an introduction to the basic use of vanishing theorems as well as a survey of results and applications available at the time. For more recent results one should consult [EV92], [Ein97], [Kol97], [Smi97], [Kov00c], [Kov02], [Kov03a], [Kov03b]. Because of the availability of those surveys, we will only recall statements that are important for the present article. Nonetheless, any discussion of vanishing theorems should start with the fundamental vanishing theorem of Kodaira.

Theorem 7.7 [Kod53]. *Let X be a smooth complex projective variety and $\mathscr{L}$ an ample line bundle on X. Then*

$$H^i(X, \omega_X \otimes \mathscr{L}) = 0 \textit{ for } i > 0.$$

This has been generalized in several ways, but as noted above we will restrict to a select few. The original statement of Kodaira was generalized to allow semi-ample and big line bundles in place of ample ones by Grauert and Riemenschneider.

Theorem 7.8 [GR70]. *Let X be a smooth complex projective variety and $\mathscr{L}$ a semi-ample and big line bundle on X. Then*

$$H^i(X, \omega_X \otimes \mathscr{L}) = 0 \textit{ for } i > 0.$$

"Semi-ample" was later replaced by "nef" in the statement by Kawamata and Viehweg.

Theorem 7.9 [Kaw82a],[Vie82]. *Let X be a smooth complex projective variety and $\mathscr{L}$ a nef and big line bundle on X. Then*

$$H^i(X, \omega_X \otimes \mathscr{L}) = 0 \textit{ for } i > 0.$$

Akizuki and Nakano extended Kodaira's vanishing theorem to include other exterior powers of the sheaf of differential forms.

Theorem 7.10 (Akizuki–Nakano [AN54]). *Let X be a smooth complex projective variety and $\mathscr{L}$ an ample line bundle on X. Then*

$$H^q(X, \Omega^p_X \otimes \mathscr{L}) = 0 \textit{ for } p + q > \dim X.$$

REMARK 7.11. Ramanujam [Ram72] gave a simplified proof of (7.10) and showed that it does not hold if one only requires $\mathscr{L}$ to be semi-ample and big.

In order to proceed we will need more delicate vanishing theorems than before. Our starting point is the theorem of Esnault and Viehweg that extends (7.10) to sheaves of logarithmic differential forms.

Theorem 7.12 [EV90, 6.4]. *Let X be a smooth complex projective variety, $\mathscr{L}$ an ample line bundle and D a normal crossing divisor on X. Then*

$$H^q\left(X, \Omega^p_X(\log D) \otimes \mathscr{L}\right) = 0 \textit{ for } p + q > \dim X.$$

REMARK 7.13. Extending the known vanishing theorems in a different direction, Navarro-Aznar *et al.* proved a version of the Kodaira-Akizuki-Nakano vanishing theorem for singular varieties that implies our previous statements: (7.7), (7.8), and (7.10) cf. [Nav88] in [GNPP88] and Theorem 8.5 in the next section.

As mentioned earlier, in order to prove $(\mathbf{WB})$ we need a suitable vanishing theorem. The following is a somewhat weaker statement than that is really needed, but shows the main idea of the proof and how to apply it.

Theorem 7.14 [Kov97b], [Kov00a]. *Let $f\colon X \to B$ be a family such that B is a smooth projective curve. Assume that $D = f^*\Delta$ is a normal crossing divisor. Let $n = \dim X_{\text{gen}}$ and $\mathscr{L}$ an ample line bundle on X such that $\mathscr{L} \otimes f^*\omega_B(\Delta)^{-n}$ is also ample. Then*

$$H^{n+1}\left(X, \mathscr{L} \otimes f^*\omega_B(\Delta)\right) = 0.$$

PROOF. After taking exterior powers of the sheaves of logarithmic differential forms, one has the following short exact sequence for each $p = 1, \dots, n+1$:

$$0 \longrightarrow \Omega^{p-1}_{X/B}(\log D) \otimes f^*\omega_B(\Delta) \longrightarrow \Omega^p_X(\log D) \longrightarrow \Omega^p_{X/B}(\log D) \longrightarrow 0.$$

