STEENBRINK VANISHING EXTENDED

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To Steven Kleiman on the occasion of his 70th birthday

ABSTRACT. The notion of *DB index*, a measure of how far a singularity of a pair is from being Du Bois, is introduced and used to generalize vanishing theorems of [Ste85], [GKKP11], and [Kov11] with simpler and more natural proofs than the originals. An argument used in one of these proofs also yields an additional theorem connecting various push forwards that lie outside of the range of the validity of the above vanishing theorems.

1. INTRODUCTION

The importance of rational singularities has been demonstrated for decades through various applications. Log terminal singularities (of all stripes) are rational and this single fact has far reaching consequences in the minimal model program. Unfortunately, not all singularities that appear in the minimal model program are rational. In particular, the class of log canonical singularities which emerges as the most important class in many applications, for instance in moduli theory, is not necessarily rational.

The class of Du Bois singularities is an enlargement of the class of rational singularities. Even though this notion was introduced several decades ago [Ste83], it has remained relatively obscure for a long time. It was recently proved that log canonical singularities are Du Bois [KK10] and this fact has started a flurry of activities and Du Bois singularities are becoming central in the minimal model program and related areas.

An important application of Du Bois singularities appeared in [GKKP11] and in some other articles that grew out of it [Dru13, Gra13]. The way Du Bois singularities were used in these articles is through a vanishing theorem that can be considered a generalization of a vanishing theorem due to Steenbrink [Ste85].

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The notion of Du Bois singularities was recently extended for pairs in [Kov11] and the purpose of the present article is to extend the vanishing theorem used in [GKKP11] to Du Bois pairs. In particular, the following is proved.

Theorem 1.1. *Let X be a normal variety and $\pi : Y \to X$ a resolution of singularities. Let $\Sigma \subseteq X$ be a subvariety and set $E = \operatorname{Exc}(\pi)$ and $\Gamma = E \cup (\pi^{-1}\Sigma)_{red}$. Assume that (X, Σ) is a Du Bois pair and that Γ is an snc divisor. Then for all p,*

$$\mathcal{R}^{\dim X - 1}\pi_*\big(\Omega^p_Y(\log\Gamma)(-\Gamma)\big) = 0.$$

In fact, a stronger version will be proved in Corollary 5.3, but stating the stronger form requires some preparation, done in §4. Theorem 1.1 is a generalization of [GKKP11, Thm. 14.1]. As noted in [GKKP11] it follows from Steenbrink's vanishing theorem [Ste85, Thm. 2(b)] in the cases $p > 1$. In §3 a generalization, Theorem 3.1, of Steenbrink's vanishing is proved after reviewing the definition and basic properties of the Deligne-Du Bois complex in §2. Arguably the proof presented here is much simpler than Steenbrink's original proof and perhaps more importantly, at least in the opinion of the author, more natural and makes it more clear why the statement is true. The same is true for the $p = 1$ case of Theorem 1.1 which is proved in §5. The final section, §6, contains a theorem that could be considered a byproduct of the proof in §5.

DEFINITIONS AND NOTATION 1.2. Unless otherwise stated, all objects are assumed to be defined over $\mathbb{C}$, all schemes are assumed to be of finite type over $\mathbb{C}$ and a morphism means a morphism between schemes of finite type over $\mathbb{C}$.

If $\phi : Y \to Z$ is a birational morphism, then $\operatorname{Exc}(\phi)$ will denote the *exceptional set* of ϕ. For a closed subscheme $W \subseteq X$, the ideal sheaf of W is denoted by $\mathscr{I}_{W\subseteq X}$ or if no confusion is likely, then simply by $\mathscr{I}_W$. For a point $x \in X$, $\kappa(x)$ denotes the residue field of $\mathscr{O}_{X,x}$.

The right derived functor of an additive functor F, if it exists, is denoted by $\mathcal{R}F$ and $\mathcal{R}^iF$ is short for $h^i \circ \mathcal{R}F$, where h^i is the i^{th} cohomology sheaf of a complex.

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2. THE DELIGNE-DU BOIS COMPLEX

The Deligne-Du Bois complex is a generalization of the de Rham complex to singular varieties. It is a complex of sheaves on X that is quasi-isomorphic to the constant sheaf $\mathbb{C}_X$. The terms of this complex are harder to describe but its properties, especially cohomological properties are very similar to the de Rham complex of smooth varieties. In fact, for a smooth variety the Deligne-Du Bois complex is quasi-isomorphic to the de Rham complex, so it is indeed a direct generalization.

