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Relative De Rham complex for non-smooth morphisms

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In Memoriam: Professor Wei-Liang Chow

Let $f : X \to S$ be a smooth morphism of algebraic varieties. It is well-known that the short exact sequence,

$$0 \longrightarrow f^*\Omega_S \longrightarrow \Omega_X \longrightarrow \Omega_{X/S} \longrightarrow 0,$$

provides a powerful tool for cohomological computations. Taking exterior powers it yields a filtration of Ω_X^p with associated graded quotients $f^*\Omega_S^r \otimes \Omega_{X/S}^{p-r}$. This filtration in turn leads to a spectral sequence,

$$E_1^{r,s} = H^{r+s}(X, f^*\Omega_S^r \otimes \Omega_{X/S}^{p-r} \otimes \mathcal{L}) \Rightarrow H^{r+s}(X, \Omega_X^p \otimes \mathcal{L}),$$

for an arbitrary line bundle $\mathcal{L}$.

The purpose of this article is to construct a similar spectral sequence for a morphism that is not necessarily smooth. More precisely, the goal is to define natural objects that relate $\Omega_X^{\cdot}$ and $f^*\Omega_S^{\cdot}$. These objects – the relative De Rham complexes – will not be single sheaves anymore, but objects of the derived category of $\mathcal{O}_X$-modules. Nevertheless their functorial and cohomological properties resemble the ones of $\Omega_{X/S}^p$. The relative De Rham complexes were already defined and constructed in [Kovács96] for the case when S is a smooth curve. The present construction is a special case of the construction of [Kovács97a].

§1 contains a brief summary of some technical material that is essential to the construction. (1.1.2) ought to be well-known, but I do not know a convenient reference other than the one quoted. §§1.2 is a simple generalization of the notion of a filtration to the derived categorical setting.

In §2 the complexes $\underline{\Omega}_{X/S}^p$ are constructed for an arbitrary morphism between smooth varieties. The guiding principles are funcoriality and the desired "filtration".

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Next, one would like to extend this definition to the case when X may be singular. If X is no longer assumed to be smooth, then in particular the sheaf of Kähler differentials of X is not locally free and the usual (non-relative) De Rham complex does not behave as nicely as in the smooth case. Hence one needs to replace it with another object.

[DuBois81] showed that one can define an object, $\underline{\Omega}^{\cdot}_X$, in the derived category of filtered complexes of $\mathcal{O}_X$-modules, that resemble the cohomological properties of the De Rham complex in the smooth case. In particular it gives a resolution of the constant sheaf $\mathbb{C}_X$. Later Du Bois' results were re-proved in [GNPP88] using somewhat different methods. They also proved some new results about $\underline{\Omega}^{\cdot}_X$, for instance the analogue of the Kodaira-Akizuki-Nakano vanishing theorem (cf. (3.1)).

An important ingredient of the construction of $\underline{\Omega}^{\cdot}_X$ is the notion of a hyperresolution. In fact, there are several different flavors of this notion that will not be discussed here. The reader is referred to [DuBois81], [GNPP88] or [Steenbrink85].

In §3, using the construction from §2 and the generalized De Rham complex of [DuBois81], [GNPP88], the relative De Rham complexes are defined for an arbitrary morphism with S being smooth. The main result is (3.3).

The rest of the article is concerned with applications. The first application is a vanishing theorem, proved in §4.

Usually one cannot expect that a notion so generally defined could be very useful without further restrictions on the morphisms allowed. A class of morphisms, so called SP-morphisms (cf. [Kovács96]), is defined and some important consequencies of the vanishing theorem of §4 are derived in §5.

§6 contains the main application. In fact, this was the initial driving force for the entire project.

THEOREM. *Let $g : Y \to S$ be a smooth family of projective varieties of general type with nef canonical bundle. Assume that Ω_S is semi-negative. Then the fibres of g are birational.*

This is a common generalization of [Migliorini95], [Kovács96], [Kovács97]. The sketch of the proof is as follows. By [DPS94] a finite étale cover of S is a smooth Fano fibre space over an abelian variety. Using Poincaré's Complete Reducibility Theorem one can assume that this abelian variety is irreducible. Considering the relative canonical model of $g : Y \to S$ and using results of [Kovács97] one easily reduces the problem to proving that the relative canonical bundle of the relative canonical model is not ample. This in turn follows from the above mentioned vanishing theorem.

