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Spectral sequences associated to morphisms of locally free sheaves

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Let

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be a short exact sequence of locally free sheaves. Then for any $r \geq 0$ there is a finite filtration of $\bigwedge^r \mathcal{F}$,

$$\bigwedge^r \mathcal{F} = F^0 \supset F^1 \supset \cdots \supset F^r \supset F^{r+1} = 0,$$

with quotients

$$F^p/F^{p+1} \simeq \bigwedge^p \mathcal{F}' \otimes \bigwedge^{r-p} \mathcal{F}''$$

for each p.

This filtration in turn leads to the spectral sequence,

$$E_1^{p,q} = H^{p+q}(X, \bigwedge^p \mathcal{F}' \otimes \bigwedge^{r-p} \mathcal{F}'') \Rightarrow H^{p+q}(X, \bigwedge^r \mathcal{F}).$$

This is a very useful tool, but requires $\mathcal{F}', \mathcal{F}, \mathcal{F}''$ be locally free. The aim of the present article is to give a construction that yields a similar spectral sequence in a more general case. More precisely, the goal is to define natural objects that will play the role of $\bigwedge^{r-p} \mathcal{F}''$ if the cokernel of the morphism $\mathcal{F}' \to \mathcal{F}$ is not locally free. These objects will not be single sheaves anymore, but objects of the derived category of $\mathcal{O}_X$-modules. Nevertheless their functorial and cohomological properties will be strong enough to make applications possible.

This work originated in [**Kovács96**]. The principal goal was to make a cohomological argument work in a more general setting and the path to that goal led through the construction of certain objects whose properties resemble those of the components of the relative De Rham complex of a smooth morphism. In particular the analogue of the above spectral sequence was constructed in a special case (cf. [**ibid.**]).

Subsequently I improved the construction and later I realized that similar arguments can be used elsewhere. The constructions needed were very similar, yet somewhat different cf. [**Kovács97b, Kovács02**]. It seems appropriate to give a

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common generalization, so one would only have to go through the construction once rather then to include it with every application, not to mention the hope that there will be new applications. The present article attempts to provide a unified treatment.

§1 contains some technical material that will be essential later. §§1.1 ought to be known, but I do not know a convenient reference, so some of the statements are proved here. §§1.2 is a simple generalization of the notion of a filtration to the derived categorical setting. (1.2.2) is the central result of this section, it is the analogue of its well-known relative about the spectral sequence associated to a filtration.

In §2 the complexes $\underline{\mathfrak{Q}}^p_{\theta_X}$ are constructed for a morphism θ_X between locally free sheaves. If θ_X is the first morphism of the above short exact sequence and its cokernel is locally free, then $\underline{\mathfrak{Q}}^p_{\theta_X} \simeq \bigwedge^p \mathcal{F}''$. The guiding principles of the construction are functoriality and the desired "filtration". The main result is (2.7).

The results in this article were first developed for their use in [**Kovács97b**] in a more specific form. Later the similar, but still different needs of another project prompted this more general presentation. Hence the applications form an important part.

The first application is a vanishing theorem, proved in §3. This is a generalization of both [**Kovács96, 2.13**] and [**Kovács97a, 1.4**]. The functorial property of $\underline{\mathfrak{Q}}^p_{\theta_X}$ is proved in §4.

§5 gives a brief account of the more explicit applications of the theory. These statements are not proved here, the reader is referred to the cited articles for more details.

Definitions and Notation. Let X be a scheme over T. Then $C(X)$ is the category of complexes of $\mathcal{O}_X$-modules and for $u \in \operatorname{Mor}(C(X))$, $M(u) \in \operatorname{Ob}(C(X))$ denotes the mapping cone of u. $K(X)$ is the category of homotopy equivalence classes of objects of $C(X)$. A diagram in $C(X)$ will be called a *predistinguished triangle,* if its image in $K(X)$ is a distinguished triangle. $D(X)$ denotes the derived category of complexes of $\mathcal{O}_X$-modules. The superscripts $+, -, b$ carry the usual meaning (bounded below, bounded above and bounded). Regarding these notions the basic reference will be [**Hartshorne66**]. S_k denotes the symmetric group of degree k.

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§1. Tools

§§**1.1 Wedge products.** Let Φ_X and Ψ_X be locally free sheaves on X of rank k and n respectively, and let $\theta_X : \Phi_X \to \Psi_X$ be a morphism.

1.1.1 Definition. Let η be a section of $\bigwedge^p \Psi_X$ over an open set and $\xi_1, \dots, \xi_k$ sections of Φ_X over the same set. Then $\eta \otimes (\xi_1 \wedge \dots \wedge \xi_k)$ is a section of $\bigwedge^p \Psi_X \otimes \det \Phi_X$. For any $\sigma \in S_k$ let

$$\xi_{\sigma,q} = \theta_X(\xi_{\sigma(1)}) \wedge \dots \wedge \theta_X(\xi_{\sigma(q)}),$$

and

$$\xi^{\sigma,q} = \xi_{\sigma(q+1)} \wedge \cdots \wedge \xi_{\sigma(k)}.$$

Further let

$$S_{k,q} = \{\sigma \in S_k | \sigma(1) < \cdots < \sigma(q) \text{ and } \sigma(q+1) < \cdots < \sigma(k)\},$$

and

$$I_\sigma = \{\sigma(1), \ldots, \sigma(q)\}.$$

It is easy to see that every $\sigma \in S_{k,q}$ is determined by I_σ. Now define

$$\curlywedge_q^\theta(\eta \otimes (\xi_1 \wedge \cdots \wedge \xi_k)) \in \bigwedge^{p+q} \Psi_X \otimes \bigwedge^{k-q} \Phi_X$$

by the formula

$$\curlywedge_q^\theta(\eta \otimes (\xi_1 \wedge \cdots \wedge \xi_k)) = \sum_{\sigma \in S_{k,q}} (-1)^{\operatorname{sgn}\sigma} (\xi_{\sigma,q} \wedge \eta) \otimes \xi^{\sigma,q},$$

and extend it linearly.

To see that

$$\curlywedge_q^\theta : \bigwedge^{p} \Psi_X \otimes \det \Phi_X \to \bigwedge^{p+q} \Psi_X \otimes \bigwedge^{k-q} \Phi_X$$

is a well-defined morphism of sheaves, it is enough to verify the multi-linear and alternating properties. This is left to the reader.

1.1.2 Lemma. *Let* id *denote* $\mathrm{id}_{\Phi_X} : \Phi_X \to \Phi_X$. *Then*

$$\begin{array}{ccc}
\bigwedge^{p} \Psi_X \otimes \det \Phi_X \otimes \det \Phi_X & \xrightarrow{\curlywedge_q^\theta} & \bigwedge^{p+q} \Psi_X \otimes \det \Phi_X \otimes \bigwedge^{k-q} \Phi_X \\
\downarrow \curlywedge_{q+r}^\theta & & \downarrow \curlywedge_r^\theta \\
\bigwedge^{p+q+r} \Psi_X \otimes \bigwedge^{k-q-r} \Phi_X \otimes \det \Phi_X & \xrightarrow[\curlywedge_q^{\mathrm{id}}]{} & \bigwedge^{p+q+r} \Psi_X \otimes \bigwedge^{k-r} \Phi_X \otimes \bigwedge^{k-q} \Phi_X
\end{array}$$

is a commutative diagram, i.e., $\curlywedge_r^\theta \circ \curlywedge_q^\theta = \curlywedge_q^{\mathrm{id}} \circ \curlywedge_{q+r}^\theta$.

Proof. Use the same notation as in (1.1.1). Then

$$\curlywedge_r^\theta \circ \curlywedge_q^\theta (\eta \otimes (\xi_1 \wedge \cdots \wedge \xi_k) \otimes (\xi_1 \wedge \cdots \wedge \xi_k)) =$$
$$= \sum_{\tau \in S_{k,r}} \sum_{\sigma \in S_{k,q}} (-1)^{\operatorname{sgn}\tau + \operatorname{sgn}\sigma} (\xi_{\tau,r} \wedge \xi_{\sigma,q} \wedge \eta) \otimes \xi^{\sigma,q} \otimes \xi^{\tau,r}.$$

Let $\sigma \in S_{k,q}$, $\tau \in S_{k,r}$. If

$$I_\tau \cap I_\sigma = \{\tau(1), \ldots, \tau(r)\} \cap \{\sigma(1), \ldots, \sigma(q)\} \neq \emptyset,$$

then $\xi_{\tau,r} \wedge \xi_{\sigma,q} = 0$. Otherwise let $\mu = \mu(\sigma, \tau) \in S_{k,q+r}$ be defined by $I_\mu = I_\tau \cup I_\sigma$ and let $\nu = \nu(\sigma, \tau) = \sigma \in S_{k,q}$.

$$\curlywedge_q^{\mathrm{id}} \circ \curlywedge_{q+r}^\theta (\eta \otimes (\xi_1 \wedge \cdots \wedge \xi_k) \otimes (\xi_1 \wedge \cdots \wedge \xi_k)) =$$
$$\sum_{\nu \in S_{k,q}} \sum_{\mu \in S_{k,q+r}} (-1)^{\operatorname{sgn}\mu + \operatorname{sgn}\nu} (\xi_{\mu,q+r} \wedge \eta) \otimes \xi^{\nu,q} \otimes (\xi_{\nu,q} \wedge \xi^{\mu,q+r})$$

and for $\nu \in S_{k,q},\ \mu \in S_{k,q+r},\ \xi_{\nu,q} \wedge \xi^{\mu,q+r} \neq 0$ let $\sigma = \sigma(\mu,\nu) = \nu$ and $\tau = \tau(\mu,\nu) \in S_{k,r}$ be defined by $I_\tau = I_\mu \setminus I_\nu$.

This gives a one-to-one correspondence between the pairs (σ,τ) and the pairs (μ,ν).

Observe that

$$(-1)^{\operatorname{sgn}\tau} \underbrace{(\xi_{\tau,r} \wedge \xi_{\sigma,q})}_{\pm\xi_{\mu,q+r}} \otimes \xi^{\tau,r} = (-1)^{\operatorname{sgn}\mu} \xi_{\mu,q+r} \otimes \underbrace{(\xi_{\nu,q} \wedge \xi^{\mu,q+r})}_{\pm\xi^{\tau,r}},$$

so

$$(-1)^{\operatorname{sgn}\tau + \operatorname{sgn}\sigma} (\xi_{\tau,r} \wedge \xi_{\sigma,q} \wedge \eta) \otimes \xi^{\sigma,q} \otimes \xi^{\tau,r} =$$
$$(-1)^{\operatorname{sgn}\mu + \operatorname{sgn}\nu} (\xi_{\mu,q+r} \wedge \eta) \otimes \xi^{\nu,q} \otimes (\xi_{\nu,q} \wedge \xi^{\mu,q+r}).$$

□

§§**1.2 Hyperfiltrations and spectral sequences.** Let $\mathfrak{A}$ be an abelian category and $D(\mathfrak{A})$ its derived category. Let $\Phi : \mathfrak{A} \to \mathfrak{Ab}$ be a left exact additive functor from $\mathfrak{A}$ to the category of abelian groups and assume that $R\Phi : D(\mathfrak{A}) \to D(\mathfrak{Ab})$, the right derived functor of Φ exists.