The original construction of this complex, $\underline{\Omega}^{\bullet}_X$, is based on simplicial resolutions. The reader interested in the details is referred to the original article [DB81]. Note also that a simplified construction was later obtained in [Car85] and [GNPP88] via the general theory of polyhedral and cubic resolutions. An easily accessible introduction can be found in [Ste85]. Other useful references are the recent book [PS08] and the survey [KS09]. We will actually not use these resolutions here. They are needed for the construction, but if one is willing to believe the listed properties (which follow in a rather straightforward way from the construction) then one should be able follow the material presented here. The interested reader should note that recently Schwede found a simpler alternative construction of (part of) the Deligne-Du Bois complex that does not need a simplicial resolution [Sch07]. For applications of the Deligne-Du Bois complex and Du Bois singularities other than the ones listed here see [Ste83], [Kol95, Chapter 12], [Kov99, Kov00, KSS10, KK10].

As mentioned in the introduction, the Deligne-Du Bois theory was recently extended for pairs in [Kov11]. In particular, we will be using the Deligne-Du Bois complex of a pair. I will not repeat the construction or all the basic properties of this complex. The interested reader may consult the original article [Kov11] or the more recent and more detailed account in [Kol13, §6]. In particular, the most important properties of the Deligne-Du Bois complex of a pair are listed in [Kol13, Theorem 6.5]. I will only recall the ones that are used here.

DEFINITION 2.1. A *reduced pair* consists of X a reduced scheme of finite type over $\mathbb{C}$ and $\Sigma \subseteq X$ a reduced closed subscheme of X. The Deligne-Du Bois complex of the reduced pair (X, Σ) is denoted by $\underline{\Omega}^{\bullet}_{X,\Sigma}$. If $\Sigma = \emptyset$, then this reduces to the Deligne-Du Bois complex of X and will be denoted by $\underline{\Omega}^{\bullet}_X$.

The shifted associated graded quotients of $\underline{\Omega}^{\bullet}_{X,\Sigma}$ are denoted and defined by the following formula:

$$\underline{\Omega}^{p}_{X,\Sigma} := Gr^{p}_{\text{filt}} \underline{\Omega}^{\bullet}_X[p].$$

BASIC PROPERTIES 2.2. The Deligne-Du Bois complex is a resolution of the sheaf $j_!\mathbb{C}_{X\setminus\Sigma}$, the constant sheaf $\mathbb{C}$ on $X\setminus\Sigma$ extended by 0 to the entire X [Kov11, Thm. 4.1], [Kol13, Thm. 6.5(1)]:

$$j_!\mathbb{C}_{X\setminus\Sigma} \simeq_{\text{qis}} \underline{\Omega}^{\bullet}_{X,\Sigma} \tag{2.2.1}$$

Here, of course, $j : X\setminus\Sigma \hookrightarrow X$ is the natural inclusion map.

If X is smooth and Σ is an snc divisor on X, then the shifted associated graded quotients have only one non-zero cohomology sheaf, h^0, and that one is given by logarithmic differentials vanishing along Σ [Kov11, (3.10)], [Kol13, Thm. 6.5(3)], that is, if (X,Σ) is an snc pair, then

$$\underline{\Omega}^p_{X,\Sigma} \simeq_{\text{qis}} \Omega^p_X(\log\Sigma)(-\Sigma) \simeq \Omega^p_X(\log\Sigma)\otimes\mathscr{I}_{\Sigma\subseteq X} \tag{2.2.2}$$

The following two distinguished triangles will be important in the sequel.

The first one connects the Deligne-Du Bois complex of a pair to that of the members of the pair. See [Kov11] or [Kol13, Thm. 6.5(7)]: For any $p\in\mathbb{N}$ there exists a distinguished triangle,

$$\underline{\Omega}^p_{X,\Sigma} \longrightarrow \underline{\Omega}^p_X \longrightarrow \underline{\Omega}^p_\Sigma \xrightarrow{+1} . \tag{2.2.3}$$

The second one relates the Deligne-Du Bois complexes of pairs connected via a birational morphism. See [DB81] or [Kol13, Thm. 6.5(10)]: Let $\pi : Y\to X$ be a projective morphism and $\Sigma\subseteq X$ a reduced closed subscheme such that π is an isomorphism over $X\setminus\Sigma$. Set $\Gamma = (\pi^{-1}\Sigma)_{\text{red}}$. Then for any $p\in\mathbb{N}$ there exists a distinguished triangle,

$$\underline{\Omega}^p_X \longrightarrow \underline{\Omega}^p_\Sigma \oplus \mathcal{R}\pi_*\underline{\Omega}^p_Y \longrightarrow \mathcal{R}\pi_*\underline{\Omega}^p_\Gamma \xrightarrow{+1} . \tag{2.2.4}$$

One more property of the Deligne-Du Bois complex will be very useful. This may be considered a Grauert-Riemenschneider-type vanishing theorem in this setting. See [GNPP88, V.6.2]: For any X a reduced scheme of finite type over $\mathbb{C}$,

$$h^q(\underline{\Omega}^p_X) = 0, \text{ for } p+q > \dim X \tag{2.2.5}$$

3. GENERALIZED STEENBRINK VANISHING

The following theorem extends Steenbrink's vanishing theorem to a more general situation.

