

STRONG NON-ISOTRIVIALITY AND RIGIDITY

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1. Introduction

Throughout the article the groundfield will always be \mathbb{C} , the field of complex numbers.

A family is called isotrivial if any two general members are isomorphic. For example the blow up of the projective plane at a single point, considered as a \mathbb{P}^1 -bundle over \mathbb{P}^1 , is an isotrivial family since all of its members are isomorphic to the projective line. Since the genus of a curve in a family is constant, and there is only one curve of genus zero, every family of rational curves is isotrivial. However, for higher genus curves, or more generally, higher dimensional varieties, one can find non-isotrivial families.

At the 1962 ICM in Stockholm, Shafarevich conjectured that there exist only finitely many isomorphism classes of non-isotrivial families of smooth projective curves of a given genus over a given base curve. Shafarevich further conjectured that if there is such a family, then the base curve satisfies a certain hyperbolic condition. For definitions and a more precise formulation, see Section §1. The conjecture was confirmed by [Parshin68] for the case of a compact base and by [Arakelov71] in general. It was recently generalized to families of higher dimensional varieties. This generalization however, is not straightforward.

It will be advantageous to work with a compactification of the family. Considering families over a compact base curve B naturally leads to a slightly different view on the problem. Instead of smooth families over a non-compact base, one may work with arbitrary families over a compact base, and consider the locus over which the family is smooth. Clearly these are equivalent situations. An arbitrary family over a compact base gives a smooth family over some open subset and a smooth family over an open curve can be extended to a (not necessarily smooth) family over the projective closure of the curve.

NOTATION 1.1. For a morphism $f : X \rightarrow B$, and a point $b \in B$, X_b will denote the scheme theoretic fibre $f^{-1}(b)$.

DEFINITION 1.2. Let B be a smooth projective variety and $\Delta \subseteq B$ a subvariety. A surjective flat morphism with connected fibers will be called a *family*. A family $f : X \rightarrow B$ is *isotrivial* if $X_a \simeq X_b$ for general points $a, b \in B$. The family $f : X \rightarrow B$ will be called *admissible with respect to* (B, Δ) if it is not isotrivial and Δ contains the discriminant locus of f , i.e., the map $f : X \setminus f^{-1}(\Delta) \rightarrow B \setminus \Delta$ is smooth.

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CONVENTIONS 1.3. Unless explicitly stated otherwise, *admissible* will always mean *admissible with respect to* (B, Δ) . Similarly, if \mathcal{E} and \mathcal{F} are \mathcal{O}_B -modules, then $\mathcal{E} \otimes_{\mathcal{O}_B} \mathcal{F}$ will be simply denoted by $\mathcal{E} \otimes \mathcal{F}$.

REMARK 1.4. A smooth projective family, all of whose fibers are isomorphic, is locally trivial in the euclidean topology by the Grauert-Fischer theorem [Fischer-Grauert65]. Similarly, an isotrivial family is generically locally trivial in the étale topology. Finally, the automorphism group of a variety of general type is finite [Kobayashi72, III.2.4]. In fact, even the birational automorphism group is finite [Litaka82, 11.12]. Therefore, if $f : X \rightarrow B$ is an isotrivial family of projective varieties of general type, then there exists an étale cover $\tilde{B} \rightarrow B \setminus \Delta$ such that the family

$$\tilde{f}: \tilde{X} = (X \setminus f^{-1}(\Delta)) \times_{B \setminus \Delta} \tilde{B} \rightarrow \tilde{B}$$

is trivial.

Our starting point is the aforementioned conjecture of Shafarevich:

Shafarevich's Conjecture 1.5. *Let B be a smooth projective curve of genus g and $\Delta \subseteq B$ a finite subset. Further let $q \in \mathbb{Z}, q \geq 2$. Then*

(1.5.1) *there exist only finitely many isomorphism classes of admissible families of curves of genus q , and*

(1.5.2) *if $2g - 2 + \#\Delta \leq 0$, then there exist no such families.*

Shafarevich showed a special case of (1.5.2): There exist no smooth families of curves of genus q over \mathbb{P}^1 . (1.5.1) was confirmed by [Parshin68] for $\Delta = \emptyset$ and by [Arakelov71] in general.

The ultimate goal is to generalize this statement to higher dimensional families. In order to work toward that goal, the statement has to be reformulated following Parshin and Arakelov.