Define $\mathscr{L}_p = \mathscr{L} \otimes f^*\omega_B(\Delta)^{1-p}$ for $p = 0, \dots, n+1$. Then the above short exact sequence yields:

$$0 \longrightarrow \Omega^{p-1}_{X/B}(\log D) \otimes \mathscr{L}_{p-1} \longrightarrow \Omega^p_X(\log D) \otimes \mathscr{L}_p \longrightarrow \Omega^p_{X/B}(\log D) \otimes \mathscr{L}_p \longrightarrow 0.$$

$\mathscr{L}_p$ is ample for $p = 1, \dots, n+1$ since either $\omega_B(\Delta)$ or $\omega_B(\Delta)^{-1}$ is nef. Then by (7.12) $H^{n+1-(p-1)}(X, \Omega^p_X(\log D) \otimes \mathscr{L}_p) = 0$ (recall that $\dim X = n+1$). Hence the map

$$H^{n+1-p}\left(X, \Omega^p_{X/B}(\log D) \otimes \mathscr{L}_p\right) \longrightarrow H^{n+1-(p-1)}\left(X.\Omega^{p-1}_{X/B}(\log D) \otimes \mathscr{L}_{p-1}\right)$$

is surjective for $p = 1, \dots, n+1$. Observe that these maps form a chain as p runs through $p = n+1, n, \dots, 1$. Hence the composite map,

$$H^0\left(X, \Omega^{n+1}_{X/B}(\log D) \otimes \mathscr{L}_{n+1}\right) \longrightarrow H^{n+1}(X, \mathscr{L}_0),$$

is also surjective. However, $\Omega^1_{X/B}(\log D)$ is of rank n, so $\Omega^{n+1}_{X/B}(\log D) = 0$, and therefore $H^{n+1}(X, \mathscr{L}_0) = H^{n+1}(X, \mathscr{L} \otimes f^*\omega_B(\Delta)) = 0$ as well. □

We are finally able to prove $(\mathbf{WB})$, at least for $\Delta = \emptyset$, by combining positivity and vanishing: (7.3) and (7.5) with $\mathscr{N} = \mathscr{O}_B$ imply that $\omega_{X/B}$ is ample. Since

$$H^{n+1}(X, \underbrace{\omega_{X/B} \otimes f^*\omega_B}_{\omega_X}) \neq 0,$$

this and (7.14) imply that $\omega_{X/B} \otimes f^*\omega_B^{-n}$ cannot be ample. Then (7.5) with $\mathscr{N} = f^*\omega_B^n$ implies that

$$\deg f_*\omega^m_{X/B} \leq \deg f^*\omega_B^{n\cdot m\cdot e\cdot r} = m \cdot e \cdot r \cdot \dim X_{\text{gen}} \cdot (2g-2).$$

REMARK 7.15. For a complete proof of $(\mathbf{WB})$ without the assumption $\Delta = \emptyset$, see [BV00], [Kov02], or [VZ02].

7.C. *Kernels of Kodaira-Spencer maps*

The germ of the method described above was first used in [Kov96] and then it was polished through several articles [Kov97b, Kov97a, Kov00a, BV00, OV01, Kov02]. Then Viehweg and Zuo [VZ01, VZ02] combined some of the ideas of this method with Zuo's discovery of the negativity of kernels of Kodaira-Spencer maps [Zuo00]. This negativity is essentially a dual phenomenon of the positivity results mentioned earlier (7.2), (7.4).

The Viehweg-Zuo method has a great advantage over the previous method. The latter uses global vanishing theorems which limits the scope of the applications, while the Viehweg-Zuo method uses local arguments and hence is more applicable. Unfortunately this method is rather technical and so we cannot present it here. However, it is discussed in many places. The interested reader should start by Viehweg's excellent survey [Vie01] and then read the full account in [VZ01, VZ02].

8. FURTHER RESULTS AND CURRENT DIRECTIONS

8.A. *More general fibers*

In the pursuit of more general results somewhat different approaches were taken in [VZ02] and [Kov02]. Both of these approaches led to several further results and these results, in accordance with the different approaches, were somewhat different. Here we discuss the latter approach and the related results. For a survey on the former, the reader is referred to [Vie01] and the references therein.