Theorem 3.1. *Let $\pi : Y\to X$ be a projective morphism of reduced schemes of finite type over $\mathbb{C}$ and $\Sigma\subseteq X$ a reduced closed subscheme such that π is an isomorphism over $X\setminus\Sigma$. Assume that Σ does not contain any irreducible components of X. Set $\Gamma = (\pi^{-1}\Sigma)_{red}$. Then for any $x\in X$,*

$$(\mathcal{R}^q\pi_*\underline{\Omega}^p_{Y,\Gamma})_x = 0, \textit{ for } p+q > \dim_x X.$$

In particular,

$$\mathcal{R}^q\pi_*\underline{\Omega}^p_{Y,\Gamma} = 0, \textit{ for } p+q > \dim X.$$

Proof. Let $x \in X$ and $n := \dim_x X$. The statement is local on X, so we may restrict to a neighborhood of x and assume that $\dim X = \dim_x X$.

Consider the long exact sequence of cohomology sheaves induced by the distinguished triangle (2.2.4):

$$\cdots \to h^q(\underline{\Omega}^p_X) \to h^q(\underline{\Omega}^p_\Sigma) \oplus \mathcal{R}^q\pi_*\underline{\Omega}^p_Y \to \mathcal{R}^q\pi_*\underline{\Omega}^p_\Gamma \to h^{q+1}(\underline{\Omega}^p_X) \to \cdots .$$

By (2.2.5) the morphism

$$\mathcal{R}^q\pi_*\underline{\Omega}^p_Y \to \mathcal{R}^q\pi_*\underline{\Omega}^p_\Gamma \tag{3.1.1}$$

is an isomorphism for $p+q > n$ and a surjection for $p+q = n$. Notice that this is the place where we use the assumption that Σ does not contain any irreducible components of X: Indeed that implies that then $\dim_x \Sigma < n$ and hence for $p+q = n$, we still have that $h^q(\underline{\Omega}^p_\Sigma) = 0$. Without this we could only conclude that $h^q(\underline{\Omega}^p_\Sigma) \oplus \mathcal{R}^q\pi_*\underline{\Omega}^p_Y \to \mathcal{R}^q\pi_*\underline{\Omega}^p_\Gamma$ is surjective, but this is not sufficient in the next step.

Next consider the long exact sequence induced by applying the functor $\mathcal{R}\pi_*$ to the distinguished triangle from (2.2.3) on Y:

$$\cdots \to \mathcal{R}^{q-1}\pi_*\underline{\Omega}^p_Y \to \mathcal{R}^{q-1}\pi_*\underline{\Omega}^p_\Gamma \to \mathcal{R}^q\pi_*\underline{\Omega}^p_{Y,\Gamma} \to \mathcal{R}^q\pi_*\underline{\Omega}^p_Y \to \mathcal{R}^q\pi_*\underline{\Omega}^p_\Gamma \to \cdots .$$

Now the desired statement follows from (3.1.1). $\square$

This way we obtained a very simple proof of Steenbrink's vanishing theorem:

Theorem 3.2. [Ste85, Thm. 2(b)] *Let X be a complex variety, $\Sigma \subseteq X$ such that $X \setminus \Sigma$ is smooth and $\pi : Y \to X$ a proper birational morphism such that Y is smooth, $E = \pi^{-1}\Sigma$ is an snc divisor on Y and π induces an isomorphism between $Y \setminus E$ and $X \setminus \Sigma$. Then*

$$\mathcal{R}^q\pi_*\Omega^p_Y(\log E)(-E) = 0, \textit{ for } p+q > \dim X.$$

Proof. By (2.2.2) this is a direct consequence of (3.1). $\square$

4. DB INDEX AND MORE VANISHING

First we will extend the notion of Du Bois pairs as follows.

DEFINITION 4.1. Let (X, Σ) be a reduced pair and $x \in X$ a point. The *local DB index of* (X, Σ) *at* x, $\mathcal{db}_x(X, \Sigma)$, is the smallest natural number above which the cohomology sheaves of the 0^{th} shifted associated graded complex of the Deligne-Du Bois complex of (X, Σ) vanish at x. In other words,

$$\mathcal{db}_x(X, \Sigma) = \min\{q \in \mathbb{N} \mid h^i(\underline{\Omega}^0_{X,\Sigma})_x = 0 \text{ for } i > q\}.$$

Notice that $\mathcal{db}_x(X\Sigma)$ is an upper semicontinuos function of x.

