SMOOTH FAMILIES OVER RATIONAL AND ELLIPTIC CURVES

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The aim of this article is to present a generalization of the following result of L. Migliorini to arbitrary dimensions.

Theorem. [Migliorini95] *A smooth family of minimal surfaces of general type over a projective curve of genus at most one is a locally trivial fiber bundle.*

Because of the absence of minimal models among smooth varieties in higher dimensions, it is not entirely clear what the proper generalization ought to be. One may replace the condition "minimal" by the property that the canonical bundle satisfies certain positivity properties and then prove that the fibers are birational. (See the end of the introduction for some definitions.)

Theorem 1. *Let $g : Y \to C$ be a smooth family of projective varieties of general type with nef canonical bundle and C a smooth projective curve of genus at most one. Then the fibers of g are birational.*

Since the minimal model of a surface of general type is unique, Migliorini's Theorem follows from Theorem 1.

One might try to generalize Migliorini's Theorem replacing "smooth family of minimal surfaces" by the property that the fibers are minimal varieties. However, the corresponding statement is not true as soon as the dimension of the fibers is three or more. Rational double points give terminal singularities in dimension 3 or higher, so a generic Lefschetz pencil with non-birational fibers provides a counterexample.

In dimension three, [Kollár-Mori92, 12.7.3] implies a stronger statement:

Corollary 2. *Let $g : Y \to C$ be a smooth family of projective threefolds of general type with nef canonical bundle and C a smooth projective curve of genus at most one. Then g is locally trivial.*

An important step in the proof is a higher dimensional Arakelov type result.

Theorem 3. *Let $f : X \to C$ be a proper morphism between complex projective varieties such that C is a smooth curve and X has only canonical singularities. Assume that the restriction of $K_{X/C}$ to every fiber is ample. Then either $K_{X/C}$ is ample or the fibers of f are isomorphic over an open set.*

Remark. In fact a little more is necessary than this statement. The precise form can be found in (2.16).

Having this statement, the main line of the proof of Theorem 1 can be illustrated nicely in the special case when the fibers have an ample canonical bundle.

Let $g : Y \to C$ be a smooth family of projective varieties with ample canonical bundle over a rational or elliptic curve. Suppose $\omega_{Y/C}$ is ample. Let $\mathcal{L}_p$ denote the ample line bundle $\omega_Y \otimes g^*\omega_C^{-p}$ for $p > 0$ and consider the following exact sequence:

$$0 \to \Omega^{p-1}_{Y/C} \otimes \mathcal{L}_{p-1} \to \Omega^p_Y \otimes \mathcal{L}_p \to \Omega^p_{Y/C} \otimes \mathcal{L}_p \to 0.$$

By the Kodaira-Akizuki-Nakano Vanishing Theorem $H^{n-(p-1)}(Y, \Omega^p_Y \otimes \mathcal{L}_p) = 0$, so

$$H^{n-p}(Y, \Omega^p_{Y/C} \otimes \mathcal{L}_p) \to H^{n-(p-1)}(Y, \Omega^{p-1}_{Y/C} \otimes \mathcal{L}_{p-1})$$

is surjective and then the composite of all these maps

$$H^1(Y, \omega_{Y/C} \otimes \mathcal{L}_{n-1}) \to H^n(Y, \omega_Y)$$

is surjective as well. This gives a contradiction since the former group is zero by the Kodaira Vanishing Theorem while the latter is not. Therefore the fibers of g are isomorphic by (2.16).

Now for a family of surfaces one can prove Migliorini's Theorem in a similar way using a refinement of the Kodaira-Akizuki-Nakano Vanishing Theorem [Migliorini95].

For higher dimensional families the possible general vanishing results do not seem to be strong enough to carry the proof through, so this article follows an alternate way. Instead of trying to require less from the line bundle, using (2.16) one can pass to the family of the canonical models where ampleness is provided, but have to allow the spaces to be singular. The inconvenience of doing so is that it becomes necessary to work in derived categories of complexes of sheaves. However, this approach has the advantage that all the necessary ingredients exist in this setting, namely [DuBois81] gave a general definition of the filtered De Rham complex for singular spaces and the Kodaira-Akizuki-Nakano Vanishing Theorem admits a good generalization as well (cf. [GNPP88, V.5.1], [Steenbrink85]).

The purpose of §1 is to define a complex of sheaves for a complex variety dominating a smooth curve such that this new complex admits a relationship to the generalized De Rham complex, similar to the one between the sheaf of relative differentials and the sheaf of differentials in the smooth case. In §2 property SP – a very technical one – is defined and the following theorem is proved:

Theorem 4. *A family of Gorenstein canonical varieties with property SP over a rational or an elliptic curve has isomorphic fibers.*

Note that this statement is not true without property SP, and also that any smooth morphism has this property trivially. This in turn will easily imply Theorem 1 as shown in §3.

Finally, as another application of the same principles, a bound on the minimal number of singular fibers for non-smooth families of varieties of general type of even dimension admitting at most double points is given in (3.3).

Definitions and Notation. Throughout the article the groundfield will always be $\mathbb{C}$, the field of complex numbers. A *complex scheme* will mean a separated scheme of finite type over $\mathbb{C}$.

A divisor D on a scheme X is called $\mathbb{Q}$-Cartier if mD is Cartier for some $m > 0$. It is called *ample* if mD is ample. A $\mathbb{Q}$-Cartier divisor D is called *nef* if $D.C \geq 0$ for every proper curve $C \subset X$. D is called *big* if X is proper and $|\, mD\,|$ gives a birational map for some $m > 0$. In particular ample implies nef and big.

A locally free sheaf $\mathcal{E}$ on a scheme X is called *semipositive* if for every smooth complete curve C and every map $\gamma : C \to X$, any quotient bundle of $\gamma^*\mathcal{E}$ has nonnegative degree. $S^k(\mathcal{E})$ denotes the kth symmetric power of $\mathcal{E}$.

A normal variety X is said to have *canonical* (resp. *terminal*) *singularities* if K_X is $\mathbb{Q}$-Cartier and for any resolution of singularities $\pi : \tilde{X} \to X$, with the collection of exceptional prime divisors $\{E_i\}$, there exist $a_i \in \mathbb{Q}$, $a_i \geq 0$ (resp. $a_i > 0$) such that $K_{\tilde{X}} = \pi^* K_X + \sum a_i E_i$ (cf. [CKM88]). X is called a *canonical variety* if it has only canonical singularities and K_X is ample. X is called a *minimal variety* if it has only terminal singularities and K_X is nef.

A singularity is called *Gorenstein* if its local ring is a Gorenstein ring. A variety is *Gorenstein* if it admits only Gorenstein singularities. In particular, the dualizing sheaf of a Gorenstein variety is locally free (cf. [Bruno-Herzog93, §3]).