Our starting point is a principle that has been applied with great success in birational geometry.

PRINCIPLE 8.1. Studying an ample line bundle on a singular variety is similar to studying a semi-ample and big line bundle on a smooth variety.

The traditional way to use this principle is the following. The goal is to prove a statement for a pair, $(X, \mathscr{L})$, where X is possibly singular, and $\mathscr{L}$ is ample on X. Instead of working on X one works on a desingularization $f\colon Y \to X$, and consider the semi-ample and big line bundle $\mathscr{K} = f^*\mathscr{L}$. A prominent example of this trick is the use of the Kawamata-Viehweg vanishing theorem (7.9) in the Minimal Model Program.

Here we will turn the situation upside-down. Our goal is a statement for $(Y, \mathscr{K})$, where Y is smooth and $\mathscr{K}$ is a semi-ample and big line bundle on Y. Instead of working on Y we construct a pair $(X, \mathscr{L})$ and a map $f\colon Y \to X$, where X is possibly singular, $\mathscr{L}$ is ample on X, f is birational, and $\mathscr{K} = f^*\mathscr{L}$.

The motivation for this approach is that we would like to extend the previous results to the case when $\omega_{X_{\rm gen}}$ is not necessarily ample but only semi-ample and big. However, a crucial ingredient of the proof is an appropriate version of the Kodaira-Akizuki-Nakano vanishing theorem (7.10), and as Ramanujam (7.11) pointed out, (7.10) fails if the line bundle in question is only assumed to be semi-ample and big instead of ample. On the other hand, Navarro-Aznar *et al.* proved a singular version of the Kodaira-Akizuki-Nakano vanishing theorem (see Remark 7.13), so one hopes that this way the proof can be made to work.

In order to state the singular version of the Kodaira-Akizuki-Nakano vanishing theorem, we need to use derived categories. The reader unfamiliar with the basics may wish to consult [Hart66] and [Con00] for definitions and details.

8.2. DU BOIS'S COMPLEX. We also need Du Bois's generalized De Rham complex. The original construction of Du Bois's complex, $\underline{\Omega}^{\bullet}_X(\log D)$, is based on simplicial resolutions. The reader interested in the details is referred to the original article [DB81]. Note also that a simplified construction was later obtained in [Car85] and [GNPP88] and via the general theory of polyhedral and cubic resolutions. An easily accessible introduction can be found in [Ste85].

Recently Schwede found an alternative construction of Du Bois's complex that does not need a simplicial resolution [Sch06], however we will use the original construction here. For more on recent applications of Du Bois's complex and Du Bois singularities see [Ste83], [Kol95, Chapter 12], [Kov99], [Kov00b], [Kov00c].

The word "hyperresolution" will refer to either simplicial, polyhedral, or cubic resolution. Formally, the construction of $\underline{\Omega}^{\bullet}_{X}(\log D)$ is the same regardless the type of resolution used and no specific aspects of either types will be used.

The following definition is included to make sense of the statements of some of the forthcoming theorems. It can be safely ignored if the reader is not interested in the detailed properties of Du Bois's complex and is willing to accept that it is a very close analogue of the De Rham complex of smooth varieties.

DEFINITION 8.3. Let X be a complex scheme and D a closed subscheme whose complement in X is dense. Then $(X_{\bullet}, D_{\bullet}) \to (X, D)$ is a *good hyperresolution* if $X_{\bullet} \to X$ is a hyperresolution, and if $U_{\bullet} = X_{\bullet} \times_X (X \setminus D)$ and $D_{\bullet} = X_{\bullet} \setminus U_{\bullet}$, then D_i is a divisor with normal crossings on X_i for all i.