DEFINITION 1.6. A *deformation (over T)* of a family $g: Y \rightarrow S$ (with the base fixed) is a family $\mathfrak{Y} \rightarrow S \times T$, such that for some $t_0 \in T$, $(\mathfrak{Y}_{t_0} \rightarrow S \times \{t_0\}) \simeq (Y \rightarrow S)$:

$$\begin{array}{ccc} Y \simeq \mathfrak{Y}_{t_0} & \longrightarrow & \mathfrak{Y} \\ g \downarrow & & \downarrow g \\ S \simeq S \times \{t_0\} & \longrightarrow & S \times T. \end{array}$$

For simplicity, $S \times \{t\}$ will be denoted by S_t . Two families $Y_1 \rightarrow S$ and $Y_2 \rightarrow S$ are said to have the same *deformation type* if they can be deformed into each other, i.e., if there exists a connected T and a deformation of families, $\mathfrak{Y} \rightarrow S \times T$ such that for some $t_1, t_2 \in T$, $(\mathfrak{Y}_{t_i} \rightarrow S_{t_i}) \simeq (Y_i \rightarrow S \times T)$ for $i = 1, 2$.

Next consider deformations of admissible families over the base $B \setminus \Delta$. Doing so potentially allows more deformations than over the original base B : it can easily happen that a deformation over $B \setminus \Delta$, that is, a family $\mathfrak{X} \rightarrow (B \setminus \Delta) \times T$, cannot be compactified to a (flat) family over $B \times T$, because the compactification may contain fibers of dimension that are higher than expected. This however, will not cause any problems because of the nature of the present inquiry. One wants to argue the opposite way. If one can prove rigidity for a family over $B \setminus \Delta$, then it will automatically apply to any family over B as well.

With regard to the Shafarevich conjecture, Parshin made the following observation. In order to prove that there are only finitely many admissible families, one can try to proceed the following way. Instead of aiming for the general statement immediately, first try to

prove that there are only finitely many deformation types. This is expected to be somewhat easier, because there are ways to parametrize deformations, and the fact that there are only finitely many types translates to the parameter space being of finite type. The next step then is to prove that admissible families are rigid, that is, they do not admit non-trivial deformations. Notice that if one proves these statements for families over $B \setminus \Delta$, then they also follow for families over B . Now, if every deformation type contains only one family, and there are only finitely many deformation types, the original statement follows.

The following is the reformulation of the Shafarevich conjecture according to the principle just outlined. This was already used by Parshin and Arakelov.

REFORMULATION 1.7. Let B be a smooth projective curve of genus g and $\Delta \subseteq B$ a finite subset. Further let $q \in \mathbb{Z}, q \geq 2$. Recall the convention from (1.3).

- (1.7.1) *Boundedness*: There exist only finitely many deformation types of admissible families, i.e., admissible families of curves of genus q are parametrized by \mathbb{T} , a scheme of finite type.
- (1.7.2) *Rigidity*: There exist no non-trivial deformations of any admissible family of curves of genus q , in particular, $\dim \mathbb{T} = 0$.
- (1.7.3) *Hyperbolicity*: There do not exist admissible families of curves of genus q if $2g - 2 + \#\Delta \leq 0$, i.e., $\mathbb{T} \neq \emptyset \Rightarrow 2g - 2 + \#\Delta > 0$.

Notice that the notion of rigidity here is somewhat different from the one commonly used. A family will be called *rigid* if its deformation space is zero-dimensional. This intention is made precise below.

DEFINITION 1.8. A deformation, $\mathfrak{g}: \mathfrak{Y} \rightarrow S \times T$, is called *essentially trivial* if there exists an open neighborhood T_0 of $t_0 \in T$ such that for all $t \in T_0$, $(\mathfrak{Y}_t \rightarrow S_t) \simeq (Y \rightarrow S)$. It is called *trivial* if $\mathfrak{Y} = Y \times_S (S \times T)$ and $\mathfrak{g} = g \times \pi_S$. Notice that if $T = \text{Spec } R$ where R is a local ring, then an essentially trivial family is also trivial.

Arakelov-Parshin rigidity is said to hold for a family $g: Y \rightarrow S$ if it does not admit a non-trivial deformation over $T = \text{Spec } R$ where R is a DVR. By a slight abuse of terminology, in this case g will be also called simply *rigid*.

The following notion will also be needed.