1.2.1 Definition. Let $K \in \operatorname{Ob}(D^b(\mathfrak{A}))$ be a bounded complex. A *bounded hyperfiltration* $\mathbb{F}$ of K consists of a set of objects $\mathbb{F}^j K \in \operatorname{Ob}(D^b(\mathfrak{A}))$ for $j = l, \dots, k+1$, where $l, k \in \mathbb{Z}$ and morphisms

$$\varphi_j \in \operatorname{Hom}_{D^b(\mathfrak{A})}(\mathbb{F}^{j+1}K, \mathbb{F}^j K) \qquad \text{for } j = l, \dots, k,$$

where $\mathbb{F}^l K \simeq K$ and $\mathbb{F}^{k+1} K \simeq 0$. $\mathbb{F}^j K$ will be denoted by $\mathbb{F}^j$ when no confusion is likely. For convenience let $\mathbb{F}^i K = K$ for $i < l$ and $\mathbb{F}^i K = 0$ for $i > k$.

The *p-th associated graded complex* of a hyperfiltration $\mathbb{F}$ is

$$\mathbb{G}^p = \mathbb{G}r^p_{\mathbb{F}} K = M(\varphi_p)$$

the mapping cone[1] of the morphism φ_p.

Then one has the following standard result:

1.2.2 Theorem. *There exists a spectral sequence E_r with $E_1^{p,q} = R^{p+q}\Phi(\mathbb{G}^p)$ abutting to $R^{p+q}\Phi(K)$.*

The proof of this theorem is relatively straightforward, but cumbersome and is not used elsewhere in this article. The key observation to make in order to prove this result is that one needs to start at the E_1-level since there is no E_0-level, so the proof has to be adjusted accordingly. An important point to notice is the following: once one passes to the E_1-level of this spectral sequence, one leaves the realm of derived categories behind and starts working with long exact sequences of abelian groups (or more generally of objects of an abelian category). This allows one to mimic the usual proof of the similar well-known result for filtrations. For the interested reader the proof is included in an appendix at the end of the article.

[1] Strictly speaking this is the class of the mapping cone of a morphism in $C(X)$ whose class in $D(X)$ is φ_p. The point is that $\mathbb{F}^{p+1} \longrightarrow \mathbb{F}^p \longrightarrow \mathbb{G}^p \xrightarrow{+1}$ forms a distinguished triangle.

1.2.2.1 Remark. All of the above makes sense and remains true if one replaces $\mathfrak{Ab}$ by an arbitrary abelian category.

§2. Filtration diagrams

Let Φ_X and Ψ_X be locally free sheaves on X of rank k and n respectively, and let $\theta_X : \Phi_X \to \Psi_X$ be a morphism.

Let $p, i \in \mathbb{N}$. We are going to define an object, $\mathfrak{F}_i^p = \mathfrak{F}_i^p(\theta_X) \in \mathrm{Ob}(C(X))$ and a (p,i)-*filtration diagram of* θ_X diagram, $\boxed{\mathfrak{F}_i^p} = \boxed{\mathfrak{F}_i^p(\theta_X)}$. This will be done recursively, starting with $i = 0$ and then increasing i.

2.1 Definition. The $(p,0)$-*filtration diagram of* θ_X is

$$\boxed{\mathfrak{F}_0^p} = \mathfrak{F}_0^p = \bigwedge^{n-p} \Psi_X.$$

A 0-*filtration morphism* for some p, q, consists of locally free sheaves $\mathcal{E}, \mathcal{F}$ and a morphism between $\bigwedge^{n-p} \Psi_X \otimes \mathcal{E}$ and $\bigwedge^{n-q} \Psi_X \otimes \mathcal{F}$.

For instance,

$$\lambda_p^\theta : \bigwedge^{n-p} \Psi_X \otimes \det \Phi_X \to \bigwedge^{n} \Psi_X \otimes \bigwedge^{k-p} \Phi_X$$

is a 0-filtration morphism. Let

$$\mathfrak{F}_1^p = M(\lambda_p^\theta)[-1].$$

2.2 Definition. The $(p,1)$-*filtration diagram of* θ_X consists of the predistinguished triangle,

$$\mathfrak{F}_1^p \to \bigwedge^{n-p} \Psi_X \otimes \det \Phi_X \to \bigwedge^{n} \Psi_X \otimes \bigwedge^{k-p} \Phi_X \xrightarrow{+1}$$

It is denoted by $\boxed{\mathfrak{F}_1^p}$. A 1-*filtration morphism* for some p, r, consists of locally free sheaves $\mathcal{E}, \mathcal{F}$ and morphisms between the corresponding terms of $\boxed{\mathfrak{F}_1^p} \otimes \mathcal{E}$ and

$\boxed{\mathfrak{F}_1^r} \otimes \mathcal{F}$ such that the resulting diagram,

$$\begin{array}{ccc}
\mathfrak{F}_1^p \otimes \mathcal{E} & \longrightarrow & \mathfrak{F}_1^r \otimes \mathcal{F} \\
\big\downarrow & & \big\downarrow \\
\bigwedge^{n-p} \Psi_X \otimes \det \Phi_X \otimes \mathcal{E} & \longrightarrow & \bigwedge^{n-r} \Psi_X \otimes \det \Phi_X \otimes \mathcal{F} \\
{\scriptstyle \curlywedge_p^\theta}\big\downarrow & & \big\downarrow{\scriptstyle \curlywedge_r^\theta} \\
\bigwedge^{n} \Psi_X \otimes \bigwedge^{k-p} \Phi_X \otimes \mathcal{E} & \longrightarrow & \bigwedge^{n} \Psi_X \otimes \bigwedge^{k-r} \Phi_X \otimes \mathcal{F} \\
\big\downarrow{\scriptstyle +1} & & \big\downarrow{\scriptstyle +1}
\end{array}$$

is commutative.

Consider the following commutative diagram (cf. (1.1.2)).

$$\begin{array}{ccc}
\mathfrak{F}_1^p \otimes \det \Phi_X & & \mathfrak{F}_1^{p-q} \otimes \bigwedge^{k-q} \Phi_X \\
\big\downarrow & & \big\downarrow \\
\bigwedge^{n-p} \Psi_X \otimes \det \Phi_X \otimes \det \Phi_X & \xrightarrow{\curlywedge_q^\theta} & \bigwedge^{n-p+q} \Psi_X \otimes \det \Phi_X \otimes \bigwedge^{k-q} \Phi_X \\
{\scriptstyle \curlywedge_p^\theta}\big\downarrow & & \big\downarrow{\scriptstyle \curlywedge_{p-q}^\theta} \\
\bigwedge^{n} \Psi_X \otimes \bigwedge^{k-p} \Phi_X \otimes \det \Phi_X & \xrightarrow[\curlywedge_q^{\mathrm{id}}]{} & \bigwedge^{n} \Psi_X \otimes \bigwedge^{k-p+q} \Phi_X \otimes \bigwedge^{k-q} \Phi_X \\
\big\downarrow{\scriptstyle +1} & & \big\downarrow{\scriptstyle +1}
\end{array} \tag{2.2.1}$$

There exists a morphism,

$$\alpha : \mathfrak{F}_1^p \otimes \det \Phi_X \to \mathfrak{F}_1^{p-q} \otimes \bigwedge^{k-q} \Phi_X,$$

that makes the above diagram commutative.

The diagram (2.2.1), combined with α gives a 1-filtration morphism

$$\boxed{\mathfrak{F}_1^p} \otimes \det \Phi_X \to \boxed{\mathfrak{F}_1^{p-q}} \otimes \bigwedge^{k-q} \Phi_X,$$

with $r = p - q, \mathcal{E} = \det \Phi_X, \mathcal{F} = \bigwedge^{k-q} \Phi_X$.

Let

$$\mathfrak{F}_2^p = M\Big(\mathfrak{F}_1^p \otimes \det \Phi_X \to \mathfrak{F}_1^1 \otimes \bigwedge^{k-p+1} \Phi_X\Big)[-1].$$

Then there exists a distinguished triangle,

$$\mathfrak{F}_2^p \to \mathfrak{F}_1^p \otimes \det \Phi_X \to \mathfrak{F}_1^1 \otimes \bigwedge^{k-p+1} \Phi_X \xrightarrow{+1}$$

2.3 Definition. The $(p,2)$-*filtration diagram of* θ_X consists of the diagram,

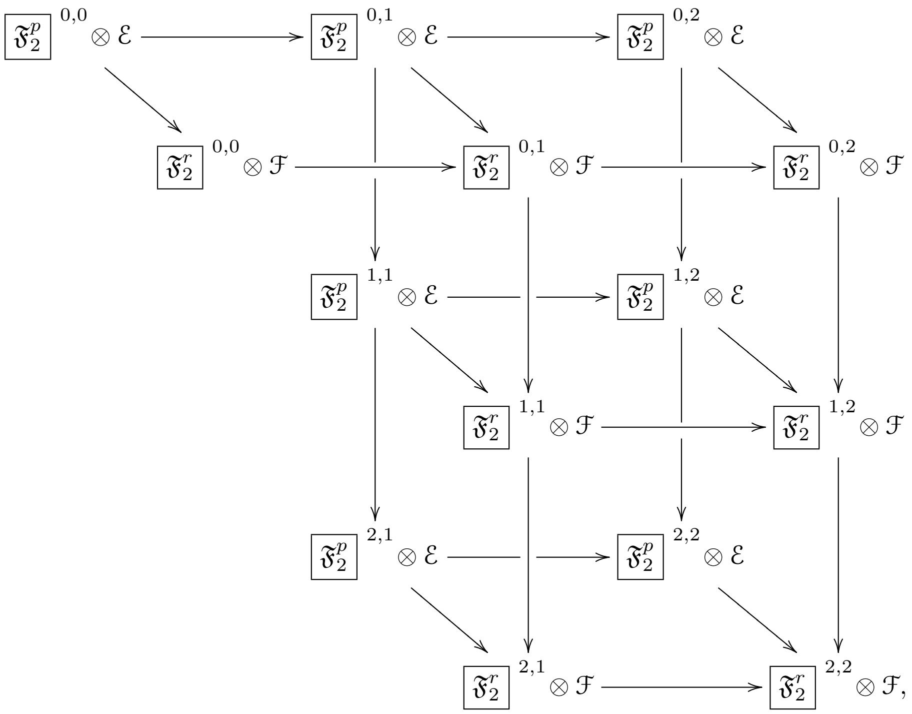
$$\mathfrak{F}_2^p \to \boxed{\mathfrak{F}_1^p} \otimes \det \Phi_X \to \boxed{\mathfrak{F}_1^1} \otimes \bigwedge^{k-p+1} \Phi_X.$$

It is denoted by $\boxed{\mathfrak{F}_2^p}$. A 2-*filtration morphism* for some p, r, consists of locally free sheaves $\mathcal{E}, \mathcal{F}$ and morphisms between the corresponding terms of $\boxed{\mathfrak{F}_2^p} \otimes \mathcal{E}$ and $\boxed{\mathfrak{F}_2^r} \otimes \mathcal{F}$ such that the resulting diagram is commutative.

