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Small saturated sets in finite projective planes

S.J. KOVÁCS

RIASSUNTO – *Un sottoinsieme Σ di un piano proiettivo di ordine q dicesi saturato se le rette che incontrano Σ in almeno due punti costituiscono un ricoprimento del piano. Attraverso l'uso di metodi probabilistici si dimostra l'esistenza di insiemi saturati di cardinalità al più* $6\sqrt{3q\log q}$.

ABSTRACT – *A subset Σ of a projective plane of order q is said to be saturated if the lines meeting Σ in at least two points cover the whole plane. Using probabilistic methods we prove the existence of saturated sets with cardinality at most* $6\sqrt{3q\log q}$.

KEY WORDS – *Finite projective planes - Saturated sets.*

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1 – Introduction

We recall some well known notions (see [3]). A k-arc is a set of k points no three of which are collinear. A k-arc is said to be *complete* if it is not contained in a $k+1$-arc. As it is well known the maximum value of k such that a projective plane of order q may contain k-arcs is $k = q+1$ or $k = q+2$ according as q is odd or even. A k-arc with this maximum number of points is called an *oval.* A *k-set* will mean a set of k points.

DEFINITION 1.1. In a recent paper U. Bartocci [1] introduced the study of *saturated* configurations: a subset Σ of a projective plane of order q is said to be saturated if the lines meeting Σ in at least two points cover the whole plane.

Clearly every set containing a saturated set is itself saturated. So we are mostly interested in *minimal* saturated sets (ie: saturated sets which do not contain proper saturated subsets). We note that every complete k-arc is a minimal saturated set.

We define the function $\sigma(q)$ as the smallest possible value for which there exists a saturated $\sigma(q)$-set in the projective plane of order q.

PROPOSITION 1.2. [7] *If there exists a minimal saturated k-set in a projective plane of order q, then k satisfies*

$$\frac{3+\sqrt{1+8q}}{2} \leq k \leq q+2\,.$$

REMARK 1.3. The lower bound here is exactly the same as in the corresponding inequality established by M. Sce for complete k-arcs.

We observe that the upper bound is effectively attained as it is shown in the following way.

EXAMPLE 1.4. ([7]). The set consisting of all points of a line L in a projective plane of order q plus another point not on the line is obviously a $q+2$-set which is saturated and minimal.

On the other hand the lower bound seems unsatisfactory, since the known examples of complete k-arcs or of saturated sets all have a number of points whose order of magnitude is too large compared to this lower bound.

EXAMPLE 1.5. ([7]). Assume that q is a square. Take all the points in three independent lines in a Baer subplane PG(2,$\sqrt{q}$) $\subset$ PG(2,q). This set is a minimal saturated $3\sqrt{q}$-set in PG(2,q).

In general, whitout assuming q being a square, no minimal saturated sets of magnitude O($\sqrt{q}$) are known. In the present paper we give some examples of size not far from the lower bound we have seen.

According to the connections between saturated sets and complete arcs, it may be interesting to mention the following result of T. Szőnyi.

PROPOSITION 1.6. *Let $\frac{3}{4} \leq \alpha < 1$ be fixed. Then there exist c,d constants, such that there is a complete k-arc in PG(2,q) with $cq^\alpha \leq k \leq dq^\alpha$ if $q > q_0(\alpha)$.*

The purpose of this paper is to show the existence of a saturated set containing at most $6\sqrt{3q\log q}$ points in an arbitrary projective plane of order q (including non-desarguesian planes, too).

2 – Small minimal saturated sets

We will make use of the following more or less known lemma from probability theory.

LEMMA 2.1. *If x_i $i = 1, 2, \dots, n$ are arbitrary events, then*

$$\mathrm{Prob}(\bigcap_{i=1}^{n} x_i) \geq \prod_{i=1}^{n} \mathrm{Prob}(x_i) - (1 - \frac{1}{n})^n \geq \prod_{i=1}^{n} \mathrm{Prob}(x_i) - \frac{1}{e}.$$

PROOF. Define the events $y_1 = x_1 - \bigcap_{i=1}^{n} x_i$ and $y_i = x_i$ for $2 \leq i \leq n$. We see that

$$\mathrm{Prob}(\bigcap_{i=1}^{n} x_i) - \prod_{i=1}^{n} \mathrm{Prob}(x_i) \geq -\prod_{i=1}^{n} \mathrm{Prob}(y_i)$$

and

$$\bigcap_{i=1}^{n} y_i = \emptyset$$

so

$$\sum_{i=1}^{n} \mathrm{Prob}(y_i) \leq n - 1.$$

The assertion follows from the inequality between geometric and arithmetic means. □

Our main result is the following.

THEOREM 2.2. *In every projective plane of order q there exists a saturated $6\sqrt{3q\log q}$-set, ie:*

$$\sigma(q) \leq 6\sqrt{3q\log q}.$$

PROOF. We will apply the probabilistic method (see [2]).