Let $g : Y \to C$ be a morphism of normal varieties, then $K_{Y/C} = K_Y - g^*K_C$, similarly if Y and C are smooth, then $\omega_{Y/C} = \omega_Y \otimes g^*\omega_C^{-1}$.

Acknowledgements. The present article is a part of my Ph.D. thesis. I am very grateful to my advisor, János Kollár, for his guidance and support. He called my attention to this problem and helped me in so many ways during my work that it would be hard to list all of them. I would also like to thank him for his infinite patience.

Prof. Fabrizio Catanese pointed out an incorrect argumentation in an earlier version of the article. I would like to thank him and the referee for helpful comments.

1. Relative De Rham complexes

Let X be a complex scheme of dimension n. $D_{filt}(X)$ denotes the derived category of filtered complexes of $\mathcal{O}_X$-modules with differentials of order ≤ 1 and $D_{filt,coh}(X)$ the subcategory of $D_{filt}(X)$ of complexes $K^{\cdot}$, such that for all i, the cohomology sheaves of $Gr^i_F K^{\cdot}$ are coherent (cf. [DuBois81], [GNPP88]). $D(X)$ and $D_{coh}(X)$ denotes the derived categories with the same definition except that the complexes are not assumed to be filtered. The superscripts $+, -, b$ carry the usual meaning (bounded below, bounded above and bounded).

$C(X)$ is the category of complexes of $\mathcal{O}_X$-modules with differentials of order ≤ 1 and for $u \in \mathcal{M}or(C(X))$, $M(u) \in \mathcal{O}bj(C(X))$ denotes the mapping cone of u (cf.

[Hartshorne66]). The isomorphism in these categories will be denoted by $\simeq_{qis}$. For the definition of a hyperresolution the reader is referred to [DuBois81], [GNPP88] or [Steenbrink85].

1.1 General Čech spectral sequence. [GNPP88] Let X be a complex scheme and $\mathcal{U} = \{\nu_i : U_i \hookrightarrow X\}_{i=0}^r$ a finite open cover of X. For $J \subset I \subset \{0, \dots, r\}$, let $U_I = \cap_{i \in I} U_i$, $\nu_I : U_I \hookrightarrow X$ and $\nu_{I,J} : U_I \hookrightarrow U_J$. Assume that $\mathcal{F}^{\cdot}_{U_I} \in \mathcal{O}bj(C(U_I))$ and compatible restriction maps $\nu^*_{I,J} : (\nu_J)_* \mathcal{F}^{\cdot}_J \to (\nu_I)_* \mathcal{F}^{\cdot}_I$ are given for all I, J. Let

$$C^p(\mathcal{U}, \mathcal{F}^{\cdot}_{U.})^{\cdot} = \underset{i_0 < \dots < i_p}{\oplus} (\nu_{\{i_0, \dots, i_p\}})_* \mathcal{F}^{\cdot}_{\{i_0, \dots, i_p\}}$$

$$\delta^p = \sum_{i_0 < \dots < i_p} \sum_{k=0}^{p} (-1)^k \nu^*_{\{i_0, \dots, i_p\}, \{i_0, \dots, i_p\} \setminus \{i_k\}} : C^p(\mathcal{U}, \mathcal{F}^{\cdot}_{U.})^{\cdot} \to C^{p+1}(\mathcal{U}, \mathcal{F}^{\cdot}_{U.})^{\cdot}.$$

By construction $\delta^{p+1} \circ \delta^p = 0$, so one can define

$$\check{C}(\mathcal{U}, \mathcal{F}^{\cdot}_{U.})^m = \underset{q \geq 0}{\oplus} C^q(\mathcal{U}, \mathcal{F}^{\cdot}_{U.})^{m-q}$$
$$\check{d} = \sum_{q \geq 0} (-1)^q (d_{C^q(\mathcal{U}, \mathcal{F}^{\cdot}_{U.})} - \delta^q).$$

$\check{C}$ is a left exact functor from the product category of $C(U_I)$ for $I \subset \{0, \dots, r\}$ to $C(X)$. Its right derived functor is denoted by $R\nu_{\cdot *} \mathcal{F}^{\cdot}_{U.}$.

Using the usual Čech resolution for sheaves one sees easily that for an $\mathcal{F}^{\cdot} \in C(X)$, $\mathcal{F}^{\cdot} \simeq_{qis} \check{C}(\mathcal{U}, \mathcal{F}^{\cdot}_{U.})^{\cdot}$, where $\mathcal{F}^{\cdot}_{U_I}$ is the restriction of $\mathcal{F}^{\cdot}$ to U_I.

The following theorem states the existence and some properties of the filtered De Rham complex for singular spaces. Note that it holds in both the algebraic and the analytic case.

1.2 Theorem. [DuBois81], [GNPP88, III.1.12, V.3.6, V.5.1] *For every complex scheme X of dimension n there exists an $\underline{\Omega}^{\cdot}_X \in \mathcal{O}bj(D_{filt}(X))$ with the following properties.*
(1.2.1) *It is functorial, i.e. if $\phi : Y \to X$ is a morphism of complex schemes, then there exists a natural map ϕ^* of filtered complexes*

$$\phi^* : \underline{\Omega}^{\cdot}_X \to R\phi_* \underline{\Omega}^{\cdot}_Y.$$

Furthermore $\underline{\Omega}^{\cdot}_X \in \mathcal{O}bj(D^b_{filt,coh}(X))$ and if ϕ is proper, then ϕ^* is a morphism in $D^b_{filt,coh}(X)$.
(1.2.2) *Let $\Omega^{\cdot}_X$ be the usual De Rham complex of Kähler differentials considered with the "filtration bête". Then there exists a natural map of filtered complexes*

$$\Omega^{\cdot}_X \to \underline{\Omega}^{\cdot}_X$$

and if X is smooth, it is a quasi-isomorphism.
(1.2.3) *Let $\{\nu_i : U_i \hookrightarrow X\}$ be a finite open cover of X. Then*

$$\underline{\Omega}^{\cdot}_X \simeq_{qis} R\nu_{\cdot *}\underline{\Omega}^{\cdot}_{U_\cdot}.$$

(1.2.4) *Let $\underline{\Omega}^p_X = Gr^p_F\,\underline{\Omega}^{\cdot}_X\,[p]$. If X is projective and $\mathcal{L}$ is an ample line bundle on X, then*

$$\mathbb{H}^q(X, \underline{\Omega}^p_X \otimes \mathcal{L}) = 0 \qquad \text{for } p+q > n.$$

If Y is a smooth complex variety, let $\mathcal{A}^{p,q}_Y$ denote the sheaf of complex valued C^∞ forms of type (p,q).