DEFINITIONS AND NOTATION. Henceforth the groundfield will be $\mathbb{C}$, the field of complex numbers. A *complex scheme* will mean a separated scheme of finite type over $\mathbb{C}$.

A divisor D on a scheme X is called $\mathbb{Q}$-Cartier if mD is Cartier for some $m > 0$. It is called *ample* if mD is ample. A $\mathbb{Q}$-Cartier divisor D is called *nef* if $D.C \geq 0$ for every proper curve $C \subset X$. D is called *big* if X is proper and $|mD|$ gives a birational map for some $m > 0$. In particular ample implies nef and big.

A normal variety X is said to have *canonical* (resp. *terminal*) *singularities* if K_X is $\mathbb{Q}$-Cartier and for any resolution of singularities $\pi : \tilde{X} \to X$, with the collection of exceptional prime divisors $\{E_i\}$, there exist $a_i \in \mathbb{Q}$, $a_i \geq 0$ (resp. $a_i > 0$) such that $K_{\tilde{X}} \equiv \pi^* K_X + \sum a_i E_i$ (cf. [CKM88]). X is called a *canonical variety* if it has only canonical singularities and K_X is ample. X is called a *minimal variety* if it has only terminal singularities and K_X is nef.

A singularity is called *Gorenstein* if its local ring is a Gorenstein ring. A variety is *Gorenstein* if it admits only Gorenstein singularities. In particular, the dualizing sheaf of a Gorenstein variety is locally free.

A locally free sheaf $\mathcal{E}$ on a scheme X is called *semi-positive* (or *nef*) if for every smooth complete curve C and every map $\gamma : C \to X$, any quotient bundle of $\gamma^*\mathcal{E}$ has non-negative degree. $\mathcal{E}$ is called *semi-negative* if $\check{\mathcal{E}}$, the dual of $\mathcal{E}$, is semi-positive. $\operatorname{Sym}^l(\mathcal{E})$ denotes the l-th symmetric power of $\mathcal{E}$, and $\det \mathcal{E}$ the determinant bundle of $\mathcal{E}$, i.e., $\det \mathcal{E} = \bigwedge^r \mathcal{E}$ if $r = \operatorname{rk} \mathcal{E}$.

Let $f : X \to S$ be a morphism of schemes, then X_s denotes the fibre of f over the point $s \in S$ and f_s denotes the restriction of f to X_s. More generally, for a morphism $Z \to S$, let $f_Z : X_Z = X \times_S Z \to Z$. If f is composed with another morphism $g : S \to T$, then X_t denotes the fibre of $g \circ f$ over the point $t \in T$, i.e., $X_t = X_{S_t}$.

f is called *isotrivial* if $X_s \simeq X_t$ for every $s, t \in S$.

A smooth projective variety X is called a *Fano variety* if $-K_X$ is ample. X is a *Fano fibre space* over S if the fibres of f are connected Fano varieties.

A proper variety X is called *rationally connected* if two arbitrary points of X can be joined by an irreducible rational curve (cf. [KMM92a], [Campana91]).

Ω_X denotes the sheaf of differentials on X, $\Omega_{X/S}$ is the sheaf of relative differentials. $\Omega^p_X = \bigwedge^p \Omega_X$, $\Omega^p_{X/S} = \bigwedge^p \Omega_{X/S}$, and $\omega_{X/S} = \omega_X \otimes g^*\omega_S^{-1}$.

Let X be a complex scheme of dimension n. Then $C(X)$ is the category of complexes of $\mathcal{O}_X$-modules with differentials of order ≤ 1 and for $u \in \operatorname{Mor}(C(X))$, $M(u) \in \operatorname{Ob}(C(X))$ denotes the mapping cone of u (cf. [Hartshorne66]). $K(X)$ is the category of homotopy equivalence classes of objects of $C(X)$. A diagram in $C(X)$ will be called a *predistinguished triangle,* if its image in $K(X)$ is a distinguished triangle.

$D_{filt}(X)$ denotes the derived category of filtered complexes of $\mathcal{O}_X$-modules with differentials of order ≤ 1 and $D_{filt,coh}(X)$ the subcategory of $D_{filt}(X)$ of complexes $K^{\cdot}$, such that for all i, the cohomology sheaves of $Gr^i_F K^{\cdot}$ are coherent (cf. [DuBois81], [GNPP88]). $D(X)$ and $D_{coh}(X)$ denotes the derived categories with the same definition except that the complexes are not assumed to be filtered. The superscripts $+, -, b$ carry the usual meaning (bounded below, bounded above and bounded).