More explicitly, the $(p,2)$-filtration diagram of θ_X is:

$$\begin{array}{ccccc}
\mathfrak{F}_2^p & \longrightarrow & \mathfrak{F}_1^p \otimes \det \Phi_X & \longrightarrow & \mathfrak{F}_1^1 \otimes \bigwedge^{k-p+1} \Phi_X \\
 & & \downarrow & & \downarrow \\
 & & \bigwedge^{n-p} \Psi_X \otimes \det \Phi_X \otimes \det \Phi_X & \xrightarrow{\text{人}^{\theta}_{p-1}} & \bigwedge^{n-1} \Psi_X \otimes \det \Phi_X \otimes \bigwedge^{k-p+1} \Phi_X \\
 & & {\scriptstyle \text{人}^{\theta}_{p}}\downarrow & & \downarrow{\scriptstyle \text{人}^{\theta}_{1}} \\
 & & \bigwedge^{n} \Psi_X \otimes \bigwedge^{k-p} \Phi_X \otimes \det \Phi_X & \xrightarrow[\text{人}^{\mathrm{id}}_{p-1}]{} & \bigwedge^{n} \Psi_X \otimes \bigwedge^{k-1} \Phi_X \otimes \bigwedge^{k-p+1} \Phi_X \\
 & & \downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1}
\end{array}$$

Similarly, a 2-filtration morphism is:



where the $(p, 2)$-filtration diagram,

$$
\begin{array}{ccccc}
\mathfrak{F}_2^p & \longrightarrow & \mathfrak{F}_1^p \otimes \det \Phi_X & \longrightarrow & \mathfrak{F}_1^1 \otimes \bigwedge^{k-p+1} \Phi_X \\
 & & \downarrow & & \downarrow \\
 & & \bigwedge^{n-p} \Psi_X \otimes \det \Phi_X \otimes \det \Phi_X & \longrightarrow & \bigwedge^{n-1} \Psi_X \otimes \det \Phi_X \otimes \bigwedge^{k-p+1} \Phi_X \\
 & & \downarrow & & \downarrow \\
 & & \bigwedge^{n} \Psi_X \otimes \bigwedge^{k-p} \Phi_X \otimes \det \Phi_X & \longrightarrow & \bigwedge^{n} \Psi_X \otimes \bigwedge^{k-1} \Phi_X \otimes \bigwedge^{k-p+1} \Phi_X \\
 & & \downarrow {\scriptstyle +1} & & \downarrow {\scriptstyle +1}
\end{array}
$$

is represented by the simplified diagram,

$$
\begin{array}{ccccc}
\boxed{\mathfrak{F}_2^p}^{\,0,0} & \longrightarrow & \boxed{\mathfrak{F}_2^p}^{\,0,1} & \longrightarrow & \boxed{\mathfrak{F}_2^p}^{\,0,2} \\
 & & \big\downarrow & & \big\downarrow \\
 & & \boxed{\mathfrak{F}_2^p}^{\,1,1} & \longrightarrow & \boxed{\mathfrak{F}_2^p}^{\,1,2} \\
 & & \big\downarrow & & \big\downarrow \\
 & & \boxed{\mathfrak{F}_2^p}^{\,2,1} & \longrightarrow & \boxed{\mathfrak{F}_2^p}^{\,2,2}.
\end{array}
$$

To define the (p,i)-filtration diagram of θ_X and the i-filtration morphisms we will iterate this construction.

2.4 Inductive Hypotheses For a given i assume that the following holds for all $p, q, r \in \mathbb{N}$.

(2.4.1) The (p,i)-*filtration diagram of* θ_X is defined and denoted by $\boxed{\mathfrak{F}_i^p}$.

(2.4.2) An i-*filtration morphism*, by definition, consists of locally free sheaves $\mathcal{E}, \mathcal{F}$ and a morphism between the corresponding terms of $\boxed{\mathfrak{F}_i^p} \otimes \mathcal{E}$ and $\boxed{\mathfrak{F}_i^r} \otimes \mathcal{F}$ such that the resulting diagram is commutative.

(2.4.3) $\boxed{\mathfrak{F}_i^p}$ has a unique object, $\mathfrak{F}_i^p$, with only one adjacent arrow pointing out.

(2.4.4) $\mathfrak{F}_i^p = 0$ for $p < i$.

(2.4.5) There exists an i-filtration morphism,

$$\curlywedge_q^{\theta,i} : \boxed{\mathfrak{F}_i^p} \otimes \det \Phi_X \to \boxed{\mathfrak{F}_i^{p-q}} \otimes \bigwedge^{k-q} \Phi_X.$$

(2.4.6) The diagram,

$$
\begin{array}{ccc}
\boxed{\mathfrak{F}_i^p} \otimes \det \Phi_X \otimes \det \Phi_X & \xrightarrow{\curlywedge_q^{\theta,i}} & \boxed{\mathfrak{F}_i^{p-q}} \otimes \det \Phi_X \otimes \bigwedge^{k-q} \Phi_X \\
{\scriptstyle \curlywedge_{q+r}^{\theta,i}} \big\downarrow & & \big\downarrow {\scriptstyle \curlywedge_r^{\theta,i}} \\
\boxed{\mathfrak{F}_i^{p-q-r}} \otimes \bigwedge^{k-q-r} \Phi_X \otimes \det \Phi_X & \xrightarrow[\curlywedge_q^{\mathrm{id}}]{} & \boxed{\mathfrak{F}_i^{p-q-r}} \otimes \bigwedge^{k-r} \Phi_X \otimes \bigwedge^{k-q} \Phi_X
\end{array}
$$

is commutative.

2.5 Lemma-Definition. *If* (2.4) *holds for* $i = 0, \dots, j$, *then* $\boxed{\mathfrak{F}^p_{j+1}}$ *can be defined so that* (2.4) *holds for* $i = j+1$.

Proof. If $j > p$, let $\boxed{\mathfrak{F}^p_{j+1}} = \mathfrak{F}^p_{j+1} = 0$. If $j \leq p$ then by (2.4.5) there exists a j-filtration morphism,

$$\curlywedge^{\theta,j}_{p-j} : \boxed{\mathfrak{F}^p_j} \otimes \det \Phi_X \to \boxed{\mathfrak{F}^j_j} \otimes \bigwedge^{k-p+j} \Phi_X.$$

Let $\curlywedge^{\theta,j}_{p-j,res} : \mathfrak{F}^p_j \otimes \det \Phi_X \to \mathfrak{F}^j_j \otimes \bigwedge^{k-p+j} \Phi_X$ be the restriction of $\curlywedge^{\theta,j}_{p-j}$ to $\mathfrak{F}^p_j \otimes \det \Phi_X$ and let

$$\mathfrak{F}^p_{j+1} = M(\curlywedge^{\theta,j}_{p-j,res})[-1].$$

Now $\mathfrak{F}^p_{j+1}$ maps to $\mathfrak{F}^p_j$ and they form the predistinguished triangle

$$\mathfrak{F}^p_{j+1} \to \mathfrak{F}^p_j \otimes \det \Phi_X \to \mathfrak{F}^j_j \otimes \bigwedge^{k-p+j} \Phi_X \xrightarrow{+1} \tag{2.5.7}$$

such that this predistinguished triangle together with $\curlywedge^{\theta,j}_{p-j}$ form a diagram that is commutative.

2.5.8 The $(p, j+1)$-*filtration diagram of* θ_X consists of the diagram

$$\mathfrak{F}^p_{j+1} \to \boxed{\mathfrak{F}^p_j} \otimes \det \Phi_X \to \boxed{\mathfrak{F}^j_j} \otimes \bigwedge^{k-p+j} \Phi_X.$$

It is denoted by $\boxed{\mathfrak{F}^p_{j+1}}$. A $j+1$-*filtration morphism* for some p, r, consists of locally free sheaves $\mathcal{E}, \mathcal{F}$ and a morphism between the corresponding terms of $\boxed{\mathfrak{F}^p_{j+1}} \otimes \mathcal{E}$ and $\boxed{\mathfrak{F}^r_{j+1}} \otimes \mathcal{F}$ such that the resulting diagram is commutative.

(2.4.1) and (2.4.2) follow from this definition, and (2.4.3) and (2.4.4) are also clear from the construction above.

Next consider the following commutative diagram (cf. (2.4.6)).

$$
\begin{array}{ccc}
\mathfrak{F}^{p}_{j+1} \otimes \det \Phi_X & & \mathfrak{F}^{p-q}_{j+1} \otimes \bigwedge^{k-q} \Phi_X \\
\downarrow & & \downarrow \\
\boxed{\mathfrak{F}^{p}_{j}} \otimes \det \Phi_X \otimes \det \Phi_X & \xrightarrow{\curlywedge^{\theta,j}_{q}} & \boxed{\mathfrak{F}^{p-q}_{j}} \otimes \det \Phi_X \otimes \bigwedge^{k-q} \Phi_X \\
\curlywedge^{\theta,j}_{p-j}\downarrow & & \downarrow \curlywedge^{\theta,j}_{p-q-j} \\
\boxed{\mathfrak{F}^{j}_{j}} \otimes \bigwedge^{k-p+j} \Phi_X \otimes \det \Phi_X & \xrightarrow[\curlywedge^{\mathrm{id}}_{q}]{} & \boxed{\mathfrak{F}^{j}_{j}} \otimes \bigwedge^{k-p+q+j} \Phi_X \otimes \bigwedge^{k-q} \Phi_X
\end{array}
$$

Then the construction of $\boxed{\mathfrak{F}^{p}_{j+1}}$ implies that there is a morphism,

$$\mathfrak{F}^{p}_{j+1} \otimes \det \Phi_X \to \mathfrak{F}^{p-q}_{j+1} \otimes \bigwedge^{k-q} \Phi_X$$

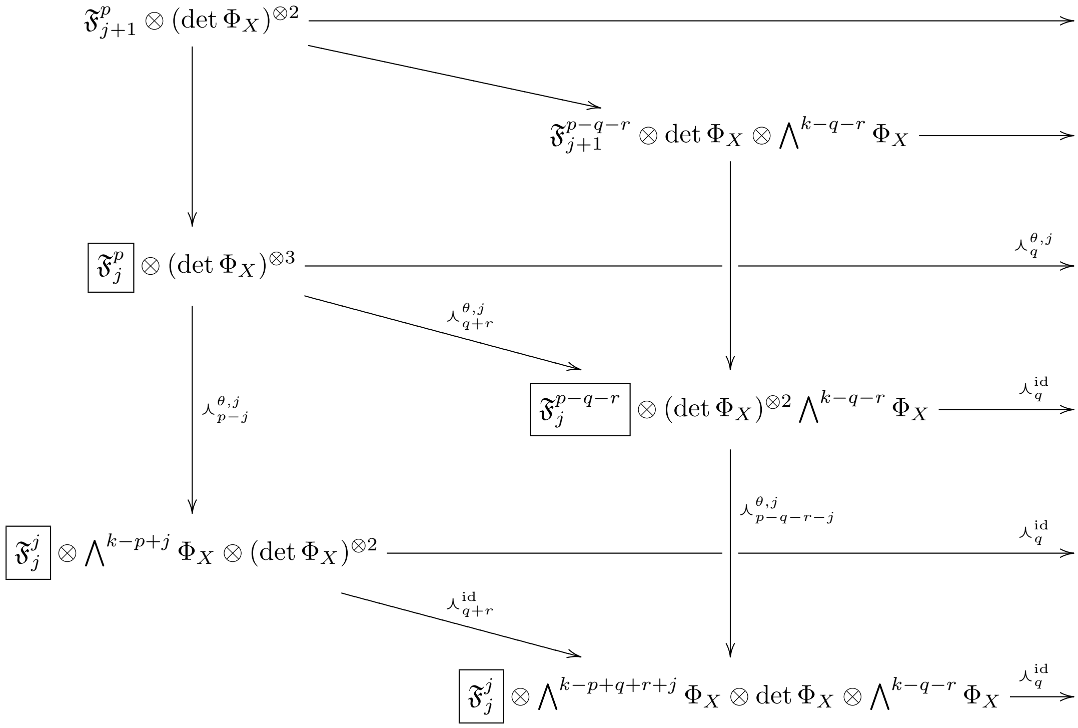
making the diagram commutative and so providing a $j+1$-filtration morphism,

$$\curlywedge^{\theta,j+1}_{q} : \boxed{\mathfrak{F}^{p}_{j+1}} \otimes \det \Phi_X \to \boxed{\mathfrak{F}^{p-q}_{j+1}} \otimes \bigwedge^{k-q} \Phi_X. \tag{2.5.9}$$