Let X be a complex variety and $\varepsilon_\cdot : X_\cdot \to X$ a hyperresolution of X (cf. [DuBois81], [GNPP88], [Steenbrink85]). Then

$$K^{\cdot}_p = \oplus \varepsilon_{i*}\mathcal{A}^{p,\cdot}_{X_i}[-i] \in \mathcal{O}bj(C(X))$$

is an incarnation of $\underline{\Omega}^p_X$. (The differential of $K^{\cdot}_p$ is cooked up from the ordinary differentiation of C^∞ forms and from pull-backs between the pieces of the hyperresolution. The construction is similar to that of $R\nu_{\cdot *}\mathcal{F}^{\cdot}_{U_\cdot}$ in (1.1). For a more detailed discussion see [Steenbrink85].)

Let $f : X \to C$ be a dominant morphism such that C is a smooth complex curve and for each natural number p construct a complex which will play the role of $\Omega^p_{X/C}$ in the singular case. First define a morphism of complexes induced by the wedge product of differential forms as follows. Let

$$\wedge_p : K^{\cdot}_p \otimes f^*\Omega^1_C \to K^{\cdot}_{p+1}$$

be the map generated by

$$\varepsilon_{i*}\eta_i \otimes \xi \mapsto \varepsilon_{i*}(\eta_i \wedge \varepsilon_i^*\xi).$$

Using the explicit form of $K^{\cdot}_p$ given above, one can see easily that $\wedge_p$ is in fact a morphism of complexes.

Let $\wedge'_p = \wedge_p \otimes id_{f^*\Omega^1_C}$. Since $f^*\Omega^1_C$ is a line bundle, $\wedge_p \circ \wedge'_{p-1} = 0$.

Let $M^{\cdot}_r = 0 \in \mathcal{O}bj(C(X))$, $w''_r = 0 \in \mathrm{Hom}_{C(X)}(K^{\cdot}_r \otimes f^*\Omega^1_C, M^{\cdot}_r \otimes f^*\Omega^1_C)$ and $w'_r = 0 \in \mathrm{Hom}_{C(X)}(M^{\cdot}_r \otimes f^*\Omega^1_C, K^{\cdot}_{r+1})$ for $r \geq n$. Assume that $p < n$ and for every $q > p$, $M^{\cdot}_q \in \mathcal{O}bj(C(X))$ is defined and there are given morphisms of complexes

$$w''_q : K^{\cdot}_q \otimes f^*\Omega^1_C \to M^{\cdot}_q \otimes f^*\Omega^1_C \qquad \text{and} \qquad w'_q : M^{\cdot}_q \otimes f^*\Omega^1_C \to K^{\cdot}_{q+1}$$

such that

$$\wedge_q = w'_q \circ w''_q \qquad \text{and} \qquad w''_q \circ \wedge'_{q-1} = 0.$$

Let

$$w_q = w''_q \otimes id_{(f^*\Omega^1_C)^{-1}} \in \operatorname{Hom}_{C(X)}(K^\cdot_r, M^\cdot_r)$$

and

$$M^\cdot_p = M(w_{p+1})[-1] \otimes (f^*\Omega^1_C)^{-1} \in \mathcal{O}bj(C(X)),$$

i.e.,

$$M^m_p \otimes f^*\Omega^1_C = K^m_{p+1} \oplus M^{m-1}_{p+1}$$

and

$$d_{M^m_p \otimes f^*\Omega^1_C} = \begin{pmatrix} d^m_{K_{p+1}} & 0 \\ -w^m_{p+1} & -d^{m-1}_{M_{p+1}} \end{pmatrix}.$$

Also let

$$w''_p = \begin{pmatrix} \wedge_p \\ 0 \end{pmatrix} : K^\cdot_p \otimes f^*\Omega^1_C \to M^\cdot_p \otimes f^*\Omega^1_C$$

and

$$w'_p = (id_{K^\cdot_{p+1}}, 0) : M^\cdot_p \otimes f^*\Omega^1_C \to K^\cdot_{p+1}.$$

w'_p is a morphism of complexes by the definition of the mapping cone and w''_p is a morphism of complexes because $w_{p+1} \circ \wedge_p = 0$. It is also obvious that $\wedge_p = w'_p \circ w''_p$ and $w''_p \circ \wedge'_{p-1} = 0$.

$\wedge_p$ is natural, i.e., if α is a morphism of hyperresolutions,

$$\begin{array}{ccc} \tilde{X}. & \xrightarrow{\alpha} & X. \\ {\scriptstyle \tilde{\varepsilon}.}\downarrow & & \downarrow{\scriptstyle \varepsilon.} \\ X & \xrightarrow[id_X]{} & X \end{array}$$

then it induces a morphism of complexes: $\alpha^* : K^\cdot_p \to \tilde{K}^\cdot_p$ and by the definition of $\wedge_p$ the following diagram is commutative:

$$\begin{array}{ccc} K^\cdot_p \otimes f^*\Omega^1_C & \xrightarrow{\wedge_p} & K^\cdot_{p+1} \\ {\scriptstyle \alpha^*}\downarrow & & \downarrow{\scriptstyle \alpha^*} \\ \tilde{K}^\cdot_p \otimes f^*\Omega^1_C & \xrightarrow[\tilde{\wedge}_p]{} & \tilde{K}^\cdot_{p+1} \end{array}$$

Now α^* is a quasi-isomorphism, so $\wedge_p$ and $\tilde{\wedge}_p$ are equivalent in $D(X)$. Then by [DuBois81, 2.1.4] or [GNPP88, I.3.10], the equivalence class of $\wedge_p$ in $D(X)$ is independent of the hyperresolution chosen.

Therefore there exists a natural map in $D(X)$ induced by the usual wedge product of differential forms:

$$\underline{\Omega}^p_X \otimes f^*\Omega^1_C \xrightarrow{\wedge_p} \underline{\Omega}^{p+1}_X$$

Also, by their definition, the equivalence classes of w_p, w'_p and w''_p in $D(X)$ are independent of the hyperresolution chosen. From now on these symbols will denote their equivalence classes in $D(X)$. A map will mean an element of $\mathcal{M}or(D(X))$, so it is possibly not represented by an actual morphism of complexes between two arbitrary representatives of the respective objects.