The *DB index of* (X,Σ) is the maximum of the local DB indices of (X,Σ) at x for all $x \in X$, that is,

$$\mathcal{db}(X,\Sigma) = \max\{\mathcal{db}_x(X,\Sigma) \mid x \in X\}.$$

Observe that if (X,Σ) is a Du Bois pair, then $\mathcal{db}(X,\Sigma) = 0$, but the converse is not true. For instance, any reduced curve C has $\mathcal{db}(C,\emptyset) = 0$, but C is Du Bois if and only if it is seminormal.

CLAIM 4.2. (cf. [Kol13, 6.7]) Let (X,Σ) be a reduced pair and $x \in X$ a point. If Σ does not contain any irreducible component of X, then

$$\mathcal{db}_x(X,\Sigma) \leq \dim_x X - 1.$$

Proof. Consider the long exact sequence of cohomology sheaves induced by the distinguished triangle (2.2.3):

$$h^{q-1}(\underline{\Omega}^0_\Sigma) \to h^q(\underline{\Omega}^0_{X,\Sigma}) \to h^q(\underline{\Omega}^0_X) \tag{4.2.1}$$

The assumption implies that $\dim_x \Sigma < \dim_x X$ and then the fact (cf. [Kol13, 6.6]) that $h^q(\underline{\Omega}^0_X) = 0$ for all $q \geq \dim X$ imply that the left and right hand side of (4.2.1) are zero for $q \geq \dim_x X$ and hence so is the middle. $\square$

Using the DB index we generalize the vanishing theorem [Kov11, 6.1].

Theorem 4.3. *Let $\pi : Y \to X$ be a projective birational morphism of reduced schemes of finite type over $\mathbb{C}$ and $\Sigma \subseteq X$ a reduced closed subscheme such that Σ does not contain any irreducible components of X. Set $E = \mathrm{Exc}(\pi)$, $\Gamma = E \cup (\pi^{-1}\Sigma)_{red}$, and $Z = \overline{\pi(E) \setminus \Sigma} \subseteq X$. Then for any $x \in X$,*

$$(\mathcal{R}^q\pi_*\underline{\Omega}^0_{Y,\Gamma})_x = 0, \text{ for } q > \max\{\mathcal{db}_x(X,\Sigma), \mathcal{db}_x(Z, Z\cap\Sigma)+1\}.$$

In particular,

$$\mathcal{R}^q\pi_*\underline{\Omega}^0_{Y,\Gamma} = 0, \text{ for } q > \max\{\mathcal{db}(X,\Sigma), \mathcal{db}(Z, Z\cap\Sigma)+1\}.$$

Proof. Let $x \in X$. The statement is local on X and $\mathcal{db}_x$ is upper semicontinuous as a function of x, so we may restrict to a neighborhood of x and assume that $\mathcal{db}(X,\Sigma) = \mathcal{db}_x(X,\Sigma)$ and $\mathcal{db}(Z, Z\cap\Sigma) = \mathcal{db}_x(Z, Z\cap\Sigma)$.

Using the long exact sequence of cohomology sheaves associated to the distinguished triangle of (2.2.3) shows that the morphism

$$\alpha^q : h^q(\underline{\Omega}^0_X) \to h^q(\underline{\Omega}^0_\Sigma) \tag{4.3.1}$$

is an isomorphism for $q > \mathcal{db}(X,\Sigma)$. and a surjection for $q = \mathcal{db}(X,\Sigma)$.

Next let $\widetilde{\Sigma} = Z \cup \Sigma$. Then [Kov11, 3.19] implies that for any $x \in X$

$$\mathcal{db}_x(\widetilde{\Sigma},\Sigma) = \mathcal{db}_x(Z, Z\cap\Sigma),$$

and hence that $h^q(\underline{\Omega}^0_{\widetilde{\Sigma},\Sigma}) = 0$ for $q > \mathcal{db}(Z, Z\cap\Sigma)$. The same way as above this implies that the morphism

$$\beta^q : h^q(\underline{\Omega}^0_{\widetilde{\Sigma}}) \to h^q(\underline{\Omega}^0_{\Sigma}) \tag{4.3.2}$$

is an isomorphism for $q > \mathcal{db}(Z, Z\cap\Sigma)$.

Now observe that the natural morphism $\underline{\Omega}^0_X \to \underline{\Omega}^0_{\Sigma}$ factors through $\underline{\Omega}^0_{\widetilde{\Sigma}}$ and hence $\alpha^q = \beta^q \circ \gamma^q$ where $\gamma^q : h^q(\underline{\Omega}^0_X) \to h^q(\underline{\Omega}^0_{\widetilde{\Sigma}})$ is the natural morphism corresponding to the embedding $\widetilde{\Sigma} \subseteq X$. It follows from (4.3.1) and (4.3.2) that

(4.3.3) $\quad \gamma^q$ is an isomorphism for $q > \max\{\mathcal{db}(X,\Sigma), \mathcal{db}(Z, Z\cap\Sigma)\}$,
and a surjection for $q > \max\{\mathcal{db}(X,\Sigma)-1, \mathcal{db}(Z, Z\cap\Sigma)\}$.