Let X be a complex scheme of dimension n. Let $D_{\mathrm{filt}}(X)$ denote the derived category of filtered complexes of $\mathscr{O}_X$-modules with differentials of order ≤ 1 and $D_{filt,coh}(X)$ the subcategory of $D_{\mathrm{filt}}(X)$ of complexes $K^{\bullet}$, such that for all i, the cohomology sheaves of $Gr^{i}_{\mathrm{filt}} K^{\bullet}$ are coherent cf. [DB81], [GNPP88]. Let $D(X)$ and $D_{\mathrm{coh}}(X)$ denote the derived categories with the same definition except that the complexes are assumed to have the trivial filtration. The superscripts $+, -, b$ carry the usual meaning (bounded below, bounded above, bounded). Isomorphism in these categories is denoted by $\simeq_{qis}$. A sheaf $\mathscr{F}$ is also considered a complex $\mathscr{F}^{\bullet}$ with $\mathscr{F}^{0} = \mathscr{F}$ and $\mathscr{F}^{i} = 0$ for $i \neq 0$. If $K^{\bullet}$ is a complex in any of the above categories, then $h^{i}(K^{\bullet})$ denotes the i-th cohomology sheaf of $K^{\bullet}$.

The right derived functor of an additive functor F, if it exists, is denoted by RF and $R^{i}F$ is short for $h^{i} \circ RF$. Furthermore, $\mathbb{H}^{i}$, $\mathbb{H}^{i}_{Z}$, and $\mathscr{H}^{i}_{Z}$ will denote $R^{i}\Gamma$, $R^{i}\Gamma_{Z}$, and $R^{i}\mathscr{H}_{Z}$ respectively, where Γ is the functor of global sections, Γ_Z is the functor of global sections with support in the closed subset Z, and $\mathscr{H}_Z$ is the functor of the sheaf of local sections with support in the closed subset Z. Note that according to this terminology, if $\phi\colon Y \to X$ is a morphism and $\mathscr{F}$ is a coherent sheaf on Y, then $R\phi_{*}\mathscr{F}$ is the complex whose cohomology sheaves give rise to the usual higher direct images of $\mathscr{F}$.

Theorem 8.4 [DB81, 6.3, 6.5]. *Let X be a proper complex scheme of finite type and D a closed subscheme whose complement is dense in X. Then there exists a unique object $\underline{\Omega}^{\bullet}_{X}(\log D) \in \operatorname{Ob} D_{\mathrm{filt}}(X)$ such that using the notation*

$$\underline{\Omega}^{p}_{X}(\log D) := Gr^{p}_{\mathrm{filt}}\, \underline{\Omega}^{\bullet}_{X}(\log D)[p],$$

it satisfies the following properties

(8.4.1) *Let $j\colon X \setminus D \to X$ be the inclusion map. Then*

$$\underline{\Omega}^{\bullet}_{X}(\log D) \simeq_{qis} Rj_{*}\mathbb{C}_{X\setminus D}.$$

(8.4.2) $\underline{\Omega}^{\bullet}_{(\_)}(\log(\_))$ *is functorial, i.e., if $\phi\colon Y \to X$ is a morphism of proper complex schemes of finite type, then there exists a natural map ϕ^{*} of filtered complexes*

$$\phi^{*}\colon \underline{\Omega}^{\bullet}_{X}(\log D) \to R\phi_{*}\underline{\Omega}^{\bullet}_{Y}(\log \phi^{*}D).$$

Furthermore, $\underline{\Omega}^{\bullet}_{X}(\log D) \in \operatorname{Ob}\left(D^{b}_{filt,coh}(X)\right)$ and if ϕ is proper, then ϕ^{*} is a morphism in $D^{b}_{filt,coh}(X)$.

(8.4.3) *Let $U \subseteq X$ be an open subscheme of X. Then*

$$\underline{\Omega}^{\bullet}_{X}(\log D)\big|_{U} \simeq_{qis} \underline{\Omega}^{\bullet}_{U}\left(\log D|_{U}\right).$$

(8.4.4) *There exists a spectral sequence degenerating at E_1 and abutting to the singular cohomology of $X \setminus D$:*

$$E_1^{pq} = \mathbb{H}^q\left(X, \underline{\Omega}^p_X(\log D)\right) \Rightarrow H^{p+q}(X \setminus D, \mathbb{C}).$$

(8.4.5) *If $\varepsilon_{\bullet} \colon (X_{\bullet}, D_{\bullet}) \to (X, D)$ is a good hyperresolution, then*

$$\underline{\Omega}^{\bullet}_X(\log D) \simeq_{qis} R\varepsilon_{\bullet *}\Omega^{\bullet}_{X_{\bullet}}(\log D_{\bullet}).$$

In particular, $h^i\left(\underline{\Omega}^p_X(\log D)\right) = 0$ for $i < 0$.