Let Π be a projective plane of order q and l_1, l_2, l_3 be three different lines in Π through a common point P and set $\Delta = l_1 \cup l_2 \cup l_3 = \{P, R_1, \dots, R_{3q}\}$. Moreover, let p be a real number, $0 < p < 1$. Let $X_1, \dots, X_{3q}$ be independent random variables with $\mathrm{Prob}(X_i = \{R_i\}) = p$ and $\mathrm{Prob}(X_i = \emptyset) = 1 - p$ for all $1 \le i \le 3q - 3$. Finally define

$$\Sigma = \bigcup_{i=1}^{3q} X_i \cup \{P, R_{3q-2}, R_{3q-1}, R_{3q}\}\,.$$

Pick a point $A \in \Pi \setminus \Delta$ and let $l^1, l^2, \dots, l^q$ be the lines of Π through A except the one containing P. We will say that A is *covered* if A lies on at least one line meeting Σ in at least two points. Thus

$$\xi = \mathrm{Prob}(\mid l^i \cap \Sigma \mid < 2) \ge (1-p)^2(1+2p)$$

and

$$\mathrm{Prob}(A \text{ is covered}) = 1 - \xi^q\,,$$

since the events $A_i\colon \mid l^i \cap \Sigma \mid < 2$ are independent for $1 \le i \le q$ by the definition of Σ. Then by Lemma 2.1

$$\mathrm{Prob}(\bigcap_{A \in \Pi} A \text{ is covered}) \ge \prod_{A \in \Pi} \mathrm{Prob}(A \text{ is covered}) - \frac{1}{e}\,.$$

Note that

$$\mathrm{Prob}(\Sigma \text{ is a saturated set}) = \mathrm{Prob}(\bigcap_{A \in \Pi} A \text{ is covered})\,.$$

It is easy to see that

$$\prod_{A \in \Delta} \mathrm{Prob}(A \text{ is covered}) = 1$$

and

$$\prod_{A \in \Pi \setminus \Delta} \mathrm{Prob}(A \text{ is covered}) = (1 - \xi^q)^{q^2 - 2q}$$

so

$$\text{Prob}(\Sigma \text{ is a saturated set}) \geq (1-\xi^q)^{q^2-2q} - \frac{1}{e}.$$

Let us choose now

$$p = \sqrt{\frac{3\log q}{q}}.$$

Substituting the value of p and ξ we obtain

$$\text{Prob}(\Sigma \text{is a saturated set}) \geq \frac{1}{2}.$$

Now let X be the size of Σ. It is a random variable with binomial distribution, hence its expectation is:

$$E(X) = 3\sqrt{3q\log q}$$

and its variaance is

$$D^2(X) = 3qp(1-p) \leq 3\sqrt{3q\log q}.$$

By the Chebyshev inequality (cf. [4]) we obtain

$$\text{Prob}(|\, X - E(X)\,| > 2D(X)) < \frac{1}{4}.$$

Therefore

$$\text{Prob}(\Sigma \text{is a saturated set and the size of} \Sigma \leq 6\sqrt{3q\log q}) > \frac{1}{4}$$

and the theorem is proved. $\square$

REMARK 2.3. The theorem is valid for non-desarguesian planes, too.

3 – Saturated sets wich are 'almost arcs'

The saturated sets we obtained in the previous section contain many points on the same line. The next theorem gives a saturated set Σ with the following property: each line of the plane meets Σ in at most four points. So Σ is not too far from being an arc. Our construction works in every projective plane having two ovals which intersect in at most $c\sqrt{q}\log q$ points with a certain constant c. For example, it works for every Galois plane, but also for some non-desarguesian planes.

THEOREM 3.1. *Let Π be a projective plane of order q. Assume there exist two ovals in Π intersecting in at most $c\sqrt{q}\log q$ points (c is a constant). Then there exists a saturated set Σ of size at most $d\sqrt{q}\log q$ (where $d < c+12$) with the property that every line of Π intersects Σ in at most four points.*

PROOF. Let Ω be an oval in Π. Similarly to the proof of Theorem 2.2 we choose each point of Ω with probability p $(0 < p < 1)$ to make up the set Σ.

Pick a point $A \in \Pi \setminus \Omega$. There are at least $k = [q/2]$ lines $l_1, l_2, \ldots, l_k$ which contain A and two different points of Ω. Thus

$$\mathrm{Prob}(|\, l_i \cap \Sigma \,| < 2) = 1 - p$$

and

$$\mathrm{Prob}(A \text{ is covered} \geq 1 - (1-p)^k\,.$$

Let us now choose

$$p = \sqrt{\frac{6\log q}{q}},$$

then the calculation gives us:

$$\mathrm{Prob}(Y) \geq \frac{1}{2},$$

where Y is the following event: each point in $\Pi \setminus \Omega$ lies on at least one line meeting Σ in at least two points.

Now let X be the size of Σ. It is a random variable with binomial distribution, so

$$E(X) \leq \sqrt{7q\log q}$$

$$D^2(X) \leq \sqrt{7q\log q}$$

and just like in the proof of Theorem 2.2 we see that there exists an $\Sigma \subset \Omega$, such that

$$|\,\Sigma\,| \leq 6\sqrt{q\log q}$$

and each point in $\Pi \setminus \Omega$ lies on at least one line meeting Σ in at least two points.

Now by the assumption there are two ovals Ω_1, Ω_2 such that

$$|\,\Omega_1 \cap \Omega_2\,| \leq c\sqrt{q\log q}\,.$$

Let $\Sigma_i \subset \Omega_i$ $(i = 1, 2)$ be the sets defined above. Then

$$\Sigma = \Sigma_1 \cup \Sigma_2 \cup (\Omega_1 \cap \Omega_2)$$

has the desired properties. $\square$

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INDIRIZZO DELL'AUTORE:

Sándor J. Kovács - Department of Mathematics - University of Utah - Salt Lake City - UT 84112, USA