DEFINITION 1.9. Let $g: Y \rightarrow S$ be a family of projective varieties of general type. By [Kollár87, 2.5] there exists an open subset $S_0 \subseteq S$ and a morphism $\nu_S: S_0 \rightarrow Z$ such that Y_s and Y_t , for $s, t \in S_0$, are birational if and only if $\nu_S(s) = \nu_S(t)$. Then the *variation of g in moduli*, denoted by $\text{Var } g$, is defined as the dimension of the image of ν_S , that is, $\text{Var } g := \dim \overline{\nu_S(S)}$. Obviously $0 \leq \text{Var } g \leq \dim S$.

REMARK 1.10. Let $g: Y \rightarrow S$ be a family of varieties of general type. Then using Kollár's birational moduli map, ν_S , it is easy to see that a deformation $\mathfrak{g}: \mathfrak{Y} \rightarrow S \times T$ of g is essentially trivial if and only if $\text{Var } \mathfrak{g} = \text{Var } g$. In particular, if $\dim T = 1$ and g has maximal variation in moduli, i.e., $\text{Var } g = \dim S$, then \mathfrak{g} is essentially trivial if and only if $\text{Var } \mathfrak{g} < \dim(S \times T)$.

This implies that a family $g: Y \rightarrow S$ with $\text{Var } g = \dim S$ is *rigid* if and only if there does not exist a deformation $\mathfrak{g}: \mathfrak{Y} \rightarrow S \times T$ of g with $\text{Var } \mathfrak{g} = \dim(S \times T)$.

Next an equivalent criterion for a family to be rigid will be given.

Lemma 1.11. *Let $g: Y \rightarrow S$ be a family of varieties of general type with $\text{Var } g = \dim S$. Let T be a smooth curve. Then g is rigid if and only if all of its deformations over T are essentially trivial. Equivalently, g is rigid if and only if for any deformation \mathfrak{g} over T , $\text{Var } \mathfrak{g} < \dim(S \times T)$.*

PROOF. Let $\mathfrak{g}: \mathfrak{Y} \rightarrow S \times T$ be a deformation of g over T . Let $t \in T$ and $R = \mathcal{O}_{T,t}$. Clearly, \mathfrak{g} is essentially trivial if and only if its pull-back to $\text{Spec } R$ is trivial. The latter is equivalent to g being rigid. \square

REMARK 1.12. As discussed above, (1.7.1) and (1.7.2) together imply (1.5.1) and (1.7.3) is clearly equivalent to (1.5.2).

Recently there has been a flurry of activity regarding higher dimensional generalizations of (1.7.1) and (1.7.3), cf. [Migliorini95], [Zhang97], [Bedulev-Viehweg00], [Oguiso-Viehweg01], [Viehweg-Zuo01,02,03], [Kovács96,97abc,00,02,03ab]. For detailed surveys one can turn to [Viehweg00] and [Kovács03c].

The focus of this note is (1.7.2): Rigidity. It should also be noted, that some of the results here have been independently obtained by [Viehweg-Zuo04], although their main emphasis is somewhat different.

Unfortunately rigidity, as stated, fails for higher dimensional families as the following simple example shows.

EXAMPLE 1.13. Let $Y \rightarrow B$ be an arbitrary non-isotrivial family of curves of genus ≥ 2 , and C a smooth projective curve of genus ≥ 2 . Then $f: X = Y \times C \rightarrow B$ is an admissible family, and a deformation of C gives a deformation of f . Therefore (1.7.2) fails as stated.

However, this example gives the feeling of cheating. The family defined here is indeed a non-isotrivial family, but only in one direction. It contains, as a term of a fibered product, an isotrivial family, and that's what makes rigidity fail. The goal of this article is to give a definition of a stronger non-isotriviality notion, that does not allow this to happen and prove that for that notion, rigidity holds. The main result is the following. For the definition of strong non-isotriviality, see (4.7).

Theorem 1.14. *Let f be an admissible family of projective varieties of general type. If f is strongly non-isotrivial over B , then Arakelov-Parshin rigidity holds for f .*

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2. Setup

Let $f: X \rightarrow B$ be a projective family where X and B are smooth varieties. Then the short exact sequence,

$$0 \rightarrow T_{X/B} \rightarrow T_X \rightarrow f^*T_B \rightarrow 0,$$

induces an \mathcal{O}_B -homomorphism,

$$\rho_f: T_B \rightarrow R^1 f_* T_{X/B},$$

the Kodaira-Spencer map of f . It is well-known that f is isotrivial if and only if ρ_f is injective.