1.3 Theorem-Definition. *Let $f : X \to C$ be a dominant morphism between complex varieties such that* $\dim X = n$ *and C is a smooth curve. For every nonnegative integer p there exists a complex $\underline{\Omega}^p_{X/C} \in \mathcal{O}bj(D(X))$ with the following properties.*
(1.3.1) *The natural map $\wedge_p$ factors through $\underline{\Omega}^p_{X/C} \otimes f^*\Omega^1_C$, i.e., there exist maps:*

$$w''_p : \underline{\Omega}^p_X \otimes f^*\Omega^1_C \to \underline{\Omega}^p_{X/C} \otimes f^*\Omega^1_C \qquad \textit{and}$$
$$w'_p : \underline{\Omega}^p_{X/C} \otimes f^*\Omega^1_C \to \underline{\Omega}^{p+1}_X$$

such that $\wedge_p = w'_p \circ w''_p$.
(1.3.2) *If $w_p = w''_p \otimes id_{(f^*\Omega^1_C)^{-1}} : \underline{\Omega}^p_X \to \underline{\Omega}^p_{X/C}$, then*

$$\underline{\Omega}^p_{X/C} \otimes f^*\Omega^1_C \xrightarrow{w'_p} \underline{\Omega}^{p+1}_X \xrightarrow{w_{p+1}} \underline{\Omega}^{p+1}_{X/C} \xrightarrow{+1}$$

is a distinguished triangle in $D(X)$.
(1.3.3) *w_p is functorial, i.e., if $\phi : Y \to X$ is a C-morphism, then there are natural maps in $D(X)$ forming a commutative diagram:*

$$\begin{array}{ccc} \underline{\Omega}^p_X & \longrightarrow & \underline{\Omega}^p_{X/C} \\ \downarrow & & \downarrow \\ R\phi_*\underline{\Omega}^p_Y & \longrightarrow & R\phi_*\underline{\Omega}^p_{Y/C} \end{array}$$

(1.3.4) *If f is smooth, then $\underline{\Omega}^p_{X/C} \simeq_{qis} \Omega^p_{X/C} = \bigwedge^p \Omega_{X/C}$ where $\Omega_{X/C}$ is the sheaf of relative Kähler differentials.*
(1.3.5) *$\underline{\Omega}^r_{X/C} = 0$ for $r \geq n$ and if f is proper, then $\underline{\Omega}^p_{X/C} \in \mathcal{O}bj(D^b_{coh}(X))$ for every p.*
(1.3.6) *Let $\{\nu_i : U_i \hookrightarrow X\}$ be a finite open cover of X. Then*

$$\underline{\Omega}^p_{X/C} \simeq_{qis} R\nu_{\cdot *}\underline{\Omega}^{\cdot}_{U_\cdot/C}$$

for every p.

Proof. Let $\underline{\Omega}^p_{X/C} \simeq_{qis} M^{\cdot}_p \in \mathcal{O}bj(D(X))$. Then (1.3.1), (1.3.2) and the first part of (1.3.5) follows. Using (1.3.2), the first part of (1.3.5) and descending induction on p, (1.3.3), (1.3.4), (1.3.6) and the rest of (1.3.5) follows from (1.2.1), (1.2.2) and (1.2.3). $\square$

2. Families with property SP

By (1.2.2) there exists a natural map $\rho : \mathcal{O}_X \to \underline{\Omega}^0_X$. This map composed with w_0 gives a natural map $\mathcal{O}_X \to \underline{\Omega}^0_{X/C}$ and it is functorial in the sense of (1.3.3).

2.1 Definition. Let $f : X \to C$ be a dominant morphism between complex varieties such that C is a smooth curve. f will be said to have *property SP* if the natural map $\rho : \mathcal{O}_X \to \underline{\Omega}^0_{X/C}$ has a left inverse, i.e., there exists a map in $D(X)$, $\tilde{\rho} : \underline{\Omega}^0_{X/C} \to \mathcal{O}_X$ such that $\tilde{\rho} \circ \rho : \mathcal{O}_X \to \mathcal{O}_X$ is a quasi-isomorphism. In particular every smooth morphism has property SP.

2.2 Remark. If $f : X \to C$ has property SP and X is projective, then it has only Du Bois singularities, i.e., $\mathcal{O}_X \simeq_{qis} \underline{\Omega}^0_X$ (cf. [Kollár93, §12]).

2.3 Proposition. *SP is a local property in the following sense: Let $f : X \to C$ be a dominant morphism between complex varieties such that C is a smooth curve and $\{\nu_i : U_i \hookrightarrow X\}$ a finite open cover of X in the complex topology such that $f : U_i \to C$ has property SP for all i. Then $f : X \to C$ has property SP as well.*

Proof. By (1.1), (1.3.6) and the assumption

$$\mathcal{O}_X \simeq_{qis} R\nu_{\cdot *}\mathcal{O}_{U_\cdot} \to R\nu_{\cdot *}\underline{\Omega}^0_{U_\cdot/C} \simeq_{qis} \underline{\Omega}^0_{X/C}$$

has a left inverse. □

2.4 Proposition. *Let $f : X \to C$ be a dominant morphism between complex varieties such that C is a smooth curve, $\phi : Y \to X$ a morphism such that $f \circ \phi$ has property SP and $\mathcal{O}_X \to R\phi_*\mathcal{O}_Y$ has a left inverse. Then f has property SP.*

Proof. By functoriality there is a commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_X & \longrightarrow & \underline{\Omega}^0_{X/C} \\ \big\downarrow & & \big\downarrow \\ R\phi_*\mathcal{O}_Y & \longrightarrow & R\phi_*\underline{\Omega}^0_{Y/C} \end{array}$$

The bottom horizontal and the left vertical arrows have a left inverse by assumption, so the natural map $\mathcal{O}_X \to \underline{\Omega}^0_{X/C}$ has a left inverse as well. □

2.5 Corollary. *Assume that X has rational singularities and there exists a resolution of singularities of X, $\phi : Y \to X$, such that $f \circ \phi$ is smooth. Then f has property SP.* □

2.6 Example. Let

$$f_{n,r} : X_{n,r} = \operatorname{Spec} \mathbb{C}\,[\,t, x_1, \dots, x_n] \,/\, \big(t^r - (x_1^2 + \ldots + x_n^2)\big) \to \operatorname{Spec} \mathbb{C}\,[\,t\,]\,.$$

These families do or do not have property SP depending on n and r.

2.7 Example. Let $n = 2k+1$, $r = 2s$ and consider

$$f_{2k+1,2s} : X_{2k+1,2s} \to \operatorname{Spec} \mathbb{C}[\, t\,] .$$

Blowing up the zero locus of $(t^s + x_1,\, x_2 + \sqrt{-1}\, x_3, \dots , x_{2k} + \sqrt{-1}\, x_{2k+1})$ gives a resolution of $X_{2k+1,2s}$ that is smooth over C, thus $f_{2k+1,2s}$ has property SP by (2.5). These are examples for non-smooth morphisms having this property as well as for morphisms having property SP non-trivially, i.e., $\mathcal{O}_X \to \underline{\Omega}^0_{X/C}$ has a left inverse, but it is not a quasi-isomorphism. Note that this left inverse exists only in the derived category. If $\underline{\Omega}^0_{X/C}$ is represented by $M^{\cdot}_0$, as in its definition, then the map $\mathcal{O}_X \to M^{\cdot}_0$ does not have a left inverse in the category of complexes.