This proves (2.4.5). Next consider (2.4.6) for $i = j+1$.

$$
\begin{array}{ccc}
\boxed{\mathfrak{F}^{p}_{j+1}} \otimes \det \Phi_X \otimes \det \Phi_X & \xrightarrow{\curlywedge^{\theta,j+1}_{q}} & \boxed{\mathfrak{F}^{p-q}_{j+1}} \otimes \bigwedge^{k-q} \Phi_X \otimes \det \Phi_X \\
\curlywedge^{\theta,j+1}_{q+r}\downarrow & & \downarrow \curlywedge^{\theta,j+1}_{r} \\
\boxed{\mathfrak{F}^{p-q-r}_{j+1}} \otimes \det \Phi_X \otimes \bigwedge^{k-q-r} \Phi_X & \xrightarrow[\curlywedge^{\mathrm{id}}_{q}]{} & \boxed{\mathfrak{F}^{p-q-r}_{j+1}} \otimes \bigwedge^{k-q} \Phi_X \otimes \bigwedge^{k-r} \Phi_X
\end{array}
\tag{2.5.10}
$$

This is shown in more detail in the following diagram. Due to the size of the diagram, it is broken up into two pieces.


$\mathfrak{F}^p_{j+1} \otimes (\det \Phi_X)^{\otimes 2}$
$\mathfrak{F}^{p-q-r}_{j+1} \otimes \det \Phi_X \otimes \bigwedge^{k-q-r} \Phi_X$
$\boxed{\mathfrak{F}^p_j} \otimes (\det \Phi_X)^{\otimes 3}$
$\curlywedge^{\theta,j}_q$
$\curlywedge^{\theta,j}_{q+r}$
$\boxed{\mathfrak{F}^{p-q-r}_j} \otimes (\det \Phi_X)^{\otimes 2} \bigwedge^{k-q-r} \Phi_X$
$\curlywedge^{\mathrm{id}}_q$
$\curlywedge^{\theta,j}_{p-j}$
$\curlywedge^{\theta,j}_{p-q-r-j}$
$\boxed{\mathfrak{F}^j_j} \otimes \bigwedge^{k-p+j} \Phi_X \otimes (\det \Phi_X)^{\otimes 2}$
$\curlywedge^{\mathrm{id}}_q$
$\curlywedge^{\mathrm{id}}_{q+r}$
$\boxed{\mathfrak{F}^j_j} \otimes \bigwedge^{k-p+q+r+j} \Phi_X \otimes \det \Phi_X \otimes \bigwedge^{k-q-r} \Phi_X$
$\curlywedge^{\mathrm{id}}_q$

$$
\begin{array}{ccccc}
\longrightarrow & \mathfrak{F}_{j+1}^{p-q} \otimes \bigwedge^{k-q} \Phi_X \otimes \det \Phi_X & & & \\
 & \downarrow & \searrow & & \\
\longrightarrow & & & \mathfrak{F}_{j+1}^{p-q-r} \otimes \bigwedge^{k-q} \Phi_X \otimes \bigwedge^{k-r} \Phi_X & \\
 & \downarrow & & \downarrow & \\
\xrightarrow{\curlywedge_q^{\theta,j}} & \boxed{\mathfrak{F}_j^{p-q}} \otimes \det \Phi_X \otimes \bigwedge^{k-q} \Phi_X \otimes \det \Phi_X & \xrightarrow{\curlywedge_r^{\theta,j}} & & \\
 & \downarrow{\scriptstyle \curlywedge_{p-q-j}^{\theta,j}} & & & \\
\xrightarrow{\curlywedge_q^{\mathrm{id}}} & & & \boxed{\mathfrak{F}_{p-q-r}^{j}} \otimes \det \Phi_X \otimes \bigwedge^{k-q} \Phi_X \otimes \bigwedge^{k-r} \Phi_X & \\
 & \downarrow & & \downarrow{\scriptstyle \curlywedge_{p-q-r-j}^{\theta,j}} & \\
\xrightarrow{\curlywedge_q^{\mathrm{id}}} & \boxed{\mathfrak{F}_j^{j}} \otimes \bigwedge^{k-p+q+j} \Phi_X \otimes \bigwedge^{k-q} \Phi_X \otimes \det \Phi_X & \xrightarrow{\curlywedge_r^{\mathrm{id}}} & & \\
\xrightarrow{\curlywedge_q^{\mathrm{id}}} & & & \boxed{\mathfrak{F}_j^{j}} \otimes \bigwedge^{k-p+q+r+j} \Phi_X \otimes \bigwedge^{k-q} \Phi_X \otimes \bigwedge^{k-r} \Phi_X &
\end{array}
$$

The commutativity of vertical faces is either obvious or follows by the commutativity of (2.5.9).

$$
\begin{array}{ccc}
\boxed{\mathfrak{F}_j^{p}} \otimes (\det \Phi_X)^{\otimes 3} & \xrightarrow{\curlywedge_q^{\theta,j}} & \boxed{\mathfrak{F}_j^{p-q}} \otimes \det \Phi_X \otimes \bigwedge^{k-q} \Phi_X \otimes \det \Phi_X \\
\downarrow{\scriptstyle \curlywedge_{q+r}^{\theta,j}} & & \downarrow{\scriptstyle \curlywedge_r^{\theta,j}} \\
\boxed{\mathfrak{F}_j^{p-q-r}} \otimes (\det \Phi_X)^{\otimes 2} \otimes \bigwedge^{k-q-r} \Phi_X & \xrightarrow[\curlywedge_q^{\mathrm{id}}]{} & \boxed{\mathfrak{F}_j^{p-q-r}} \otimes \det \Phi_X \otimes \bigwedge^{k-q} \Phi_X \otimes \bigwedge^{k-r} \Phi_X
\end{array}
$$

is commutative by (2.4.6), and

$$
\begin{array}{ccc}
\boxed{\mathfrak{F}_j^{j}} \otimes \bigwedge^{k-p+j} \Phi_X \otimes (\det \Phi_X)^{\otimes 2} & \xrightarrow{\curlywedge_q^{\mathrm{id}}} & \boxed{\mathfrak{F}_j^{j}} \otimes \bigwedge^{k-p+q+j} \Phi_X \otimes \bigwedge^{k-q} \Phi_X \otimes \det \Phi_X \\
\downarrow{\scriptstyle \curlywedge_{q+r}^{\mathrm{id}}} & & \downarrow{\scriptstyle \curlywedge_r^{\mathrm{id}}} \\
\boxed{\mathfrak{F}_j^{j}} \otimes \bigwedge^{k-p+q+r+j} \Phi_X \otimes \det \Phi_X \otimes \bigwedge^{k-q-r} \Phi_X & \xrightarrow[\curlywedge_q^{\mathrm{id}}]{} & \boxed{\mathfrak{F}_j^{j}} \otimes \bigwedge^{k-p+q+r+j} \Phi_X \otimes \bigwedge^{k-q} \Phi_X \otimes \bigwedge^{k-r} \Phi_X
\end{array}
$$

is commutative by (1.1.2). Now it is easy to see that (2.5.10) is commutative. □

2.6. We are ready to define $\underline{\mathfrak{Q}}^p_{\theta_X} \in \mathrm{Ob}(D(X))$ for $p \in \mathbb{Z}, p \geq -k$. Let $\underline{\mathfrak{Q}}^p_{\theta_X}$ be the class of

$$\mathfrak{F}^{n-k-p}_{n-k-p} \otimes (\det \Phi_X)^{-(n-k-p+1)}$$

in $\mathrm{Ob}(D(X))$ for $-k \leq p \leq n-k$, and let $\underline{\mathfrak{Q}}^p_{\theta_X} = 0$ for $p > n-k$. It is easy to see that

$$\underline{\mathfrak{Q}}^{n-k}_{\theta_X} = \det \Psi_X \otimes (\det \Phi_X)^{-1}$$

and that there is a distinguished triangle:

$$\underline{\mathfrak{Q}}^{n-k-1}_{\theta_X} \otimes \det \Phi_X \to \bigwedge^{n-1} \Psi_X \to \underline{\mathfrak{Q}}^{n-k}_{\theta_X} \otimes \bigwedge^{k-1} \Phi_X \xrightarrow{+1}$$

In general for $j \geq p-n+k$ let $\mathbb{F}^j \bigwedge^p \Psi_X$ be the class of

$$\mathfrak{F}^{n-p}_{n-k-p+j} \otimes (\det \Phi_X)^{-(n-k-p+j)}$$

in $\mathrm{Ob}(D(X))$. The predistinguished triangle (2.5.7),

$$\mathfrak{F}^{n-p}_{n-k-p+j+1} \to \mathfrak{F}^{n-p}_{n-k-p+j} \otimes \det \Phi_X \to \mathfrak{F}^{n-k-p+j}_{n-k-p+j} \otimes \bigwedge^{j} \Phi_X \xrightarrow{+1}$$

gives the distinguished triangle,

$$\mathbb{F}^{j+1} \bigwedge^p \Psi_X \to \mathbb{F}^j \bigwedge^p \Psi_X \to \underline{\mathfrak{Q}}^{p-j}_{\theta_X} \otimes \bigwedge^j \Phi_X \xrightarrow{+1} .$$

Now $\mathbb{F}^{k+1} \bigwedge^p \Psi_X = 0$ by (2.4.4) and by definition $\mathbb{F}^{p-n+k} \bigwedge^p \Psi_X = \bigwedge^p \Psi_X$.

Observe that if $p-n+k<0$, then

$$\mathbb{F}^0\bigwedge^p\Psi_X \simeq \mathbb{F}^{-1}\bigwedge^p\Psi_X \simeq \cdots \simeq \mathbb{F}^{p-n+k}\bigwedge^p\Psi_X = \bigwedge^p\Psi_X,$$

since $\bigwedge^j\Phi_X = 0$ for $j<0$. If $p-n+k\geq 0$, define $\mathbb{F}^j\bigwedge^p\Psi_X = \bigwedge^p\Psi_X$ for $j=0,\ldots,p-n+k$.