Next consider the long exact sequence of cohomology sheaves induced by the distinguished triangle of (2.2.4) for the pairs $(X, \widetilde{\Sigma})$ and (Y, Γ):

$$\cdots \to h^q(\underline{\Omega}^0_X) \to h^q(\underline{\Omega}^0_{\widetilde{\Sigma}}) \oplus \mathcal{R}^q\pi_*\underline{\Omega}^0_Y \to \mathcal{R}^q\pi_*\underline{\Omega}^0_{\Gamma} \to h^{q+1}(\underline{\Omega}^0_X) \to \cdots .$$

By (4.3.3) it follows that the natural morphism

$$\mathcal{R}^q\pi_*\underline{\Omega}^0_Y \to \mathcal{R}^q\pi_*\underline{\Omega}^0_{\Gamma}$$

is

an isomorphism for $q > \max\{\mathcal{db}(X,\Sigma), \mathcal{db}(Z, Z\cap\Sigma)\}$, and
a surjection for $q > \max\{\mathcal{db}(X,\Sigma)-1, \mathcal{db}(Z, Z\cap\Sigma)\}$

As before, using the long exact sequence induced by (2.2.3) we obtain that

$$\mathcal{R}^q\pi_*\underline{\Omega}^0_{Y,\Gamma} = 0 \text{ for } q > \max\{\mathcal{db}(X,\Sigma), \mathcal{db}(Z, Z\cap\Sigma)+1\}.$$

□

This theorem has several interesting consequences.

Corollary 4.4. *Let X be an irreducible variety and $\pi : Y \to X$ a projective birational morphism. Let $\Sigma \subseteq X$ be a subvariety and set $E = \mathrm{Exc}(\pi)$, $Z = \overline{\pi(E)\setminus\Sigma}$, and $\Gamma = E \cup (\pi^{-1}\Sigma)_{red}$. Then*

$$\mathcal{R}^q\pi_*\underline{\Omega}^0_{Y,\Gamma} = 0, \textit{ for } q > \max(\mathcal{db}(X,\Sigma), \dim Z).$$

Proof. By the definition of Z, Σ cannot contain any irreducible component of Z and hence (4.2) implies that $\mathcal{db}(Z, Z\cap\Sigma) \le \dim Z - 1$. Therefore the statement follows from (4.3). □

Finally we obtain a Steenbrink-type theorem for $p = 0$ with some assumption on the singularities of the pair (X, Σ).

Corollary 4.5. *Let X be an irreducible variety and $\pi : Y \to X$ a resolution of singularities. Let $\Sigma \subseteq X$ be a subvariety and set $E = \operatorname{Exc}(\pi)$, $Z = \overline{\pi(E) \setminus \Sigma}$, and $\Gamma = E \cup (\pi^{-1}\Sigma)_{red}$. Assume that $\mathcal{db}(X,\Sigma) \le \dim Z$ and that Γ is an snc divisor. Then*

$$\mathcal{R}^q\pi_*\mathscr{I}_{\Gamma\subseteq Y} = 0, \text{ for } q > \dim Z. \tag{4.5.1}$$

In particular, if X is normal of dimension $n \ge 2$, then

$$\mathcal{R}^{n-1}\pi_*\mathscr{I}_{\Gamma\subseteq Y} = 0. \tag{4.5.2}$$

Proof. Follows from (2.2.2) and (4.4). □

REMARK 4.6. Note that for (4.5.2) one only needs that $\mathcal{db}(X,\Sigma) \le n-2$. This should be considered a mild assumption given that $\mathcal{db}(X,\Sigma) \le n-1$ always holds.

5. VANISHING FOR $p = 1$

Notice that the assumption $p+q > \dim X$ in Steenbrink's theorem (3.2), as well as its generalization (3.1) means that these theorems are vacuous in the cases $p \le 1$, since $\mathcal{R}^q\pi_* = 0$ for $q \ge \dim X$ anyway.

We obtained an extension of this theorem under additional conditions for the case $p = 0$ in 4.3. It turns out that by a simple argument one may extend this vanishing also for the case $p = 1$ and $q = n-1$.