(8.4.6) *There exists a natural map, $\mathscr{O}_X \to \underline{\Omega}^0_X(\log D)$, compatible with (8.4.2).*

(8.4.7) *If X is smooth and D is a normal crossing divisor, then*

$$\underline{\Omega}^{\bullet}_X(\log D) \simeq_{qis} \Omega^{\bullet}_X(\log D).$$

In particular,

$$\underline{\Omega}^p_X(\log D) \simeq_{qis} \Omega^p_X(\log D).$$

(8.4.8) *If $\phi \colon Y \to X$ is a resolution of singularities, then*

$$\underline{\Omega}^{\dim X}_X(\log D) \simeq_{qis} R\phi_*\omega_Y(\phi^* D).$$

Naturally, one may choose $D = \emptyset$ and then it is simply omitted from the notation. The same applies to $\underline{\Omega}^p_X := Gr^p_{\text{filt}}\, \underline{\Omega}^{\bullet}_X[p]$. We are now able to state the aforementioned singular version of the Kodaira-Akizuki-Nakano vanishing theorem.

Theorem 8.5 [Nav88], [GNPP88]. *Let X be a complex projective variety and $\mathscr{L}$ an ample line bundle on X. Then*

$$\mathbb{H}^q(X, \underline{\Omega}^p_X \otimes \mathscr{L}) = 0 \text{ for } p + q > \dim X.$$

Since Du Bois's complex agrees with the De Rham complex for smooth varieties, this theorem reduces to the Kodaira-Akizuki-Nakano theorem in the smooth case. However, this theorem is still not strong enough in our original situation if $\Delta \neq \emptyset$. We need a singular version of Esnault-Viehweg's logarithmic vanishing theorem (7.12).

Theorem 8.6 [Kov02]. *Let X be a complex projective variety and $\mathscr{L}$ an ample line bundle on X. Further let D be a normal crossing divisor on X. Then*

$$\mathbb{H}^q(X, \underline{\Omega}^p_X(\log D) \otimes \mathscr{L}) = 0 \text{ for } p + q > \dim X.$$

To adapt the proof of $(\mathbf{WB})$ to the singular case we need a singular version of (7.14). Besides the above vanishing theorem we also need an analogue of the sheaf of relative logarithmic differentials.

THEOREM-DEFINITION 8.7 [Kov02], cf. [Kov96, Kov97c, Kov05a]. *Let $f \colon X \to B$ be a morphism between complex varieties such that $\dim X = n + 1$ and B is a smooth curve. Let $\Delta \subseteq B$ be a finite set and $D = f^*\Delta$. For every non-negative integer p there exists a natural map $\wedge_p \colon \underline{\Omega}^p_X(\log D) \otimes f^*\omega_B(\Delta) \to \underline{\Omega}^{p+1}_X(\log D)$ and a complex $\underline{\Omega}^p_{X/B}(\log D) \in \operatorname{Ob}(D(X))$ with the following properties.*

(8.7.1) *The natural map $\wedge_p$ factors through $\underline{\Omega}^p_{X/B}(\log D) \otimes f^*\omega_B(\Delta)$, i.e., there exist maps:*

$$\begin{aligned} w''_p &: \underline{\Omega}^p_X(\log D) \otimes f^*\omega_B(\Delta) \to \underline{\Omega}^p_{X/B}(\log D) \otimes f^*\omega_B(\Delta) \quad \textit{and} \\ w'_p &: \underline{\Omega}^p_{X/B}(\log D) \otimes f^*\omega_B(\Delta) \to \underline{\Omega}^{p+1}_X(\log D) \end{aligned}$$

such that $\wedge_p = w'_p \circ w''_p$.