GOALS AND EXPECTATIONS 2.1. One would like to find a condition, call it *strong non-isotriviality*, that strengthens the notion of “non-isotriviality” so that rigidity holds for every strongly non-isotrivial family. Naturally, the desired condition should be reasonably weak but still have this property.

As in the case of non-isotriviality, one expects that strong non-isotriviality will only depend on the general behavior of the family. For $f: X \rightarrow B$, and a non-empty open subset $B_0 \subseteq B$, let $f_0: X_0 = f^{-1}(B_0) \rightarrow B_0$. Then one expects that f is strongly non-isotrivial if and only if so is f_0 . Notice that if rigidity holds for f_0 , then it also holds for f .

REMARK 2.2. The expectation that strong non-isotriviality should be invariant under restriction to a smaller base and the resulting guiding philosophy means that one is seeking a criterion for rigidity in a local sense. It is also worth studying rigidity with the base fixed, but that will not be addressed here.

3. Case study: Rigidity for products

EXAMPLE 3.1. Let B be a smooth projective curve, and let $g : Y \rightarrow B$ and $h : Z \rightarrow B$ be two families of smooth projective curves of genus at least two. Finally, let $X = Y \times_B Z$.

$$\begin{array}{ccc}
 & X & \\
 h_Y \swarrow & & \searrow g_Z \\
 Y & & Z \\
 g \searrow & f \downarrow & \swarrow h \\
 & B &
 \end{array}$$

Observe that if either g or h is isotrivial, then after an étale base change it becomes trivial and hence rigidity fails for the pull-back of f . Because of this, f should not be called strongly non-isotrivial if either g or h is isotrivial.

CLAIM 3.2. Using the notation from (3.1),

$$H^1(X, T_{X/B}) \simeq H^1(Y, T_{Y/B}) \oplus H^1(Z, T_{Z/B}).$$

PROOF. $T_{X/B} \simeq h_Y^* T_{Y/B} \oplus g_Z^* T_{Z/B}$ implies that

$$H^1(X, T_{X/B}) \simeq H^1(X, h_Y^* T_{Y/B}) \oplus H^1(X, g_Z^* T_{Z/B}).$$

Since $X = Y \times_B Z$, $R^1(h_Y)_* \mathcal{O}_X \simeq H^1(Z, \mathcal{O}_Z) \otimes \mathcal{O}_Y$, hence

$$H^0(Y, R^1(h_Y)_* \mathcal{O}_X \otimes T_{Y/B}) \simeq H^1(Z, \mathcal{O}_Z) \otimes H^0(Y, \omega_{Y/B}^{-1}) = 0.$$

Then by the Leray spectral sequence, $H^1(Y, T_{Y/B}) \simeq H^1(X, h_Y^* T_{Y/B})$, and similarly $H^1(Z, T_{Z/B}) \simeq H^1(X, g_Z^* T_{Z/B})$. \square

Proposition 3.3. *Still using the notation from (3.1), if g and h are both non-isotrivial, then $H^1(X, T_{X/B}) = 0$. In particular rigidity holds for f .*

PROOF. If g is non-isotrivial, then $\omega_{Y/B}$ is ample by [Kovács96, 2.16], cf. [Kovács03, 11.15]. Then by Kodaira vanishing

$$H^1(Y, T_{Y/B}) = H^1(Y, \omega_{Y/B}^{-1}) = 0.$$

Similarly $H^1(Z, T_{Z/B}) = 0$, and hence by (3.2) $H^1(X, T_{X/B}) = 0$. \square

CONCLUSION 3.4. For a fibered product of smooth projective families of curves the product family, $f = g \times_B h$ will be called *strongly non-isotrivial* if both g and h are non-isotrivial. It follows that for such an f rigidity holds. Thus one should define 'strong non-isotriviality' in such a way that for a product it coincides with this definition.

4. Iterated Kodaira-Spencer maps

Start with the previous situation, that is, $g : Y \rightarrow B$ and $h : Z \rightarrow B$ are families of curves of genus at least two and $f = g \times_B h : X = Y \times_B Z \rightarrow B$ their fibered product.

Consider the Kodaira-Spencer maps associated with g and h : $\rho_g : T_B \rightarrow R^1 g_* T_{Y/B}$, $\rho_h : T_B \rightarrow R^1 h_* T_{Z/B}$, and their tensor product:

$$\rho_g \otimes \rho_h : T_B^{\otimes 2} \rightarrow R^1 g_* T_{Y/B} \otimes R^1 h_* T_{Z/B}.$$

From (3.3) one obtains the following.