First we need the following:

Lemma 5.1 (Topological vanishing). [GKKP11, Lemma 14.4]
Let $\pi : Y \to X$ be a projective morphism of reduced schemes of finite type over $\mathbb{C}$ and $\Sigma \subseteq X$ a reduced closed subscheme such that π is an isomorphism over $X \setminus \Sigma$. Set $\Gamma = (\pi^{-1}\Sigma)_{red}$, $j : Y \setminus \Gamma \hookrightarrow Y$ the inclusion map, and $j_!\mathbb{C}_{Y\setminus\Gamma}$ the constant sheaf $\mathbb{C}$ on $Y \setminus \Gamma$ extended by 0 to the entire Y. Then $\mathcal{R}^q\pi_*\big(j_!\mathbb{C}_{Y\setminus\Gamma}\big) = 0$ for all $q > 0$.

Proof. The proof of [GKKP11, Lemma 14.4] works verbatim. □

Theorem 5.2. *Let $\pi : Y \to X$ be a projective birational morphism of reduced schemes of finite type over $\mathbb{C}$ and $\Sigma \subseteq X$ a reduced closed subscheme such that Σ does not contain any irreducible components of X. Set $E = \operatorname{Exc}(\pi)$, $\Gamma = E \cup (\pi^{-1}\Sigma)_{red}$, and $Z = \overline{\pi(E) \setminus \Sigma} \subseteq X$. Assume that $\mathcal{db}_x(X,\Sigma) \le \dim_x X - 2$ for all $x \in X$ and that $\operatorname{codim}_X Z \ge 2$. Then for any $x \in X$,*

$$(\mathcal{R}^{\dim_x X-1}\pi_*\underline{\Omega}^1_{Y,\Gamma})_x = 0.$$

In particular,

$$\mathcal{R}^{\dim X-1}\pi_*\underline{\Omega}^1_{Y,\Gamma} = 0.$$

REMARK 5.2.1. Notice that the assumption on the DB index implies that $\dim X \ge \dim_x X \ge 2$ for any $x \in X$.

Proof. The statement is local on X, so we may restrict to a neighborhood of any $x \in X$ and assume that $\mathcal{db}(X, \Sigma) = \mathcal{db}_x(X, \Sigma)$ and $\dim X = \dim_x X$. Let $n = \dim X$, which is at least 2 by (5.2.1).

The quasi-isomorphism of (2.2.1) and the filtration of $\underline{\Omega}^{\bullet}_{X,\Sigma}$ induces a spectral sequence computing $\mathcal{R}^i\pi_*\big(j_!\mathbb{C}_{Y\setminus\Gamma}\big)$:

$$E_1^{p,q} = \mathcal{R}^q\pi_*\underline{\Omega}^p_{X,\Sigma} \Rightarrow E_\infty^{p,q} = \mathcal{R}^{p+q}\pi_*\big(j_!\mathbb{C}_{Y\setminus\Gamma}\big) \tag{5.2.2}$$

By (5.1), $E_\infty^{p,q} = 0$ for $p+q > 0$, so all $E_1^{p,q}$ in that range have to be killed in the spectral sequence.

Next consider the differentials in the spectral sequence that map to or from $E_r^{1,n-1}$ for some $r > 0$:

$$\begin{aligned} d_r &: E_r^{1-r,n+r-2} \to E_r^{1,n-1} \\ d_r &: \qquad E_r^{1,n-1} \to E_r^{r+1,n-r}. \end{aligned}$$

Observe that $E_r^{1-r,n+r-2} = 0$ trivially if $r > 1$ and $E_1^{0,n-1} = 0$ by (4.3) (cf. 4.4). Furthermore, $E_r^{r+1,n-r} = 0$ by (3.1) and hence the only way $E_\infty^{1,n-1} = 0$ can happen is if already $E_1^{1,n-1} = 0$ which is exactly the desired statement. $\square$

This way we obtain the promised generalization and simplified proof of [GKKP11, Thm. 14.1]

Corollary 5.3. *Let X be a normal variety and $\pi : Y \to X$ a resolution of singularities. Let $\Sigma \subseteq X$ be a subvariety and set $E = \operatorname{Exc}(\pi)$ and $\Gamma = E \cup (\pi^{-1}\Sigma)_{red}$. Assume that $\mathcal{db}(X, \Sigma) \le \dim X - 2$ and that Γ is an snc divisor. Then for all p,*

$$\mathcal{R}^{\dim X - 1}\pi_*\big(\Omega^p_Y(\log\Gamma)(-\Gamma)\big) = 0.$$

Proof. This is a direct consequence of the combination of (3.1) for $p > 1$, (4.3) for $p = 0$, and (5.2) for $p = 1$. $\square$

6. An accidental exact sequence

As a sort of byproduct of the argument used to prove (5.2) we also obtain the following.