Therefore we proved:

2.7 Theorem. *Let Φ_X and Ψ_X be locally free sheaves on X of rank k and n respectively, and let $\theta_X:\Phi_X\to\Psi_X$ be a morphism. Then there exists a $\underline{\mathfrak{Q}}^r_{\theta_X}\in\mathrm{Ob}(D(X))$ for all $r\in\mathbb{Z}, r\geq -k$ with the following property. For any $p\in\mathbb{N}$ there exists a hyperfiltration $\mathbb{F}^j\bigwedge^p\Psi_X$ of $\bigwedge^p\Psi_X$ with $j=0,\ldots,k+1$, such that*

$$\mathbb{F}^0\bigwedge^p\Psi_X \simeq \bigwedge^p\Psi_X,$$

$$\mathbb{F}^{k+1}\bigwedge^p\Psi_X \simeq 0$$

and

$$\mathbb{G}^j\bigwedge^p\Psi_X \simeq \underline{\mathfrak{Q}}^{p-j}_{\theta_X}\otimes\bigwedge^j\Phi_X.$$

Furthermore, for $r>n-k$,

$$\underline{\mathfrak{Q}}^r_{\theta_X}\simeq 0.$$

Now using (1.2.2) one obtains the following corollary:

2.8 Corollary. *Let Φ_X and Ψ_X be locally free sheaves on X of rank k and n respectively, and let $\theta_X:\Phi_X\to\Psi_X$ be a morphism. Then for any locally free sheaf $\mathcal{E}$ and for any $p\in\mathbb{N}$ there exists a spectral sequence,*

$$E_1^{r,s} = \mathbb{H}^{r+s}(X,\underline{\mathfrak{Q}}^{p-r}_{\theta_X}\otimes\bigwedge^r\Phi_X\otimes\mathcal{E}) \Rightarrow H^{r+s}(X,\bigwedge^p\Psi_X\otimes\mathcal{E}).$$

2.9 Proposition. *If the cokernel sheaf, Ξ_X, of θ_X is locally free, then $\underline{\mathfrak{Q}}^p_{\theta_X}$ is simply the p-th exterior power of Ξ_X, and the spectral sequence above is the well-known spectral sequence associated to the filtration,*

$$\bigwedge^p\Psi_X = F^0\supset F^1\supset\cdots\supset F^p\supset F^{p+1}=0,$$

with quotients

$$F^j/F^{j+1}\simeq\bigwedge^{p-j}\Xi_X\otimes\bigwedge^j\Phi_X$$

for each j.

Proof. By definition one has that

$$\underline{\mathfrak{Q}}^p_{\theta_X} \simeq \det \Psi_X \otimes (\det \Phi_X)^{-1} \simeq \det \Xi_X .$$

Then the statement follows using descending induction, the filtration associated to the short exact sequence of locally free sheaves and the distinguished triangle,

$$\mathbb{F}^{j+1} \bigwedge^p \Psi_X \to \mathbb{F}^j \bigwedge^p \Psi_X \to \underline{\mathfrak{Q}}^{p-j}_{\theta_X} \otimes \bigwedge^j \Phi_X \xrightarrow{+1} .$$

□

2.10 Example. Let $k = 1$, i.e., assume that Φ_X is a line bundle. Then the hyperfiltration in (2.7) is simply a distinguished triangle,

$$\underline{\mathfrak{Q}}^{p-1}_{\theta_X} \otimes \Phi_X \to \bigwedge^p \Psi_X \to \underline{\mathfrak{Q}}^p_{\theta_X} \xrightarrow{+1} .$$

and then the spectral sequence in (2.8) reduces to a long exact hypercohomology sequence:

$$\ldots \to \mathbb{H}^s(X, \underline{\mathfrak{Q}}^{p-1}_{\theta_X} \otimes \Phi_X \otimes \mathcal{E}) \to H^s(X, \bigwedge^p \Psi_X \otimes \mathcal{E}) \to$$

$$\to \mathbb{H}^s(X, \underline{\mathfrak{Q}}^p_{\theta_X} \otimes \Phi_X \otimes \mathcal{E}) \to \mathbb{H}^{s+1}(X, \underline{\mathfrak{Q}}^{p-1}_{\theta_X} \otimes \Phi_X \otimes \mathcal{E}) \to \ldots$$

§3. A vanishing theorem

Throughout this section X is an algebraic veriety over a field k.

The following well-known fact is included for ease of reference.

3.1 Fact. *Let $\mathcal{E}$ be a locally free sheaf of rank r. Assume that there is a filtration*

$$\mathcal{E} = F^0 \supset F^1 \supset \cdots \supset F^r = 0$$

of $\mathcal{E}$ such that

$$F^{i-1}/F^i = \mathcal{L}_i$$

is a line bundle for all $i = 1, \ldots, r$. Then for every $1 \leq t \leq r$ there is a filtration

$$\bigwedge^t \mathcal{E} = F^0_t \supset F^1_t \supset \cdots \supset F^{\binom{r}{t}}_t = 0$$

of $\bigwedge^t \mathcal{E}$ such that

$$F^{i-1}_t / F^i_t = \mathcal{L}_{i_1} \otimes \mathcal{L}_{i_2} \otimes \cdots \otimes \mathcal{L}_{i_t}$$

for all $i = 1, \ldots, \binom{r}{t}$ and a suitable set of indicies $1 \leq i_1 < i_2 < \cdots < i_t \leq r$. □

3.2 Definition. A locally free sheaf $\mathcal{E}$ on X is called *semi-positive* or *nef* if for every smooth proper curve C and every map $\gamma : C \to X$, $\deg \mathcal{Q}\big|_C \geq 0$ for any quotient bundle $\mathcal{Q}$ of $\gamma^*\mathcal{E}$.

3.3 Definition. Let $\mathcal{E}$ be a locally free sheaf of rank r. $\mathcal{E}$ will be called *semi-negative of splitting type* if $\mathcal{E}$ has a filtration

$$\mathcal{E} = F^0 \supset F^1 \supset \cdots \supset F^r = 0$$

such that

$$F^{i-1}/F^i = \mathcal{L}_i$$

is a semi-negative line bundle, i.e., $\mathcal{L}_i^{-1}$ is nef.

3.4 Definition. A class, $\mathfrak{C}$, of line bundles on X will be called *nef-invariant*, if for all $\mathcal{L} \in \mathfrak{C}$ and $\mathcal{N}$ nef line bundle on X, $\mathcal{L} \otimes \mathcal{N} \in \mathfrak{C}$.

3.5 Example. If X is projective, then the class of ample line bundles on X is nef-invariant.

3.6 Lemma. *Let Φ_X and Ψ_X be locally free sheaves on X of rank k and n respectively, and let $\theta_X : \Phi_X \to \Psi_X$ be a morphism. Let $\mathfrak{C}$ be a nef-invariant class of line bundles on X. Let $c(n) \in \mathbb{N}$ be a constant such that $H^p(X, \bigwedge^q \Psi_X \otimes \mathcal{L}) = 0$ for every line bundle $\mathcal{L} \in \mathfrak{C}$ and $p + q > c(n)$. Let $i, j \in \mathbb{N}$ be natural numbers such that $i + j + k \geq c(n)$. Assume that Φ_X is semi-negative of splitting type and that for every natural number $l > j$ and line bundle $\mathcal{L} \in \mathfrak{C}$ on X,*

$$\mathbb{H}^i(X, \underline{\mathfrak{Q}}^l_{\theta_X} \otimes \det \Phi_X \otimes \mathcal{L}) = 0.$$

Then

$$\mathbb{H}^{i+1}(X, \underline{\mathfrak{Q}}^j_{\theta_X} \otimes \det \Phi_X \otimes \mathcal{L}) = 0$$

for every line bundle $\mathcal{L} \in \mathfrak{C}$.

Proof. By (2.8) there exists a spectral sequence

$$E_1^{r,s} = \mathbb{H}^{r+s}(X, \underline{\mathfrak{Q}}^{j+k-r}_{\theta_X} \otimes \bigwedge^r \Phi_X \otimes \mathcal{L}) \Rightarrow H^{r+s}(X, \bigwedge^{j+k} \Psi_X \otimes \mathcal{L}).$$

Since $i+j+k \geq c(n)$, $H^{i+1}(X, \bigwedge^{j+k} \Psi_X \otimes \mathcal{L}) = 0$ by assumption. Hence $E_\infty^{r,i+1-r} = 0$ for all r. In particular $E_\infty^{k,i+1-k} = 0$. Suppose now that

$$E_1^{k,i+1-k} = \mathbb{H}^{i+1}(X, \underline{\mathfrak{Q}}^j_{\theta_X} \otimes \det \Phi_X \otimes \mathcal{L}) \neq 0.$$

Observe that $E_w^{u,v} = 0$ for every $u > k$ and arbitrary v, w, so in order to have $E_\infty^{k,i+1-k} = 0$, there must be a $t \geq 1$ such that $E_t^{k-t,i-k+t} \neq 0$. Then $E_1^{k-t,i-k+t} \neq 0$ for the same t, so $l = j + t > j$ is such that

$$E_1^{k-t,i-k+t} = \mathbb{H}^i(X, \underline{\mathfrak{Q}}^l_{\theta_X} \otimes \bigwedge^{k-t} \Phi_X \otimes \mathcal{L}) \neq 0.$$

Now by assumption Φ_X has a filtration

$$\Phi_X = F^0 \supset F^1 \supset \cdots \supset F^k = 0$$

such that $F^{i-1}/F^i = \mathcal{L}_i$ and $\mathcal{L}_i^{-1}$ is a nef line bundle. Then by (3.1) there exist $i_1, \ldots, i_{k-t}$ such that

$$\mathbb{H}^i(X, \underline{\mathfrak{Q}}^l_{\theta_X} \otimes \mathcal{L}_{i_1} \otimes \cdots \otimes \mathcal{L}_{i_{k-t}} \otimes \mathcal{L}) \neq 0.$$

Therefore

$$\mathbb{H}^i(X, \underline{\mathfrak{Q}}^l_{\theta_X} \otimes \det \Phi_X \otimes \mathcal{L} \otimes \mathcal{L}^{-1}_{i_{k-t+1}} \otimes \cdots \otimes \mathcal{L}^{-1}_{i_k}) \neq 0,$$

for $\{i_{k-t+1}, \ldots, i_k\} = \{1, \ldots, k\} \setminus \{i_1, \ldots, i_{k-t}\}$.

Since $\mathcal{L} \otimes \mathcal{L}^{-1}_{i_{k-t+1}} \otimes \cdots \otimes \mathcal{L}^{-1}_{i_k} \in \mathfrak{C}$, this non-vanishing violates the assumption. Hence the statement follows. $\square$

3.7 Theorem. *Let Φ_X and Ψ_X be locally free sheaves on X of rank k and n respectively, and let $\theta_X : \Phi_X \to \Psi_X$ be a morphism. Let $\mathfrak{C}$ be a nef-invariant class of line bundles on X. Assume that $H^p(X, \bigwedge^q \Psi_X \otimes \mathcal{L}) = 0$ for every line bundle $\mathcal{L} \in \mathfrak{C}$ and $p + q > n$. Assume further that Φ_X is semi-negative of splitting type. Then for every line bundle $\mathcal{L} \in \mathfrak{C}$,*

$$\mathbb{H}^i(X, \underline{\mathfrak{Q}}^l_{\theta_X} \otimes \det \Phi_X \otimes \mathcal{L}) = 0 \qquad \text{for } i + l > n - k.$$

Proof. By (2.7) $\underline{\mathfrak{Q}}^l_{\theta_X} = 0$ for $l > n - k$, so

$$\mathbb{H}^i(X, \underline{\mathfrak{Q}}^l_{\theta_X} \otimes \det \Phi_X \otimes \mathcal{L}) = 0$$

for $l > n - k$ and $i \geq 0$. Then

$$\mathbb{H}^{i+1}(X, \underline{\mathfrak{Q}}^{n-k}_{\theta_X} \otimes \det \Phi_X \otimes \mathcal{L}) = 0$$

for $i \geq 0$ by (3.6). Hence

$$\mathbb{H}^i(X, \underline{\mathfrak{Q}}^l_{\theta_X} \otimes \det \Phi_X \otimes \mathcal{L}) = 0$$

for $l > n - k - 1$ and $i \geq 1$. Then again

$$\mathbb{H}^{i+1}(X, \underline{\mathfrak{Q}}^{n-k-1}_{\theta_X} \otimes \det \Phi_X \otimes \mathcal{L}) = 0$$

for $i \geq 1$ by (3.6). Hence again

$$\mathbb{H}^i(X, \underline{\Omega}^l_{\theta_X} \otimes \det \Phi_X \otimes \mathcal{L}) = 0$$

for $l > n - k - 2$ and $i \geq 2$.