2.8 Example. Let $n = 2$ and $r \geq 1$ arbitrary. Then

$$f_{2,r} : X_{2,r} = \operatorname{Spec} \mathbb{C}[\, t, x_1, x_2] \,/\, \big(t^r - (x_1^2 + x_2^2)\big) \to \operatorname{Spec} \mathbb{C}[\, t\,]$$

does not have property SP. This can be seen as in (2.10).

2.9 Remark. It seems likely, that $f_{2k,r}$ in general does not have property SP. The way I can prove this for $n = 2$ does not work in higher dimensions (cf. (2.12)).

The fibers having only ordinary double points does not imply property SP as the following examples show.

2.10 Example. Let

$$Z = \operatorname{Spec} \mathbb{C}[\, t, x_1, x_2, \dots , x_n] \,/\, \big(t - (x_1^2 + x_2^2 + \dots + x_n^2)\big) ,$$
$$f_{n,1} : Z \to \operatorname{Spec} \mathbb{C}[\, t\,] = C.$$

Z is smooth, so $\underline{\Omega}^{\cdot}_Z \simeq_{qis} \Omega^{\cdot}_Z$ and then by definition

$$\underline{\Omega}^0_{Z/C} = \dots \to 0 \to \Omega^1_Z \xrightarrow{\wedge dt} \Omega^2_Z \xrightarrow{\wedge dt} \dots \xrightarrow{\wedge dt} \Omega^n_Z \to 0 \to \dots .$$

Let

$$e_{i_1,\dots ,i_{n-p}} = \left(\frac{1}{2}\right)^{n-p} (-1)^{\sum_{j=1}^{n-p} i_j + \frac{1}{2}p(p+1)}\, dx_{j_1} \wedge \dots \wedge dx_{j_p},$$

where $i_1 < \dots < i_{n-p}$, $j_1 < \dots < j_p$ and $\{i_1, \dots , i_{n-p}, j_1, \dots , j_p\} = \{1, 2, \dots , n\}$. Clearly,

$$\Omega^p_Z = \big\langle e_{i_1,\dots ,i_{n-p}} \mid i_1 < \dots < i_{n-p},\, i_r \in \{1, 2, \dots , n\}\big\rangle.$$

Furthermore, $dt = 2 \sum_{i=1}^n x_i dx_i$, so

$$e_{i_1,\dots ,i_{n-p}} \wedge dt = \sum_{j=1}^{n-p} (-1)^{j-1} x_j e_{i_1,\dots ,i_{j-1},i_{j+1},\dots ,i_{n-p}}.$$

Hence the complex

$$\mathcal{O}_Z \xrightarrow{\wedge dt} \Omega^1_Z \xrightarrow{\wedge dt} \Omega^2_Z \xrightarrow{\wedge dt} \dots \xrightarrow{\wedge dt} \Omega^{n-1}_Z \to \mathfrak{m}$$

is isomorphic to the Koszul complex of $\mathfrak{m}$, the maximal ideal of the origin P. Therefore $\mathcal{H}^0(\underline{\Omega}^0_{Z/C}) = \mathcal{O}_Z$, $\mathcal{H}^i(\underline{\Omega}^0_{Z/C}) = 0$ for $0 < i < n$ and $\mathcal{H}^n(\underline{\Omega}^0_{Z/C}) = \mathbb{C}_P$, where $\mathbb{C}_P$ denotes the skyscraper sheaf $\mathbb{C}$ at the point P. Then $f_{n,1}$ does not have property SP as follows from (2.11).

2.11 Example. Let Z be a smooth projective variety and $h : Z \to \mathbb{P}^1$ a proper morphism such that $\mathcal{H}^0(\underline{\Omega}^0_{Z/C}) = \mathcal{O}_Z$, $\mathcal{H}^i(\underline{\Omega}^0_{Z/C}) = 0$ for $0 < i < n$ and $\mathcal{H}^n(\underline{\Omega}^0_{Z/C}) = \sum_{i=1}^r \mathbb{C}_{P_i}$. For instance let h be not smooth and the special fibers have singularities isomorphic to the one in the previous example. Now let

$$\mathcal{K}_i = \operatorname{im}\left[\Omega^i_Z \otimes h^*\Omega^1_{\mathbb{P}^1} \xrightarrow{\wedge} \Omega^{i+1}_Z\right] \otimes (h^*\Omega^1_{\mathbb{P}^1})^{-1}.$$

Choose a line bundle $\mathcal{N}$ such that $H^i(Z, \Omega^p_Z \otimes (h^*\Omega^1_{\mathbb{P}^1})^{n-1-p} \otimes \mathcal{N}) = 0$ for all p and $i < n$ and consider the following short exact sequence:

$$0 \to \mathcal{K}_{n-1} \otimes \mathcal{N} \to \Omega^n_Z \otimes (h^*\Omega^1_{\mathbb{P}^1})^{-1} \otimes \mathcal{N} \to \sum_{i=1}^r \mathbb{C}_{P_i} \to 0.$$

Let $\omega_p = (h^*\Omega^1_{\mathbb{P}^1})^{n-1-p} \otimes \mathcal{N}$. By definition $\underline{\Omega}^{n-1}_{Z/C} \simeq_{qis} \Omega^n_Z \otimes (h^*\Omega^1_{\mathbb{P}^1})^{-1}$, so the previous short exact sequence and vanishing can be written as:

$$0 \to \mathcal{K}_{n-1} \otimes \omega_{n-1} \to \underline{\Omega}^{n-1}_{Z/C} \otimes \omega_{n-1} \to \sum_{i=1}^r \mathbb{C}_{P_i} \to 0$$

and

$$H^i(Z, \Omega^p_Z \otimes \omega_p) = 0 \qquad \text{for all } p \text{ and } i < n.$$

In particular $\mathbb{H}^0(Z, \underline{\Omega}^{n-1}_{Z/C} \otimes \omega_{n-1}) = H^0(Z, \Omega^n_Z \otimes \omega_n) = 0$, so

$$H^1(Z, \mathcal{K}_{n-1} \otimes \omega_{n-1}) \to \mathbb{H}^1(Z, \underline{\Omega}^{n-1}_{Z/C} \otimes \omega_{n-1})$$

is not injective.