Theorem 6.1. *Let X and Y be irreducible varieties and $\pi : Y \to X$ be a projective birational morphism. Let $\Sigma \subseteq X$ be a subvariety and set $E = \operatorname{Exc}(\pi)$, $Z = \overline{\pi(E) \setminus \Sigma}$, and $\Gamma = E \cup (\pi^{-1}\Sigma)_{red}$. Assume that $\mathcal{db}(X, \Sigma) \le \dim X - 3$ and that $\operatorname{codim}_X Z \ge 3$. Then there exists a 5-term exact sequence,*

$$\mathcal{R}^{n-3}\pi_*\underline{\Omega}^2_{Y,\Gamma} \to \mathcal{R}^{n-3}\pi_*\underline{\Omega}^3_{Y,\Gamma} \to \mathcal{R}^{n-2}\pi_*\underline{\Omega}^1_{Y,\Gamma} \to \mathcal{R}^{n-2}\pi_*\underline{\Omega}^2_{Y,\Gamma} \to 0.$$

REMARK 6.1.1. Note that the morphisms $\mathcal{R}^{n-3}\pi_*\underline{\Omega}^2_{Y,\Gamma} \to \mathcal{R}^{n-3}\pi_*\underline{\Omega}^3_{Y,\Gamma}$ and $\mathcal{R}^{n-2}\pi_*\underline{\Omega}^1_{Y,\Gamma} \to \mathcal{R}^{n-2}\pi_*\underline{\Omega}^2_{Y,\Gamma}$ in the above sequence are natural maps induced by the filtration on $\underline{\Omega}^\bullet_{Y,\Gamma}$. However, the map $\mathcal{R}^{n-3}\pi_*\underline{\Omega}^3_{Y,\Gamma} \to \mathcal{R}^{n-2}\pi_*\underline{\Omega}^1_{Y,\Gamma}$ is actually the inverse of a natural map from a subsheaf of $\mathcal{R}^{n-2}\pi_*\underline{\Omega}^1_{Y,\Gamma}$ to a quotient sheaf of $\mathcal{R}^{n-3}\pi_*\underline{\Omega}^3_{Y,\Gamma}$ that turns out to be an isomorphism.

Also note that $\mathcal{R}^{n-2}\pi_*\underline{\Omega}^1_{Y,\Gamma} \to \mathcal{R}^{n-2}\pi_*\underline{\Omega}^2_{Y,\Gamma}$ is already surjective under the assumptions of Theorem 5.2.

Proof. We will use the spectral sequence and notation introduced in the proof of (5.2). In particular, first consider the differentials

$$d_r : E_r^{t,n-t} \to E_r^{r+t,n-t-r+1}, \tag{6.1.2}$$

for any t, and observe that (as above) $E_r^{r+t,n-t-r+1} = 0$ for all $r \geq 1$ by (3.1). Next consider the differentials

$$d_r : E_r^{2-r,n+r-3} \to E_r^{2,n-2},$$

and observe that (as above) that $E_r^{2-r,n+r-3} = 0$ trivially if $r > 2$ and $E_2^{0,n-1} = 0$ by (4.3) (cf. 4.4). Therefore, the only way $E_\infty^{2,n-2} = 0$ can happen is if

$$d_1 : E_1^{1,n-2} \to E_1^{2,n-2} \tag{6.1.3}$$

is surjective. By (4.3) the differential $d_1 : E_1^{0,n-2} \to E_1^{1,n-2}$ is 0 (this is where we need the stronger assumptions on $\mathcal{db}(X,\Sigma)$ and $\operatorname{codim}_X Z$), so the kernel of the morphism in (6.1.3) is equal to $E_2^{1,n-2}$, i.e., we have an exact sequence

$$E_2^{1,n-2} \xrightarrow{\ker d_1} E_1^{1,n-2} \xrightarrow{d_1} E_1^{2,n-2} \longrightarrow 0, \tag{6.1.4}$$

and again by (3.1) and (4.3) it follows that in order for $E_\infty^{1,n-2} = 0$ to hold the next differential

$$d_2 : E_2^{1,n-2} \hookrightarrow E_2^{3,n-3} \tag{6.1.5}$$

has to be injective.