Iterating this process one sees that

$$\mathbb{H}^i(X, \underline{\Omega}^l_{\theta_X} \otimes \det \Phi_X \otimes \mathcal{L}) = 0$$

for $i + l > n - k$. □

3.8 Example. If X is a smooth projective variety over $\mathbb{C}$ and $\mathfrak{C}$ is the class of ample line bundles on X, then $\Psi_X = \Omega_X$ satisfies the assumption by the Kodaira-Akizuki-Nakano vanishing theorem.

3.9 Corollary. [**Kovács97a, 1.4**] *Let $f : X \to S$ be a smooth morphism of smooth projective varieties over $\mathbb{C}$ of dimension n and k respectively. Let $\mathcal{L}$ be an ample line bundle and assume that $f^*\Omega_S$ is semi-negative of splitting type. Then*

$$H^i(X, \Omega^l_{X/S} \otimes f^*\omega_S \otimes \mathcal{L}) = 0 \qquad \text{for } i + l > n - k.$$

Proof. (3.8), (3.7), (2.9.1). □

§4. Functoriality

Let S be a scheme and $\mathfrak{Sch}_S$ the category of S-schemes. Let $\mathfrak{LocFree}_S$ be the category of pairs $(X, \mathcal{E})$ where $X \in \mathrm{Ob}(\mathfrak{Sch}_S)$ and $\mathcal{E}$ is a locally free sheaf on X. A morphism in $\mathrm{Hom}_{\mathfrak{LocFree}_S}((Y, \mathcal{F}), (X, \mathcal{E}))$ consists of pairs (ϕ, φ) where $\phi \in \mathrm{Hom}_{\mathfrak{Sch}_S}(Y, X)$ and $\varphi \in \mathrm{Hom}_X(\mathcal{E}, \phi_*\mathcal{F}) \simeq \mathrm{Hom}_Y(\phi^*\mathcal{E}, \mathcal{F})$.

A functor $\Phi = (\Phi_1, \Phi_2) : \mathfrak{Sch}_S \to \mathfrak{LocFree}_S$ will be called *honest* if Φ_1 is the identity. In this case Φ_X will denote $\Phi_2(X)$, i.e., Φ_X is a locally free sheaf on X.

4.1 Theorem. *Let Φ and Ψ be honest functors from $\mathfrak{Sch}_S$ to $\mathfrak{LocFree}_S$, and $\theta : \Phi \to \Psi$ a natural transformation. Then the definition of $\mathbb{F}^q \bigwedge^p \Psi_X$ and $\underline{\Omega}^p_{\theta_X}$ is functorial in the following sense, if $\phi : Y \to X$ is a morphism of S-schemes, such that $\mathrm{rk}\, \Phi_X = \mathrm{rk}\, \Phi_Y = k$, then there are natural maps in $D(X)$ forming the commutative diagram:*

$$\begin{array}{ccccccc}
\mathbb{F}^{q+1}\bigwedge^p \Psi_X & \longrightarrow & \mathbb{F}^{q}\bigwedge^p \Psi_X & \longrightarrow & \mathbb{G}^{q}\bigwedge^p \Psi_X & \xrightarrow{+1} & \\
\downarrow & & \downarrow & & \downarrow & & \\
R\phi_*\mathbb{F}^{q+1}\bigwedge^p \Psi_Y & \longrightarrow & R\phi_*\mathbb{F}^{q}\bigwedge^p \Psi_Y & \longrightarrow & R\phi_*\mathbb{G}^{q}\bigwedge^p \Psi_Y & \xrightarrow{+1} & .
\end{array}$$

In particular, there are natural maps in $D(X)$ forming the commutative diagram:

$$\begin{array}{ccc} \bigwedge^{p} \Psi_X & \longrightarrow & \underline{\Omega}^p_{\theta_X} \\ \big\downarrow & & \big\downarrow \\ R\phi_* \bigwedge^{p} \Psi_Y & \longrightarrow & R\phi_* \underline{\Omega}^p_{\theta_Y}. \end{array}$$

Proof. Notice that θ induces a natural transformation, $\theta^p : \bigwedge^p \Phi \to \bigwedge^p \Psi$, between the functors $\bigwedge^p \Phi : X \longmapsto (X, \bigwedge^p \Phi_X)$ and $\bigwedge^p \Psi : X \longmapsto (X, \bigwedge^p \Psi_X)$. Then add the following to (2.4.5):

(2.4.5′) $\curlywedge_q^{\theta,i}$ is natural, i.e., there exists a commutative diagram:

$$\begin{array}{ccc} \boxed{\mathfrak{F}_i^p(\theta_X)} \otimes \det \Phi_X & \xrightarrow{\curlywedge_q^{\theta,i}} & \boxed{\mathfrak{F}_i^{p-q}(\theta_X)} \otimes \bigwedge^{k-q} \Phi_X \\ \big\downarrow \theta^k(\phi) & & \big\downarrow \theta^{k-q}(\phi) \\ \boxed{\mathfrak{F}_i^p(\theta_Y)} \otimes \det \Phi_Y & \xrightarrow{\curlywedge_q^{\theta,i}} & \boxed{\mathfrak{F}_i^{p-q}(\theta_Y)} \otimes \bigwedge^{k-q} \Phi_Y. \end{array}$$

Now repeating the proof of (2.5) for X and Y simultaneously it is easy to see that the first statement is a simple consequence of (2.4.5′) and the construction of $\mathbb{F}^q \bigwedge^p \Psi_X$. Then the second statement follows directly from (2.7). □

4.1.1 Remark. In the above statements "natural" should be understood the following way: In the definition of $\underline{\Omega}^p_{\theta_X}$ one needs to make non-canonical choices when defining objects as the third vertex of a distinguished triangle. Here "natural" means that once those choices are made, the construction of the above maps is natural.

§5. Applications

The common theme of the applications is that certain arguments that involve ample line bundles and vanishing theorems on smooth varieties can be extended to certain singular varieties. These results about singular varieties in turn play an important role toward proving other results about smooth varieties and the key line bundles being only nef and big instead of ample.

5.1 Definition. A line bundle $\mathcal{K}$ on Y is called *big* if Y is proper and $\mathcal{K}^{\otimes m}$ gives a birational map for some $m > 0$.

5.2 Example. Let $f : Y \to X$ be a birational morphism between projective varieties and let $\mathcal{L}$ be an ample line bundle on X. Then $f^*\mathcal{L}$ is nef and big on Y.

For the rest of this section we assume that everything is defined over $\mathbb{C}$.

§§**5.3.** Let S be a smooth variety. Let Ψ be the functor of Kähler differentials, i.e., for any S-scheme X, smooth over $\mathbb{C}$, $\Psi_X = \Omega_X$, i.e., $\Psi(X) = (X, \Omega_X)$, and let Φ be the functor on $\mathfrak{Sch}_S$ that is the pullback of Ψ from S, i.e., for $f : X \to S$, $\Phi_X = f^*\Omega_S$, or in other words $\Phi(X) = (X, f^*\Omega_S)$. Then f^* induces a natural transformation $\theta : \Phi \to \Psi$.

Let $f : X \to S$ be a morphism of algebraic varieties of dimension n and k respectively, such that S is smooth. Then one obtains the definition of the p-th relative De Rham complexes[2], $\underline{\Omega}^p_{X/S} = \underline{\mathfrak{Q}}^p_{\theta_X}$, for non-smooth morphisms. The germ of this construction first appeared in [**Kovács96**], but there the base was required to be a smooth curve. The initial motivation to define the relative De Rham complexes was to prove the following statement:

5.3.1 THEOREM. [**Kovács96**] *Let $g : Y \to C$ be a smooth family of projective varieties of general type with a nef canonical bundle and C a smooth projective curve of genus at most one. Then the fibers of g are birational.*

5.3.1.1 REMARK. The case of $\dim Y = 2$ is well-known and the case of $\dim Y = 3$ was first proved by [**Migliorini95**].

The construction presented here allows the generalization of the p-th relative De Rham complexes to the case when the base of the family is higher dimensional (but still smooth). This is discussed and the following generalization of (5.3.1) is proved in [**Kovács97b**].

5.3.2 THEOREM. [**Kovács97b**] *Let $g : Y \to S$ be a smooth family of projective varieties of general type with a nef canonical bundle and S a smooth projective variety. Assume that Ω_S is semi-negative. Then the fibres of g are birational.*

Further generalizations were obtained in [**Kovács02**], where logarithmic analogues of the above were constructed. For details the reader is referred to the original article.

§6. The 'nef & big on smooth' ≈ 'ample on singular' principal

It may not be entirely clear at this point how the results of the present article are applied to prove the above theorems, or why they are necessary at all. The statements are about smooth varieties whereas the results in this paper are only interesting for singular ones.

The main point is that without these results the arguments that are used in the applications would only prove the statements for *ample* line bundles. By working with singular varieties we are able to extend the argument to include *nef and big* line bundles as explained below.

The essence of the applications may be summarized as the

"Nef & big line bundle on smooth ≈ ample line bundle on singular" principle.

In fact, it is a new approach to this principle:

[2] Note that the word *complex* here refers to an object in a derived category. These are analogues of the *components* of the usual De Rham complex and not the total complex. For more details please refer to [**Kovács97b**].

The usual approach sets its goal to obtain a statement for a possibly singular X and an ample line bundle $\mathcal{L}$ on X. This is achieved by working with a desingularization, $f : Y \to X$, and a nef and big line bundle, $f^*\mathcal{L}$, on Y.

Our "upside down" approach aims for a statement for a smooth Y and a nef and big line bundle $\mathcal{K}$ on Y. If $\mathcal{K}^{\otimes m}$ is generated by global sections, then we may work with the morphism, $f : Y \to X$, induced by $\mathcal{K}^{\otimes m}$, so $\mathcal{K}^{\otimes m} = f^*\mathcal{L}$ for some $\mathcal{L}$ ample on X. The main idea is that working on X we may assume that our line bundle is ample. The price to pay is that we no longer can assume that the variety is smooth, but fortunately this is not a fatal problem.

The detailed plan is the following:

6.1 Plan

(6.1.2) First prove the statement for Y smooth, $\mathcal{K}$ ample.

(6.1.3) Check if $\mathcal{K}^{\otimes m}$ is generated by global sections. This gives $f : Y \to X$ induced by $\mathcal{K}^{\otimes m}$, so $\mathcal{K}^{\otimes m} = f^*\mathcal{L}$ for some $\mathcal{L}$ ample on X.