Next consider the following commutative diagrams where the rows are distinguished triangles:

$$\begin{array}{ccccccc}
\mathcal{K}_p \otimes \omega_p & \longrightarrow & \Omega^{p+1}_Z \otimes \omega_{p+1} & \longrightarrow & \mathcal{K}_{p+1} \otimes \omega_{p+1} & \xrightarrow{+1} & \\
\downarrow & & \downarrow \simeq & & \downarrow & & \\
\underline{\Omega}^p_{Z/C} \otimes \omega_p & \longrightarrow & \Omega^{p+1}_Z \otimes \omega_{p+1} & \longrightarrow & \underline{\Omega}^{p+1}_{Z/C} \otimes \omega_{p+1} & \xrightarrow{+1} &
\end{array}$$

Since $H^i(Z, \Omega_Z^{p+1} \otimes \omega_{p+1}) = 0$, if

$$H^i(Z, \mathcal{K}_{p+1} \otimes \omega_{p+1}) \to \mathbb{H}^i(Z, \underline{\Omega}_{Z/C}^{p+1} \otimes \omega_{p+1})$$

is not injective, then

$$H^{i+1}(Z, \mathcal{K}_p \otimes \omega_p) \to \mathbb{H}^{i+1}(Z, \underline{\Omega}_{Z/C}^{p} \otimes \omega_p)$$

is not injective either. Hence by induction

$$H^n(Z, (h^*\Omega_{\mathbb{P}^1}^1)^{n-1} \otimes \mathcal{N}) \to \mathbb{H}^n(Z, \underline{\Omega}_{Z/C}^0 \otimes (h^*\Omega_{\mathbb{P}^1}^1)^{n-1} \otimes \mathcal{N})$$

is not injective, therefore $\mathcal{O}_Z \to \underline{\Omega}_{Z/C}^0$ does not have a left inverse, so h does not have property SP.

2.12 Remark. If in the definition of h one replaces t by t^2, as it is in the first example, then Z will no longer be smooth. However, Z_0 will still be toric, so $\underline{\Omega}_Z^p$ will be quasi-isomorphic to $i_*\Omega_U^p$, where U is the smooth locus of Z (cf. [GNPP88, V.4]).

At first the above argument seems to work in this case, too, that is to prove that h does not have property SP. However, it breaks down, because $i_*\Omega_U^1$ is not coherent, so Serre's vanishing does not apply. Indeed, if $\dim Z \geq 3$, then it is not a Cohen-Macaulay module and $H^{n-1}(Z, i_*\Omega_U^1 \otimes \omega_1) \neq 0$, because the local cohomology group will always contribute.

The next theorem makes use of essentially the same ideas as the proof of the special case did in the introduction as well as of the properties of the complexes $\underline{\Omega}_{X/C}^p$. It provides an important step in the proof of Theorem 2.

2.13 Theorem. *Assume $f : X \to C$ is a proper morphism with property SP, $\mathcal{L}$ is an ample line bundle on X and C is a smooth projective curve of genus at most one. Then $H^n(X, \mathcal{L} \otimes f^*\omega_C) = 0$.*

Proof. Let $\mathcal{L}_p = \mathcal{L} \otimes f^*\omega_C^{-p+1}$. Since $\mathcal{L}$ is ample, so is $\mathcal{L}_p$ for $p > 0$ and then by (1.3.2) there is a distinguished triangle in $D(X)$:

$$\underline{\Omega}_{X/C}^{p-1} \otimes \mathcal{L}_{p-1} \to \underline{\Omega}_X^p \otimes \mathcal{L}_p \to \underline{\Omega}_{X/C}^p \otimes \mathcal{L}_p \xrightarrow{+1}$$

$\mathbb{H}^{n-(p-1)}(X, \underline{\Omega}_X^p \otimes \mathcal{L}_p) = 0$ by (1.2.4), so the map

$$\mathbb{H}^{n-p}(X, \underline{\Omega}_{X/C}^p \otimes \mathcal{L}_p) \to \mathbb{H}^{n-(p-1)}(X, \underline{\Omega}_{X/C}^{p-1} \otimes \mathcal{L}_{p-1})$$

is surjective and then the composite of all these maps

$$\mathbb{H}^0(X, \underline{\Omega}_{X/C}^n \otimes \mathcal{L}_n) \to \mathbb{H}^n(X, \underline{\Omega}_{X/C}^0 \otimes \mathcal{L} \otimes f^*\omega_C)$$

is surjective as well. By definition $\underline{\Omega}_{X/C}^n = 0$, so $\mathbb{H}^n(X, \underline{\Omega}_{X/C}^0 \otimes \mathcal{L} \otimes f^*\omega_C) = 0$.

f has property SP, so $\mathcal{O}_X \to \underline{\Omega}_{X/C}^0$ has a left inverse and then $\mathcal{L} \otimes f^*\omega_C \to \underline{\Omega}_{X/C}^0 \otimes \mathcal{L} \otimes f^*\omega_C$ has a left inverse as well, so

$$H^n(X, \mathcal{L} \otimes f^*\omega_C) = \mathbb{H}^n(X, \mathcal{L} \otimes f^*\omega_C) \to \mathbb{H}^n(X, \underline{\Omega}_{X/C}^0 \otimes \mathcal{L} \otimes f^*\omega_C) = 0$$

is injective. $\square$

2.14 Corollary. *With the same hypothesis as in* (2.13)*, further assume that X is normal, Gorenstein. Then $K_{X/C}$ is not ample.*

Proof. $H^n(X, \mathcal{O}_X(K_{X/C}) \otimes f^*\omega_C) = H^n(X, \mathcal{O}_X(K_X)) = \mathbb{C}$ □

2.15 Lemma. *Let $f : X \to C$ be a proper morphism between complex projective varieties such that C is a smooth curve. Let $\mathcal{L}$ be a line bundle on X and assume that $R^1 f_*\mathcal{L}^k$ is locally free for $k \gg 0$ and one of the following holds.*
(2.15.1) *For any fiber F of f, $\mathcal{L}^k \otimes \mathcal{O}_F$ is generated by global sections and $f_*\mathcal{L}^k$ is an ample locally free sheaf on C for $k \gg 0$.*
(2.15.2) *$\mathcal{L} \simeq \mathcal{K} \otimes f^*\mathcal{M}$, where $\mathcal{M}$ is an ample line bundle on C and $\mathcal{K}$ is a line bundle on X such that $f_*\mathcal{K}^k$ is semipositive for $k \gg 0$.*

Let F_1 and F_2 be two arbitrary fibers of f. Then there exists a $q_0 > 0$ such that for all $q \geq q_0$ and $m \gg 0$

$$H^0(X, \mathcal{L}^{mq}) \to H^0(F_1, \mathcal{L}^{mq} \otimes \mathcal{O}_{F_1}) \oplus H^0(F_2, \mathcal{L}^{mq} \otimes \mathcal{O}_{F_2})$$

is surjective. In case (2.15.2) *$q_0 = 1$ works.*

Proof. Let $P_1, P_2 \in C$ be such that $F_1 = f^{-1}(P_1)$ and $F_2 = f^{-1}(P_2)$. First assume that there exists a $q > 0$ such that

(2.15.3) $$H^1(C, f_*\mathcal{L}^{mq} \otimes \mathcal{O}_C(-P_1 - P_2)) = 0 \qquad \text{for } m \gg 0.$$

Then by the Leray spectral sequence, there is a commutative diagram:

$$\begin{array}{ccc} H^1(X, \mathcal{L}^{mq} \otimes \mathcal{O}_X(-F_1 - F_2)) & \xrightarrow{\alpha} & H^1(X, \mathcal{L}^{mq}) \\ {\scriptstyle\gamma}\downarrow & & \downarrow \\ H^0(C, R^1 f_*\mathcal{L}^{mq} \otimes \mathcal{O}_C(-P_1 - P_2)) & \xrightarrow[\beta]{} & H^0(C, R^1 f_*\mathcal{L}^{mq}) \end{array}$$

such that γ and β are injective. Then $\beta \circ \gamma$ is injective and hence so is α. Therefore the statement follows.