Next, by (6.1.2) we see that $E_1^{3,n-3}$ has to be killed by differentials mapping *to* it, that is, by the differentials

$$d_r : E_r^{3-r,n+r-4} \to E_r^{3,n-3}.$$

As before, $E_r^{3-r,n+r-4} = 0$ for $r > 3$, and $E_r^{0,n-1} = 0$ by (4.3), so there are two differentials, d_1 and d_2 that can kill $E_1^{3,n-3}$. It follows that $E_2^{3,n-3}$ is the cokernel of $d_1 : E_1^{2,n-3} \to E_1^{3,n-3}$, i.e., we have an exact sequence

$$E_1^{2,n-3} \xrightarrow{d_1} E_1^{3,n-3} \xrightarrow{\operatorname{coker} d_1} E_2^{3,n-3}, \tag{6.1.6}$$

and that $d_2 : E_2^{1,n-2} \to E_2^{3,n-3}$ has to be surjective. However, we have already seen in (6.1.5) that this d_2 is injective and hence it must be an isomorphism:

$$d_2 : E_2^{1,n-2} \xrightarrow{\simeq} E_2^{3,n-3}. \tag{6.1.7}$$

Putting together (6.1.4), (6.1.6), and (6.1.7) gives the desired exact sequence:

$$E_1^{2,n-3} \xrightarrow{d_1} E_1^{3,n-3} \xrightarrow{(\ker d_1)\circ(d_2^{-1})\circ(\operatorname{coker} d_1)} E_1^{1,n-2} \xrightarrow{d_1} E_1^{2,n-2} \longrightarrow 0.$$

□

REMARK 6.2. It is left for the reader to formulate the consequence of this theorem in the style of (5.3).

REFERENCES

[Car85] J. A. CARLSON: *Polyhedral resolutions of algebraic varieties*, Trans. Amer. Math. Soc. **292** (1985), no. 2, 595–612. MR808740 (87i:14008)

[Dru13] S. DRUEL: *The Zariski-Lipman conjecture for log canonical spaces*, 2013. arXiv:1301.5910 [math.AG]

[DB81] P. DU BOIS: *Complexe de de Rham filtré d'une variété singulière*, Bull. Soc. Math. France **109** (1981), no. 1, 41–81. MR613848 (82j:14006)

[Gra13] P. GRAF: *An optimal extension theorem for 1-forms and the Lipman-Zariski conjecture*, 2013. arXiv:1301.7315 [math.AG]

[GKKP11] D. GREB, S. KEBEKUS, S. J. KOVÁCS, AND T. PETERNELL: *Differential forms on log canonical spaces*, Publ. Math. Inst. Hautes Études Sci. (2011), no. 114, 87–169. 2854859

[GNPP88] F. GUILLÉN, V. NAVARRO AZNAR, P. PASCUAL GAINZA, AND F. PUERTA: *Hyperrésolutions cubiques et descente cohomologique*, Lecture Notes in Mathematics, vol. 1335, Springer-Verlag, Berlin, 1988, Papers from the Seminar on Hodge-Deligne Theory held in Barcelona, 1982. MR972983 (90a:14024)

[Kol95] J. KOLLÁR: *Shafarevich maps and automorphic forms*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 1995. MR1341589 (96i:14016)

[Kol13] J. KOLLÁR: *Singularities of the minimal model program*, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, 2013, with the collaboration of SÁNDOR J KOVÁCS.

[KK10] J. KOLLÁR AND S. J. KOVÁCS: *Log canonical singularities are Du Bois*, J. Amer. Math. Soc. **23** (2010), no. 3, 791–813. 2629988 (2011m:14061)

[Kov99] S. J. KOVÁCS: *Rational, log canonical, Du Bois singularities: on the conjectures of Kollár and Steenbrink*, Compositio Math. **118** (1999), no. 2, 123–133. MR1713307 (2001g:14022)

[Kov00] S. J. KOVÁCS: *Rational, log canonical, Du Bois singularities. II. Kodaira vanishing and small deformations*, Compositio Math. **121** (2000), no. 3, 297–304. MR1761628 (2001m:14028)

[Kov11] S. J. KOVÁCS: *Du Bois pairs and vanishing theorems*, Kyoto J. Math. **51** (2011), no. 1, 47–69. 2784747 (2012d:14028)

[KS09] S. J. Kovács and K. E. Schwede: *Hodge theory meets the minimal model program: a survey of log canonical and Du Bois singularities*, preprint, 2009. arXiv:0909.0993v1 [math.AG]

[KSS10] S. J. Kovács, K. E. Schwede, and K. E. Smith: *The canonical sheaf of Du Bois singularities*, Adv. Math. **224** (2010), no. 4, 1618–1640. 2646306 (2011m:14062)

[PS08] C. A. M. Peters and J. H. M. Steenbrink: *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 52, Springer-Verlag, Berlin, 2008. MR2393625

[Sch07] K. Schwede: *A simple characterization of Du Bois singularities*, Compos. Math. **143** (2007), no. 4, 813–828. MR2339829 (2008k:14034)

[Ste83] J. H. M. Steenbrink: *Mixed Hodge structures associated with isolated singularities*, Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 513–536. MR713277 (85d:32044)

[Ste85] J. H. M. Steenbrink: *Vanishing theorems on singular spaces*, Astérisque (1985), no. 130, 330–341, Differential systems and singularities (Luminy, 1983). MR804061 (87j:14026)

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