(6.1.4) Extend the argument (from (6.1.2)) for X singular, $\mathcal{L}$ ample.

(6.1.5) "Pull back" the result.

Next we will sketch how this plan is executed in the applications mentioned in the previous section. Let us remark that the line bundle $\mathcal{K}$ is chosen to be the canonical bundle, ω_Y.

6.2. The key ingredient for (6.1.2) is the spectral sequence from the introduction. For more details please refer to the actual applications.

6.3. (6.1.3) follows easily from the Basepoint-free Theorem [**CKM88, 9.3**]. It provides the needed global generation for a power of the canonical bundle.

6.4. To complete (6.1.4) the necessary ingredients are the argument from (6.1.2), (2.7) and (2.8). In addition we use that if $g : Y \to S$ is as in (5.3.2), then $\underline{\Omega}^j_{Y/S} \simeq \Omega^j_{Y/S}$ by (2.9).

6.5. Functoriality becomes important for (6.1.5):

For the morphism $f : Y \to X$ induced by $\omega_Y^{\otimes m}$ and for all j there exists a natural morphism,

$$\phi^* : \underline{\Omega}^j_{X/S} \to R\phi_*\underline{\Omega}^j_{Y/S}.$$

One can also prove that there exists a natural morphism $\rho : \mathcal{O}_X \to \underline{\Omega}^0_{X/S}$ cf. [**Kovács97b**].

Next we need the following:

6.6 Definition. [**Kovács97b**] $f : X \to S$ is an *SP-morphism* if

$$\rho : \mathcal{O}_X \to \underline{\Omega}^0_{X/S}$$

has a *left inverse* in $D(X)$, i.e., $\exists$ a morphism $\rho' : \underline{\Omega}^0_{X/S} \to \mathcal{O}_X$ such that $\rho' \circ \rho : \mathcal{O}_X \to \mathcal{O}_X$ is a quasi-isomorphism.

6.7 Observation. *If $f: X \to S$ is an SP-morphism, then*

$$H^i(X, \mathcal{L}) = \mathbb{H}^i(X, \mathcal{O}_X \otimes \mathcal{L}) \subseteq \mathbb{H}^i(X, \underline{\Omega}^0_{X/S} \otimes \mathcal{L})$$

for all i and for all locally free sheaves $\mathcal{L}$.

6.8 Example. If f is smooth it is SP.

The importance of this notion is shown by the next simple lemma.

6.9 Lemma.

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & X \\ g\big\downarrow & & \big\downarrow f \\ S & = & S \end{array}$$

Let S and g be smooth, ϕ birational and assume that X has rational singularities. Then f is an SP-morphism

Proof.

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{\rho} & \underline{\Omega}^0_{X/S} \\ \big\downarrow\simeq & & \big\downarrow\phi^* \\ R\phi_*\mathcal{O}_Y & \xrightarrow[R\phi_*\rho]{\simeq} & R\phi_*\Omega^0_{Y/S}\ (\simeq R\phi_*\underline{\Omega}^0_{Y/S}) \end{array}$$

□

Now (3.7) implies the following:

6.10 Theorem. *Let $f: X \to S$ be a morphism of projective algebraic varieties of dimension n and k respectively, S an Abelian variety and $\mathcal{L}$ an ample line bundle on X. Then*

$$\mathbb{H}^p(X, \underline{\Omega}^q_{X/S} \otimes \mathcal{L}) = 0 \qquad p + q > n - k.$$

6.11 Corollary. *If in addition f is an SP-morphism, then*

$$H^p(X, \mathcal{L}) = 0 \qquad p > n - k.$$

This in turn implies that ω_X is not ample and that can be used to show that the fibres of $f: X \to S$ are isomorphic. Hence the fibres of g are birational.

§A. Appendix: The proof of (1.2.2)

The definition of $\mathbb{G}^p$ implies that there exists an exact sequence,

$$... \to R^{p+q}\Phi(\mathbb{F}^{p+1}) \xrightarrow{\alpha_{p,q}} R^{p+q}\Phi(\mathbb{F}^p) \xrightarrow{\beta_{p,q}} R^{p+q}\Phi(\mathbb{G}^p) \xrightarrow{\gamma_{p,q}} R^{p+q+1}\Phi(\mathbb{F}^{p+1}) \to ....$$

Therefore one has a morphism,

$$\beta_{p+1,q} \circ \gamma_{p,q} : R^{p+q}\Phi(\mathbb{G}^p) \to R^{p+q+1}\Phi(\mathbb{G}^{p+1}),$$

giving
$$d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q},$$
the first differential of E_r. Here $\gamma \circ \beta = 0$ shows that indeed $d^2 = 0$.

Next let
$$x \in \ker d_1^{p,q} \subseteq E_1^{p,q} = R^{p+q}\Phi(\mathbb{G}^p).$$
Since $\beta_{p+1,q}(\gamma_{p,q}(x)) = 0$, there exists a
$$y^1_{p,q}(x) \in R^{p+q+1}\Phi(\mathbb{F}^{p+2})$$
such that
$$\alpha_{p+1,q}(y^1_{p,q}(x)) = \gamma_{p,q}(x).$$
If
$$z^1_{p,q}(x) \in R^{p+q+1}\Phi(\mathbb{F}^{p+2})$$
is another element such that
$$\alpha_{p+1,q}(z^1_{p,q}(x)) = \gamma_{p,q}(x),$$
then there exists a
$$w \in R^{p+q}\Phi(\mathbb{G}^{p+1}) = E_1^{p+1,q-1}$$
such that
$$z^1_{p,q}(x) = y^1_{p,q}(x) + \gamma_{p+1,q-1}(w).$$
Thus
$$\beta_{p+2,q-1}(z^1_{p,q}(x)) = \beta_{p+2,q-1}(y^1_{p,q}(x)) + \underbrace{\beta_{p+2,q-1} \circ \gamma_{p+1,q-1}(w)}_{d_1^{p+1,q-1}(w)}.$$
Therefore
$$\beta_{p+2,q-1}(y^1_{p,q}(x)) + \operatorname{im} d_1^{p+1,q-1}$$
does not depend on the choice of $y^1_{p,q}(x)$. Fix such a $y^1_{p,q}(x)$ for each x. Then one has a well-defined natural morphism,
$$d_2^{p,q} : E_2^{p,q} = \ker d_1^{p,q} \big/ \operatorname{im} d_1^{p-1,q} \to E_2^{p+2,q-1} = \ker d_1^{p+2,q-1} \big/ \operatorname{im} d_1^{p+1,q-1}.$$
Now let
$$K_2^{p,q} = \{u \in \ker d_1^{p,q} \mid u + \operatorname{im} d_1^{p-1,q} \in \ker d_2^{p,q}\}$$
and let $x \in K_2^{p,q}$. Then there exists a
$$y^1_{p,q}(x) \in R^{p+q+1}\Phi(\mathbb{F}^{p+2})$$
such that
$$\begin{aligned} \alpha_{p+1,q}(y^1_{p,q}(x)) &= \gamma_{p,q}(x) \\ \beta_{p+2,q-1}(y^1_{p,q}(x)) &= \beta_{p+2,q-1} \circ \gamma_{p+1,q-1}(x') \end{aligned} \tag{A.1}$$
for some $x' \in R^{p+q}\Phi(\mathbb{G}^{p+1})$, and so
$$y^1_{p,q}(x) - \gamma_{p+1,q-1}(x') \in \ker \beta_{p+2,q-1} = \operatorname{im} \alpha_{p+2,q-1}.$$
Then there exists a
$$y^2_{p,q}(x) \in R^{p+q+1}\Phi(\mathbb{F}^{p+3})$$
such that
$$\alpha_{p+2,q-1}(y^2_{p,q}(x)) = y^1_{p,q}(x) - \gamma_{p+1,q-1}(x'). \tag{A.2}$$

Now

$$\beta_{p+3,q-2}(y^2_{p,q}(x)) \in \ker d_1^{p+3,q-2}$$

and the class,

$$\beta_{p+3,q-2}(y^2_{p,q}(x)) + \operatorname{im} d_1^{p+2,q-2} \in \ker d_1^{p+3,q-2} \Big/ \operatorname{im} d_1^{p+2,q-2} = E_2^{p+3,q-2}$$

does not depend on the choice of $y^2_{p,q}(x)$ as long as it satisfies $(A.2)$. Furthermore, since $\gamma \circ \beta = 0$,

$$\beta_{p+3,q-2}(y^2_{p,q}(x)) + \operatorname{im} d_1^{p+2,q-2} \in \ker d_2^{p+3,q-2}.$$

Thus one can define

$$d_3^{p,q} : E_3^{p,q} = \ker d_2^{p,q} \Big/ \operatorname{im} d_2^{p-2,q+1} \to E_3^{p+3,q-2} = \ker d_2^{p+3,q-2} \Big/ \operatorname{im} d_2^{p+1,q-1}$$

by the formula

$$d_3^{p,q}\left(x + \operatorname{im} d_1^{p-1,q} + \operatorname{im} d_2^{p-2,q+1}\right) = \beta_{p+3,q-2}(y^2_{p,q}(x)) + \operatorname{im} d_1^{p+2,q-2} + \operatorname{im} d_2^{p+1,q-1}.$$

Claim. $d_3^{p,q}$ is well-defined.

Proof. One needs to prove that $d_3^{p,q}$ does not depend on the choice of $y^1_{p,q}(x)$ and x'. Let $z^1_{p,q}(x) \in R^{p+q+1}\Phi(\mathbb{F}^{p+2})$ and $z^2_{p,q}(x) \in R^{p+q+1}\Phi(\mathbb{F}^{p+3})$ be such that

$$\begin{aligned} \alpha_{p+1,q}(z^1_{p,q}(x)) &= \gamma_{p,q}(x) \\ \beta_{p+2,q-1}(z^1_{p,q}(x)) &= \beta_{p+2,q-1} \circ \gamma_{p+1,q-1}(x'') \end{aligned} \tag{A.3}$$

for some $x'' \in R^{p+q}\Phi(\mathbb{G}^{p+1})$ and

$$\alpha_{p+2,q-1}(z^2_{p,q}(x)) = z^1_{p,q}(x) - \gamma_{p+1,q-1}(x'').$$

By $(A.1)$ and $(A.3)$ there exists an $x''' \in R^{p+q}\Phi(\mathbb{G}^{p+1})$ such that

$$z^1_{p,q}(x) - \gamma_{p+1,q-1}(x'') = y^1_{p,q}(x) - \gamma_{p+1,q-1}(x') + \gamma_{p+1,q-1}(x''').$$

Then

$$\gamma_{p+1,q-1}(x''') = \alpha_{p+2,q-1}(z^2_{p,q}(x) - y^2_{p,q}(x)),$$

hence by definition,

$$d_2^{p+1,q-1}(x''' + \operatorname{im} d_1^{p,q-1}) = \beta_{p+3,q-2}(z^2_{p,q}(x) - y^2_{p,q}(x)) + \operatorname{im} d_1^{p+2,q-2}.$$

In other words

$$\beta_{p+3,q-2}(z^2_{p,q}(x) - y^2_{p,q}(x)) + \operatorname{im} d_1^{p+2,q-2} \in \operatorname{im} d_2^{p+1,q-1}$$

which means that $d_3^{p,q}$ is indeed well-defined. □

In the general case one has to work in a similar way. Assume that for a fixed $r \in \mathbb{N}$ and for all p, q and $1 \le i \le r$,

$$d_i^{p,q} : E_i^{p,q} \to E_i^{p+i,q-i+1}$$

is defined and

$$E_i^{p,q} \simeq \ker d_{i-1}^{p,q} \Big/ \operatorname{im} d_{i-1}^{p-i+1,q+i-2}.$$