In the first case if $q \gg 0$, then the natural map

$$S^m(f_*\mathcal{L}^q) \to f_*\mathcal{L}^{mq}$$

is surjective for $m > 0$ by global generation on the fibers. Then (2.15.3) holds since $f_*\mathcal{L}^q$ is ample.

In the second case (2.15.3) holds trivially for $q \geq 1$. □

2.16 Theorem. *Let $f : X \to C$ be a proper morphism between complex projective varieties with irreducible fibers such that C is a smooth curve and X has only canonical singularities. Assume that $K_{X/C}$ restricted to every fiber is ample. Then either $K_{X/C}$ is ample or the fibers of f are isomorphic over an open set. If in addition the fibers are reduced with canonical singularities and $K_{X/C}$ is not ample, then all the fibers of f are isomorphic.*

Proof. First assume that there is no open subset of C over which the fibers are isomorphic. X has canonical singularities, so if $\pi : Y \to X$ is a resolution of singularities, then for some $r > 0$, $\mathcal{O}_X(rK_X) \simeq \pi_*\omega_Y^r$ is a line bundle. By [Kollár87, Theorem, p.363]

$$f_*\mathcal{O}_X(mK_{X/C}) \simeq f_*\pi_*\omega_{Y/C}^m$$

is an ample locally free sheaf on C for some $m > 0$. Then $\mathcal{L} = \mathcal{O}_X(pK_{X/C})$ satisfies (2.15.1) for a sufficiently divisible $p > 0$ by [Viehweg83, 5.4, 6.2]. Therefore $K_{X/C}$ is ample.

Next assume that there is an open subset $C_0 \subset C$ such that the fibers of f over C_0 are isomorphic and every fiber is reduced with canonical singularities. The automorphism group of the fibers is finite (cf. [Kobayashi72, III.2.1]), so there is a finite base change $\sigma : \tilde{C} \to C$ such that $\tilde{X} = X \times_C \tilde{C}$ becomes trivial over $\tilde{C}_0 = \sigma^{-1}(C_0)$:

$$\begin{array}{ccccc} \tilde{C}_0 \times F \simeq \tilde{X}_0 & \subset & \tilde{X} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{C}_0 & \subset & \tilde{C} & \longrightarrow & C. \end{array}$$

By the proofs of [Matsusaka-Mumford64, Theorems 1,2] $\tilde{X}_0$ has only one compactification with non-ruled fibers, so $\tilde{X} \simeq \tilde{C} \times F$. Therefore in this case all the fibers of f are isomorphic. □

2.17 Corollary. *Theorem 4 follows.*

Proof. X has only canonical singularities by [Stevens88, Prop. 7.], so by (2.14) $K_{X/C}$ is not ample. Then by (2.16) the fibers are isomorphic. □

3. Smooth families and Lefschetz pencils

3.1 Proposition. *Let $g : Y \to C$ be a morphism between smooth projective varieties with connected fibers, C a curve. Assume that for every fiber F of g, $\omega_{Y/C}^m \otimes \mathcal{O}_F$ is generated by global sections and $R^1 f_*\omega_{Y/C}^m$ is locally free for $m \gg 0$. Then there exist a projective variety X and surjective morphisms $\phi : Y \to X$ and $f : X \to C$ such that $g = f \circ \phi$, $f^{-1}(P)$ is the canonical model of $g^{-1}(P)$ for every $P \in C$ and ϕ induces the stable pluricanonical map on the fibers. Furthermore $\mathcal{O}_X \to R\phi_*\mathcal{O}_Y$ has a left inverse.*

Proof. Let $Q \in C$ be an arbitrary point and $\mathcal{L} = \omega_{Y/C} \otimes g^*\mathcal{O}_C(Q)$. $g_*\omega_{Y/C}^m$ is semipositive by [Kawamata82, Theorem 1], so by (2.15) a suitable power of $\omega_{Y/C}$ is generated by global sections, hence it defines a morphism ϕ. This ϕ has the required properties.

Let $X=\phi(Y)$. By the construction, there is an ample line bundle $\mathcal{H}$ on X such that $\mathcal{L}=\phi^*\mathcal{H}$ and then $\omega_Y^{-1}=\phi^*(\mathcal{H}\otimes f^*(\omega_C(-Q)))^{-1}$. By [Kollár86, Theorem 3.1] $R\phi_*\omega_Y=\sum R^i\phi_*\omega_Y[-i]$, so

$$\begin{aligned}R\phi_*\mathcal{O}_Y &= (\mathcal{H}\otimes f^*(\omega_C(-Q)))^{-1}\otimes R\phi_*\omega_Y = \\ &= (\mathcal{H}\otimes f^*(\omega_C(-Q)))^{-1}\otimes \sum R^i\phi_*\omega_Y[-i] = \sum R^i\phi_*\mathcal{O}_Y[-i]\end{aligned}$$

and since $\mathcal{O}_X=\phi_*\mathcal{O}_Y$, this finishes the proof. □

A smooth projective variety F will be called a *$\mathcal{G}$-variety* if ω_F^m is generated by global sections and $h^1(F,\omega_F^m)=0$ for $m\gg 0$, and its canonical model is Gorenstein.

Note that the canonical model of a $\mathcal{G}$-variety has canonical singularities by [Nakayama88] and that every smooth projective variety of general type with nef canonical bundle is a $\mathcal{G}$-variety. Indeed, by the Base Point Free Theorem (cf. [CKM88, (9.3)]) all the large enough multiples of the canonical divisor are base point free, hence pull-backs of Cartier divisors on the canonical model. Then the canonical divisor of the canonical model is a difference of two Cartier divisors, thus itself is Cartier (cf. [Reid87]).

3.2 Corollary. *Let $g: Y\to C$ be a smooth family of projective $\mathcal{G}$-varieties over a smooth projective curve C of genus at most one. Then the fibers of g have isomorphic canonical models. In particular, Theorem 1 follows.*

Proof. Let $f: X\to C$ be the fiber space given by (3.1). By (2.4) f has property SP, so by Theorem 2 it is locally trivial. □

The following application of the theory presented in this article was pointed out to me by János Kollár. It is an even dimensional analog of [Beauville81].