(8.7.2) *If* $w_p = w_p'' \otimes id_{f^*\omega_B(\Delta)^{-1}} \colon \underline{\Omega}^p_X(\log D) \to \underline{\Omega}^p_{X/B}(\log D)$*, then*

$$\underline{\Omega}^p_{X/B}(\log D) \otimes f^*\omega_B(\Delta) \xrightarrow{w_p'} \underline{\Omega}^{p+1}_X(\log D) \xrightarrow{w_{p+1}} \underline{\Omega}^{p+1}_{X/B}(\log D) \xrightarrow{+1}$$

is a distinguished triangle in $D(X)$*.*

(8.7.3) w_p *is functorial, i.e., if* $\phi\colon Y \to X$ *is a B-morphism, then there are natural maps in* $D(X)$ *forming a commutative diagram:*

$$\begin{array}{ccc} \underline{\Omega}^p_X(\log D) & \longrightarrow & \underline{\Omega}^p_{X/B}(\log D) \\ \downarrow & & \downarrow \\ R\phi_*\underline{\Omega}^p_Y(\log \phi^* D) & \longrightarrow & R\phi_*\underline{\Omega}^p_{Y/B}(\log \phi^* D). \end{array}$$

(8.7.4) $\underline{\Omega}^r_{X/B}(\log D) = 0$ *for* $r > n$*.*

(8.7.5) *If* f *is proper, then* $\underline{\Omega}^p_{X/B}(\log D) \in \mathrm{Ob}\left(D^b_{\mathrm{coh}}(X)\right)$ *for every* p*.*

(8.7.6) *If* f *is smooth over* $B \setminus \Delta$*, then* $\underline{\Omega}^p_{X/B}(\log D) \simeq_{qis} \Omega^p_{X/B}(\log D)$*.*

Using these objects one can make the proof work to obtain the following theorem. It is in a non-explicit form. For more precise statements see [Kov02, (7.8), (7.10), (7.11), (7.13)].

Theorem 8.8. *Fix* B*,* $\Delta \subset B$*. Then weak boundedness holds for families of canonically polarized varieties with rational Gorenstein singularities and fixed Hilbert polynomial admitting a simultaneous resolution of singularities over* $B \setminus \Delta$*. In particular,* $2g - 2 + \#\Delta > 0$ *for these families by* (3.8).

As a corollary, one obtains weak boundedness for non-birationally-isotrivial families of minimal varieties of general type.

8.B. *Iterated Kodaira-Spencer maps and strong non-isotriviality*

Let us finish by revisiting rigidity. We have seen in (4.A) that $(\mathbf{R})$ fails as stated in the original conjecture and we asked

QUESTION 8.9 = QUESTION 4.3. Under what additional conditions does $(\mathbf{R})$ hold?

This question was partially answered in [VZ03a] and [Kov05b]. Both papers gave essentially the same answer that we will discuss below. However, one must note that this is not the only case when rigidity holds as it was shown in [VZ05a]. In other words we do not have a sufficient and necessary criterion for rigidity.

8.10. ITERATED KODAIRA-SPENCER MAPS, CASE I: ONE-DIMENSIONAL BASES. Let $f\colon X \to B$ be a smooth projective family of varieties of general type of dimension n, B a smooth (not necessarily projective) curve and let $T^m_X := \wedge^m T_X$ and $T^m_{X/B} := \wedge^m T_{X/B}$.

Let $1 \leq p \leq n$ and consider the short exact sequence,

$$0 \to T^p_{X/B} \otimes f^*T_B^{\otimes(n-p)} \to T^p_X \otimes f^*T_B^{\otimes(n-p)} \to T^{p-1}_{X/B} \otimes f^*T_B^{\otimes(n-p+1)} \to 0.$$

This induces an edge map,

$$\rho_f^{(p)}\colon R^{p-1}f_*T^{p-1}_{X/B} \otimes T_B^{\otimes(n-p+1)} \to R^p f_* T^p_{X/B} \otimes T_B^{\otimes(n-p)}.$$

DEFINITION 8.11 [Kov05b]. Let $\rho_f := \rho_f^{(n)} \circ \rho_f^{(n-1)} \circ \cdots \circ \rho_f^{(1)} \colon T_B^{\otimes n} \longrightarrow R^n f_* T^n_{X/B}$ and call f *strongly non-isotrivial* if $\rho_f \neq 0$.