Let $K_0^{p,q} = E_1^{p,q}$ and

$$K_i^{p,q} = \{x \in E_1^{p,q} \mid x \in \ker d_1, x + \operatorname{im} d_1 \in \ker d_2, x + \operatorname{im} d_1 + \operatorname{im} d_2 \in \ker d_3, \dots$$
$$\dots, x + \operatorname{im} d_1 + \operatorname{im} d_2 + \dots + \operatorname{im} d_{i-1} \in \ker d_i\}.$$

Then $K_{i-1}^{p,q}$ maps surjectively onto $E_i^{p,q}$. Let the kernel of this surjection be $I_{i-1}^{p,q}$, i.e.,

$$E_i^{p,q} \simeq K_{i-1}^{p,q} \Big/ I_{i-1}^{p,q}.$$

Then $I_{i-1}^{p,q}$ is the preimage of $\operatorname{im} d_{i-1}^{p-i+1,q+i-2}$ in $K_{i-1}^{p,q}$. Let

$$\alpha_{p,q}^i = \alpha_{p+1,q} \circ \alpha_{p+2,q-1} \circ \dots \circ \alpha_{p+i,q-i+1}.$$

Further assume that there exist (set-theoretic) functions,

$$y_{p,q}^{i-1} : K_{i-1}^{p,q} \to R^{p+q+1}\Phi(\mathbb{F}^{p+i})$$

such that

$$\alpha_{p,q}^{i-1}(y_{p,q}^{i-1}(x)) = \gamma_{p,q}(x) \in R^{p+q+1}\Phi(\mathbb{F}^{p+1}). \tag{A.4}$$

Further notice, that

$$d_i^{p,q} : E_i^{p,q} \to E_i^{p+i,q-i+1}$$

is given by

$$d_i^{p,q}\left(x + I_{i-1}^{p,q}\right) = \beta_{p+i,q-i+1}(y_{p,q}^{i-1}(x)) + I_{i-1}^{p+i,q-i+1} \tag{A.5}$$

and $d_i^{p,q}$ does not depend on the choice of $y_{p,q}^j(x)$ as long as it satisfies $(A.4)$.

Note also that $y_{p,q}^0(x) = \gamma_{p,q}(x)$. Let $I_r^{p,q}$ be the preimage of $\operatorname{im} d_r^{p-r,q+r-1}$ in $K_r^{p,q}$ and let

$$E_{r+1}^{p,q} \simeq K_r^{p,q} \big/ I_r^{p,q}.$$

Suppose now that $x \in K_r^{p,q}$, i.e., $x + I_{r-1}^{p,q} \in \ker d_r^{p,q}$ or equivalently

$$\beta_{p+r,q-r+1}(y_{p,q}^{r-1}(x)) \in I_{r-1}^{p+r,q-r+1}.$$

By $(A.5)$ there exists an $x_j \in K_j^{p+r-j-1,q-r+j+1}$ for $j = 0, \dots, r-2$ such that

$$\beta_{p+r,q-r+1}\Big(y_{p,q}^{r-1}(x) - \sum_{j=0}^{r-2} y_{p+r-j-1,q-r+j+1}^j(x_j)\Big) = 0.$$

Therefore there exists a

$$y_{p,q}^r(x) \in R^{p+q+1}\Phi(\mathbb{F}^{p+r+1})$$

such that

$$\alpha_{p+r,q-r+1}(y_{p,q}^r(x)) = y_{p,q}^{r-1}(x) - \sum_{j=0}^{r-2} y_{p+r-j-1,q-r+j+1}^j(x_j).$$

By $(A.4)$, and since $\alpha \circ \gamma = 0$,

$$\begin{aligned}
\alpha_{p,q}^r(y_{p,q}^r(x)) &= \alpha_{p,q}^{r-1} \circ \alpha_{p+r,q-r+1}(y_{p,q}^r(x)) \\
&= \alpha_{p,q}^{r-1}\Big(y_{p,q}^{r-1}(x) - \sum_{j=0}^{r-2} y_{p+r-j-1,q-r+j+1}^j(x_j)\Big) \qquad \text{(A.6)} \\
&= \alpha_{p,q}^{r-1}(y_{p,q}^{r-1}(x)) = \gamma_{p,q}(x).
\end{aligned}$$

Now one can define

$$d_{r+1}^{p,q} : E_{r+1}^{p,q} = K_r^{p,q} \big/ I_r^{p,q} \to E_{r+1}^{p+r+1,q-r} = K_r^{p+r+1,q-r} \Big/ I_r^{p+r+1,q-r}$$

by

$$d_{r+1}^{p,q}\,(x + I_r^{p,q}) = \beta_{p+r+1,q-r}(y_{p,q}^r(x)) + I_r^{p+r+1,q-r}.$$

Claim. $d_{r+1}^{p,q}$ is well-defined.

Proof. One needs to prove that any set of functions $z_{p,q}^i$ for p, q and $0 \le i \le r$ satisfying $(A.4)$, $(A.5)$ and $(A.6)$ gives the same $d_{r+1}^{p,q}$. This is true for $d_i^{p,q}$, $1 \le i \le r$, by induction.

Since

$$\alpha_{p+1,q} \circ \alpha_{p+1,q-1}^{i-1}(z_{p,q}^i(x)) = \alpha_{p,q}^i(z_{p,q}^i(x)) = \gamma_{p,q}(x),$$

one finds that

$$\alpha_{p+1,q-1}^i(z_{p,q}^i(x)) = z_{p,q}^1(x) + \gamma_{p+1,q-1}(x_i)$$

for some $x_i \in E_1^{p+1,q-1}$ for all $i \ge 2$. Now replace $z_{p,q}^i(x)$ by $z_{p,q}^i(x) - z_{p+1,q-1}^{i-1}(x_i)$ for $i \ge 2$. Then $\alpha_{p,q}^i(z_{p,q}^i(x)) = \gamma_{p,q}(x)$ remains true and $\beta_{p+i+1,q-i}(z_{p,q}^i(x))$ is only changed by an element of $I_i^{p+i+1,q-i}$. Hence one may assume that

$$\alpha_{p+1,q-1}^i(z_{p,q}^i(x)) = z_{p,q}^1(x).$$

Repeating this argument for $j = 2, \ldots, r-1$, using $\alpha_{p+j,q-j}^{i-j}$ in place of $\alpha_{p+1,q-1}^{i-1}$, one may assume that

$$\alpha_{p+j,q-j}^{i-j}(z_{p,q}^i(x)) = z_{p,q}^j(x) \quad \text{for all } i > j.$$

One may also assume the same for the y's, i.e., for all p, q and $0 \le j \le r-1$,

$$\alpha_{p+j,q-j}^{i-j}(y_{p,q}^i(x)) = y_{p,q}^j(x) \quad \text{for all } i > j. \tag{A.7}$$

Henceforth it will be assumed that any set of y's satisfies $(A.7)$. Observe that

$$\alpha_{p+1,q}(z_{p,q}^1(x)) = \gamma_{p,q}(x) = \alpha_{p+1,q}(y_{p,q}^1(x)),$$

so there exists an $x' \in E_1^{p+1,q-1}$ such that

$$z_{p,q}^1(x) = y_{p,q}^1(x) + \gamma_{p+1,q-1}(x').$$

Then replace $y_{p,q}^i(x)$ by $y_{p,q}^i(x) + y_{p+1,q-1}^{i-1}(x')$ for $i \ge 1$. This way

$$\alpha_{p,q}^i(y_{p,q}^i(x)) = \gamma_{p,q}(x)$$

and $(A.7)$ remain true and $\beta_{p+i+1,q-i}(y_{p,q}^i(x))$ is only changed by an element of $I_i^{p+i+1,q-i}$. Hence one may assume that

$$z_{p,q}^1(x) = y_{p,q}^1(x).$$

Next observe that

$$\alpha_{p+2,q-1}(z_{p,q}^2(x)) = z_{p,q}^1(x) = y_{p,q}^1(x) = \alpha_{p+2,q-1}(y_{p,q}^2(x)),$$

hence there exists an $x'' \in E_1^{p+2,q-2}$ such that

$$z_{p,q}^2(x) = y_{p,q}^2(x) + \gamma_{p+2,q-2}(x'').$$

Now replace $y^i_{p,q}(x)$ by $y^i_{p,q}(x) + y^{i-2}_{p+2,q-2}(x'')$ for $i \geq 2$. Again

$$\alpha^i_{p,q}(y^i_{p,q}(x)) = \gamma_{p,q}(x)$$

and $(A.7)$ remain true and $\beta_{p+i+1,q-i}(y^i_{p,q}(x))$ is only changed by an element of $I_i^{p+i+1,q-i}$. Hence one can assume that

$$z^2_{p,q}(x) = y^2_{p,q}(x)$$

also.

Iterating this procedure one finds that $d^{p,q}_{r+1}$ is the same defined either via the y's or the z's. □

Now let $z \in R^{p+q}\Phi(\mathbb{F}^p)$. Then $\beta_{p,q}(z) \in K_i^{p,q}$ for all i, since $\gamma \circ \beta = 0$. On the other hand if $x \in K^{p,q}_{k-p}$, then $y^{k-p}_{p,q}(x) = 0$, since $\mathbb{F}^{k+1} = 0$. Then

$$\gamma_{p,q}(x) = \alpha^{k-p}_{p,q}(y^{k-p}_{p,q}(x)) = 0,$$

so there exists a $z \in R^{p+q}\Phi(\mathbb{F}^p)$ such that $x = \beta_{p,q}(z)$.

Therefore $\beta_{p,q}$ induces a surjective morphism

$$\bar{\beta}_{p,q} : R^{p+q}\Phi(\mathbb{F}^p) \twoheadrightarrow E^{p,q}_\infty.$$

Let

$$F^pR^{p+q} = \alpha^{p-l}_{l-1,p+q-l}(R^{p+q}\Phi(\mathbb{F}^p)) \subseteq R^{p+q}\Phi(\mathbb{F}^l).$$

From the construction of the spectral sequence it is clear that $\bar{\beta}_{p,q}(x) = 0$ for any $x \in R^{p+q}\Phi(\mathbb{F}^p)$ such that $\alpha^{p-l}_{l-1,p+q-l}(x) = 0$, hence $\bar{\beta}_{p,q}$ factors through F^pR^{p+q} inducing a surjective morphism

$$\hat{\beta}_{p,q} : F^pR^{p+q} \twoheadrightarrow E^{p,q}_\infty.$$

Now $\ker \beta_{p,q} = \operatorname{im} \alpha_{p,q}$ implies that $\ker \hat{\beta}_{p,q} = F^{p+1}R^{p+q}$. Hence for any m

$$R^m\Phi(\mathbb{F}^l) = F^lR^m \supseteq F^{l+1}R^m \supseteq \cdots \supseteq F^kR^m \supseteq F^{k+1}R^m = 0$$

is a filtration of $R^m\Phi(K) = R^m\Phi(\mathbb{F}^l)$ such that

$$E^{p,m-p}_\infty \simeq F^pR^m \big/ F^{p+1}R^m.$$

This completes the proof of (1.2.2). □

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