Corollary 4.1. *If $\rho_g \otimes \rho_h \neq 0$ then rigidity holds for f .*

4.2 Iterated Kodaira-Spencer maps: products. Let $\wedge^m T_X$ be denoted by T_X^m , and $\wedge^m T_{X/B}$ by $T_{X/B}^m$. Since $g_* T_{Y/B} = 0$ and $h_* T_{Z/B} = 0$ the Künneth formula implies that

$$R^1 g_* T_{Y/B} \otimes R^1 h_* T_{Z/B} \simeq R^2 f_* T_{X/B}^2.$$

Consider the short exact sequence

$$0 \rightarrow T_{X/B} \rightarrow T_X \rightarrow f^* T_B \rightarrow 0.$$

As above this induces the Kodaira-Spencer map of f : $\rho_f : T_B \rightarrow R^1 f_* T_{X/B}$. Let us define

$$\rho_f^{(1)} = \rho_f \otimes \text{id}_{T_B} : T_B^{\otimes 2} \rightarrow R^1 f_* T_{X/B} \otimes T_B.$$

By taking exterior powers one obtains the short exact sequence,

$$0 \rightarrow T_{X/B}^2 \rightarrow T_X^2 \rightarrow T_{X/B} \otimes f^* T_B \rightarrow 0,$$

which in turn induces the map

$$\rho_f^{(2)} : R^1 f_* T_{X/B} \otimes T_B \rightarrow R^2 f_* T_{X/B}^2.$$

Notice that one can compose $\rho_f^{(1)}$ and $\rho_f^{(2)}$ to obtain a map,

$$\rho_f^{(2)} \circ \rho_f^{(1)} : T_B^{\otimes 2} \rightarrow R^2 f_* T_{X/B}^2.$$

Observation 4.3. $\rho_f^{(2)} \circ \rho_f^{(1)} = \rho_g \otimes \rho_h$.

This statement is not too hard to prove, but unfortunately I do not know a short proof. By writing out explicitly what each side means one can conclude that they are indeed the same. It did not seem practical to include a cumbersome proof of a fact that is only included for motivation, so checking this equality is regretfully left to the reader.

Corollary 4.4. *If $\rho_f^{(2)} \circ \rho_f^{(1)} \neq 0$, then rigidity holds for f .*

REMARK 4.5. Notice that this statement no longer makes reference to the product structure. This fact will be important in dealing with more general families.

4.6 Iterated Kodaira-Spencer maps: one-dimensional base. Let $f : X \rightarrow B$ be a smooth projective family of varieties of general type of dimension n , B a smooth (not necessarily projective) curve and keep using the notation T_X^m for $\wedge^m T_X$ and $T_{X/B}^m$ for $\wedge^m T_{X/B}$.

Let $1 \leq p \leq n$ and consider the short exact sequence,

$$0 \rightarrow T_{X/B}^p \otimes f^* T_B^{\otimes(n-p)} \rightarrow T_X^p \otimes f^* T_B^{\otimes(n-p)} \rightarrow T_{X/B}^{p-1} \otimes f^* T_B^{\otimes(n-p+1)} \rightarrow 0.$$

This induces an edge map,

$$\rho_f^{(p)} : R^{p-1} f_* T_{X/B}^{p-1} \otimes T_B^{\otimes(n-p+1)} \rightarrow R^p f_* T_{X/B}^p \otimes T_B^{\otimes(n-p)}.$$

DEFINITION 4.7. Let $\rho_f := \rho_f^{(n)} \circ \rho_f^{(n-1)} \circ \dots \circ \rho_f^{(1)} : T_B^{\otimes n} \longrightarrow R^n f_* T_{X/B}^n$ and call f *strongly non-isotrivial* if $\rho_f \neq 0$.

EXAMPLE 4.8. Let $Y_i \rightarrow B$ be non-isotrivial families of smooth projective curves for $i = 1, \dots, r$. Then $X = Y_1 \times_B \dots \times_B Y_r \rightarrow B$ is strongly non-isotrivial.

REMARK 4.9. Since T_B is a line bundle and $R^n f_* T_{X/B}^n$ is locally free, $\rho_f \neq 0$ if and only if it is injective.

4.10 Iterated Kodaira-Spencer maps: higher-dimensional base. Let $f : X \rightarrow B$ be a smooth projective family of varieties of general type of dimension n , B a smooth (not necessarily projective) variety.