3.3 Theorem. *Let Y be a projective variety of odd dimension and $g: Y\to\mathbb{P}^1$ a morphism with connected fibers having at worst ordinary double point singularities. Assume that $K_{Y/\mathbb{P}^1}$ restricted to any fiber is nef and big. If g is not smooth, then it has at least five singular fibers.*

Proof. Let

$$U=\left\{P\in\mathbb{P}^1 \mid g \text{ has property SP in an analytic neighborhood of } g^{-1}(P)\right\}.$$

Let $S=\mathbb{P}^1\setminus U$. It is enough to prove that either g is smooth or S has at least five elements. Suppose $|S|\le 4$. Then one can find an elliptic curve E and a double cover $\rho: E\to\mathbb{P}^1$ such that the branch locus of ρ contains S. Let $Y_E=Y\times_{\mathbb{P}^1}E$ and $g_E: Y_E\to E$.

By the construction, for every fiber F of g_E, either g_E has property SP in an analytic neighbourhood of F or F is locally analytically isomorphic to the special fiber of the morphism in (2.7). Therefore by (2.3) g_E has property SP. Then (2.4) and (3.1) imply that there exist a projective variety X_E and morphisms $\phi_E: Y_E\to X_E$ and $f_E: X_E\to E$ such that X_E is a family of the canonical models of the fibers of g_E and f_E has property SP. Hence by Theorem 2, the fibers of f_E are isomorphic. Then the singularities of the fibers can be resolved simultaneously, so by [Matsusaka-Mumford64, Corollary 1] the fibers of g_E are isomorphic and then the same is true for g. Therefore g is smooth. $\square$

3.4 Remark. If $\dim Y = 3$, the assumption about $K_{Y/\mathbb{P}^1}$ being nef on the fibers is not necessary. In fact, for surfaces, rational double points are exactly the canonical singularities (cf. [Reid87, (1.2)]), so using the Minimal Model Program one can reduce to the case of the theorem, because the singularities of the fibers are preserved.

Also note that the assumptions allow a somewhat larger class of varieties, than Lefschetz pencils.

References

[Beauville81] A. Beauville, *Le nombre minimum de fibres singulières d'une courbe stable sur* $\mathbb{P}^1$, Astérisque **86** (1981), 97–108.

[Bruno-Herzog93] W. Bruno, J. Herzog, *Cohen-Macaulay rings*, Cambridge Univ. Press, 1993.

[CKM88] H. Clemens, J. Kollár, S. Mori, *Higher dimensional complex geometry*, Astérisque **166** (1988).

[DuBois81] Ph. Du Bois, *Complexe de De Rham filtré d'une variété singulière*, Bull. Soc. Math. France **109** (1981), 41–81.

[GNPP88] F. Guillén, V. Navarro Aznar, P. Pascual-Gainza, F. Puerta, *Hyperrésolutions cubiques et descente cohomologique*, Lecture Notes in Math, vol. 1335, Springer-Verlag, 1988.

[Hartshorne66] R. Hartshorne, *Residues and Duality*, Lecture Notes in Math, vol. 20, Springer-Verlag, 1966.

[Kawamata82] Y. Kawamata, *Kodaira dimension of algebraic fibre spaces over curves*, Invent. Math. **66** (1982), 57–71.

[Kobayashi72] S. Kobayashi, *Transformation groups in differential geometry*, Ergebuisse 70, Springer-Verlag, Heidelberg, 1972.

[Kollár86] J. Kollár, *Higher direct images of dualizing sheaves II.*, Ann. of Math. **124** (1986), 171–202.

[Kollár87] J. Kollár, *Subadditivity of the Kodaira dimension: Fibers of general type*, Advanced Studies in Pure Math. 10, Algebraic Geometry Sendai 85, Kinokuniya-North Holland, 1987, pp. 361–398.

[Kollár93] J. Kollár, *Shafarevich maps and automorphic forms*, Princeton Univ. Press, to appear.

[Kollár-Mori92] J. Kollár, S. Mori, *Classification of three-dimensional flips*, Jour. AMS **5** (1992), 533–703.

[Matsusaka-Mumford64] T. Matsusaka, D. Mumford, *Two fundamental theorems on deformations of polarized varieties*, Amer. J. Math. **86** (1964), 668–684.

[Migliorini95] L. Migliorini, *A smooth family of minimal surfaces of general type over a curve of genus at most one is trivial*, J. Algebraic Geometry **4** (1995), 353–361.

[Nakayama88] N. Nakayama, *The singularity of the canonical model of compact Kähler manifolds*, Math. Ann. **280** (1988), 509–512.

[Reid87] M. Reid, *Young person's guide to canonical singularities*, Proc. Symp. Pure Math., vol. 46, 1987, pp. 345–414.

[Steenbrink85] J.H.M. Steenbrink, *Vanishing theorems on singular spaces*, Astérisque **130** (1985), 330–341.

[Stevens88] J. Stevens, *On canonical singularities as total spaces of deformations*, Abh. Math. Sem. Univ. Hamburg **58** (1988), 275–283.

[Viehweg83] E. Viehweg, *Weak positivity and the additivity of Kodaira dimension for certain fibre spaces*, Advanced Studies in Pure Math. 1, Algebraic Varieties and Analytic Varieties, 1983, pp. 329–353.

Erratum

There is a hypothesis missing from the statement of (3.3) without that the presented proof is incomplete. The missing assumption is that if $\dim Y \geq 5$, then $\omega_{Y/\mathbb{P}^1}$ restricted to any fibre is required to be ample. The correct statement is as follows:

Theorem. *Let Y be a projective variety of odd dimension and $g : Y \to \mathbb{P}^1$ a morphism with connected fibres having at worst ordinary double point singularities. Assume that $\omega_{Y/\mathbb{P}^1}$ restricted to any fibre is nef and big and if $\dim Y \geq 5$, then $\omega_{Y/\mathbb{P}^1}$ restricted to any fibre is ample. If g is not smooth, then it has at least 5 singular fibres.*

The presented proof works unchanged with the additional remark, that the isotriviality of f_E implies the isotriviality of g_E by the extra assumption.

On the other hand this extra assumption is expected to be superfluous as [Kollár-Mori92, 12.7.3] is conjectured to be true in all dimensions, thus (expected to be) providing the missing step to the originally stated form, i.e., proving that the isotriviality of f_E implies the isotriviality of g_E.

Finally note that (3.3) is not used elsewhere in the article, so the rest, including the main results, holds true unchanged.

References

[Kollár-Mori92] J. Kollár, S. Mori, *Classification of three-dimensional flips*, Jour. AMS **5** (1992), 533–703.

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