EXAMPLE 8.12. Let $Y_i \to B$ be admissible families of curves for $i = 1, \dots, r$. Then $X = Y_1 \times_B \cdots \times_B Y_r \to B$ is strongly non-isotrivial.

REMARK 8.13. Since T_B is a line bundle and $R^n f_* T^n_{X/B}$ is locally free, $\rho_f \neq 0$ if and only if it is injective. We use this observation in the definition of strong non-isotriviality for higher dimensional bases.

8.14. ITERATED KODAIRA-SPENCER MAPS, CASE II: HIGHER-DIMENSIONAL BASES. Let $f : X \to B$ be a smooth projective family of varieties of general type of dimension n, B a smooth (not necessarily projective) variety.

For an integer p, $1 \leq p \leq n$, there exists a filtration

$$T^p_X = \mathscr{F}^0 \supseteq \mathscr{F}^1 \supseteq \cdots \supseteq \mathscr{F}^p \supseteq \mathscr{F}^{p+1} = 0,$$

such that

$$\mathscr{F}^i \Big/ \mathscr{F}^{i+1} \simeq T^i_{X/B} \otimes f^* T^{p-i}_B.$$

In particular,

$$\mathscr{F}^p \simeq T^p_{X/B}$$

and

$$\mathscr{F}^{p-1} / \mathscr{F}^p \simeq T^{p-1}_{X/B} \otimes f^* T_B.$$

Therefore one has a short exact sequence,

$$0 \to T^p_{X/B} \otimes f^* T_B^{\otimes(n-p)} \to \mathscr{F}^{p-1} \otimes f^* T_B^{\otimes(n-p)} \to T^{p-1}_{X/B} \otimes f^* T_B^{\otimes(n-p+1)} \to 0,$$

that induces a map

$$\rho^{(p)}_f : R^{p-1} f_* T^{p-1}_{X/B} \otimes T_B^{\otimes(n-p+1)} \to R^p f_* T^p_{X/B} \otimes T_B^{\otimes(n-p)}.$$

DEFINITION 8.15 [Kov05b]. Let $\rho_f := \rho^{(n)}_f \circ \rho^{(n-1)}_f \circ \cdots \circ \rho^{(1)}_f : T_B^{\otimes n} \longrightarrow R^n f_* T^n_{X/B}$ and call f *strongly non-isotrivial over* B if ρ_f is injective.

EXAMPLE 8.16. Let $Y_i \to B$ be non-isotrivial families of smooth projective curves for $i = 1, \ldots, r$. Then $X = Y_1 \times_B \cdots \times_B Y_r \to B$ is strongly non-isotrivial over B.

REMARK 8.17. One could consider various refinements of this notion. For instance, consider maps for which the composition of fewer $\rho^{(p)}$'s is injective or non-zero. These appear for example in the study of moduli spaces of varieties that are products with one rigid term. One could also combine this condition with $\operatorname{Var} f$, the variation of f in moduli. This is a mostly unexplored area at the moment.

Therefore a possible answer to Question 4.3 is given by the following theorem:

Theorem 8.18 [VZ03a], [Kov05b]. *Let $f : X \to B$ be a smooth projective family of varieties of general type, B a smooth variety. If f is strongly non-isotrivial over B, then rigidity holds for f.*

This, combined with Theorem 4.8, leads to a statement resembling the original Shafarevich conjecture. In fact, for families of curves it simply reduces to that.

Theorem 8.19 [KL06]. *Let B, Δ and h be fixed. Then there exist only finitely many strongly non-isotrivial families of canonically polarized varieties with Hilbert polynomial h with respect to B, Δ.*

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