For an integer p , $1 \leq p \leq n$, there exists a filtration

$$T_X^p = \mathcal{F}^0 \supseteq \mathcal{F}^1 \supseteq \dots \supseteq \mathcal{F}^p \supseteq \mathcal{F}^{p+1} = 0,$$

such that

$$\mathcal{F}^i / \mathcal{F}^{i+1} \simeq T_{X/B}^i \otimes f^* T_B^{p-i}.$$

In particular,

$$\mathcal{F}^p \simeq T_{X/B}^p$$

and

$$\mathcal{F}^{p-1} / \mathcal{F}^p \simeq T_{X/B}^{p-1} \otimes f^* T_B.$$

Therefore

$$0 \rightarrow T_{X/B}^p \otimes f^* T_B^{\otimes(n-p)} \rightarrow \mathcal{F}^{p-1} \otimes f^* T_B^{\otimes(n-p)} \rightarrow T_{X/B}^{p-1} \otimes f^* T_B^{\otimes(n-p+1)} \rightarrow 0$$

induces a map

$$\rho_f^{(p)} : R^{p-1} f_* T_{X/B}^{p-1} \otimes T_B^{\otimes(n-p+1)} \rightarrow R^p f_* T_{X/B}^p \otimes T_B^{\otimes(n-p)}.$$

DEFINITION 4.11. Let $\rho_f := \rho_f^{(n)} \circ \rho_f^{(n-1)} \circ \dots \circ \rho_f^{(1)} : T_B^{\otimes n} \longrightarrow R^n f_* T_{X/B}^n$ and call f *strongly non-isotrivial over B* if ρ_f is injective.

EXAMPLE 4.12. Let $Y_i \rightarrow B$ be non-isotrivial families of smooth projective curves for $i = 1, \dots, r$. Then $X = Y_1 \times_B \dots \times_B Y_r \rightarrow B$ is strongly non-isotrivial over B .

REMARK 4.13. One could consider various refinements:

- Consider maps for which the composition of fewer $\rho^{(p)}$'s is injective or non-zero. This is important in particular to study moduli spaces of varieties that are products with one rigid term.
- Combine this condition with $\text{Var } f$, the variation of f in moduli.

Theorem 4.14. *Let $f : X \rightarrow B$ be a smooth projective family of varieties of general type, B a smooth variety. If f is strongly non-isotrivial over B , then rigidity holds for f .*

PROOF. Let $f : \mathcal{X} \rightarrow B \times T$ be a deformation of f , and $t_0 \in T$ such that $f_{t_0} : \mathcal{X}_{t_0} \rightarrow B_{t_0}$ is isomorphic to $f : X \rightarrow B$. Here $B_{t_0} = B \times \{t_0\}$. Notice that without loss of generality one may assume that $\dim T = 1$

Using Grothendieck-Serre duality it is easy to see that $\rho_f : T_B^{\otimes n} \rightarrow R^n f_* T_{X/B}^n$ being injective implies that $f_* \omega_{X/B}^{\otimes 2} \rightarrow \Omega_B^{\otimes n}$ is generically surjective. From that and Nakayama's lemma one obtains that

$$(4.14.1) \quad f_* \omega_{\mathcal{X}/B \times T}^{\otimes 2} \rightarrow \Omega_{B \times T/T}^{\otimes n}$$

is generically surjective in a neighborhood of B_{t_0} .

By (1.11), rigidity fails for f if and only if there exists an f such that $\text{Var } f = \dim(B \times T)$. Assume that $\text{Var } f = \dim(B \times T)$ and try to derive a contradiction.

By [Kollár87, Theorem p.363] and [Esnault-Viehweg91, 0.1] $f_*\omega_{\mathbb{A}^1/B \times T}^{\otimes 2}$ is big and then by (4.14.1) so is $\Omega_{B \times T/T}^{\otimes n}$. However, $\Omega_{B \times T/T}^{\otimes n} = \pi_B^*\Omega_B^{\otimes n}$, which means that it cannot be big. This is a contradiction, so the statement is proven. \square

Corollary 4.15. *Theorem 1.14 follows.*

PROOF. If $f : X \rightarrow B$ is admissible with respect to (B, Δ) , then let $B^\circ := B \setminus \Delta$, $X^\circ := f^{-1}(B^\circ)$, and $f^\circ := f|_{X^\circ}$. Next apply (4.14) for $f^\circ : X^\circ \rightarrow B^\circ$. Finally, notice that if f° is rigid, then so is f . \square

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