S_k denotes the symmetric group of degree k.

§1. Wedge products and hyperfiltrations

§§**1.1 Wedge products.** Let $f : X \to S$ be a morphism of smooth algebraic varieties of dimension n and k respectively. Let Ψ be the functor of Kähler differentials on smooth varieties, i.e., for any smooth variety X, $\Psi_X = \Omega_X$. Further let Φ be the functor on the category of S-schemes that is the pullback of Ψ from S, i.e., for $f : X \to S$, $\Phi_X = f^*\Omega_S$. Then f^* induces a natural transformation $\theta : \Phi \to \Psi$.

Following the notation of [Kovács97a] let $\curlywedge_q^f = \curlywedge_q^\theta$. Let us recall the definition of $\curlywedge_q^\theta$ in this special case.

1.1.1 Definition. Let η be a section of Ω_X^p over an open set and $\xi_1, \dots, \xi_k$ a basis for $f^*\Omega_S$ over the same set. Then $\eta \otimes (\xi_1 \wedge \dots \wedge \xi_k)$ is a section of $\Omega_X^p \otimes f^*\omega_S$. For any $\sigma \in S_k$ let $\xi_{\sigma,q} = \xi_{\sigma(1)} \wedge \dots \wedge \xi_{\sigma(q)}$ and $\xi^{\sigma,q} = \xi_{\sigma(q+1)} \wedge \dots \wedge \xi_{\sigma(k)}$. Now define

$$\curlywedge_q^f(\eta \otimes (\xi_1 \wedge \dots \wedge \xi_k)) \in \Omega_X^{p+q} \otimes f^*\Omega_S^{k-q}$$

by the formula

$$\curlywedge_q^f(\eta \otimes (\xi_1 \wedge \dots \wedge \xi_k)) = \frac{1}{q!(k-q)!} \sum_{\sigma \in S_k} (-1)^{\operatorname{sgn}\sigma} (\xi_{\sigma,q} \wedge \eta) \otimes \xi^{\sigma,q},$$

and extend it linearly.

1.1.2 Lemma. *$\curlywedge_q^f : \Omega_X^p \otimes f^*\omega_S \to \Omega_X^{p+q} \otimes f^*\Omega_S^{k-q}$ is a well-defined morphism of sheaves and*

$$\begin{array}{ccc}
\Omega_X^p \otimes f^*\omega_S \otimes f^*\omega_S & \xrightarrow{\curlywedge_q^f} & \Omega_X^{p+q} \otimes f^*\Omega_S^{k-q} \otimes f^*\omega_S \\
{\scriptstyle \curlywedge_{q+r}^f}\big\downarrow & & \big\downarrow{\scriptstyle \curlywedge_r^f} \\
\Omega_X^{p+q+r} \otimes f^*\omega_S \otimes f^*\Omega_S^{k-q-r} & \xrightarrow[\curlywedge_q^{\mathrm{id}}]{} & \Omega_X^{p+q+r} \otimes f^*\Omega_S^{k-q} \otimes f^*\Omega_S^{k-r}
\end{array}$$

is a commutative diagram, i.e., $\curlywedge_r^f \circ \curlywedge_q^f = \curlywedge_q^{\mathrm{id}} \circ \curlywedge_{q+r}^f$.

Proof. [Kovács97a, 1.1.2 and 1.1.3] $\square$

§§**1.2 Hyperfiltrations and spectral sequences.** Let $\mathfrak{A}$ be an abelian category and $D(\mathfrak{A})$ its derived category. Let $\Gamma : \mathfrak{A} \to \mathfrak{Ab}$ be a left exact additive functor from $\mathfrak{A}$ to the category of abelian groups and assume that $R\Gamma : D(\mathfrak{A}) \to D(\mathfrak{Ab})$, the right derived functor of Γ, exists.

1.2.1 Definition. Let $K \in \mathrm{Ob}(D^b(\mathfrak{A}))$ be a bounded complex. A *hyperfiltration* $\mathbb{F}$ of K consists of a set of objects $\mathbb{F}^j K \in \mathrm{Ob}(D^b(\mathfrak{A}))$ for $j = l, \dots, k+1$, where $l, k \in \mathbb{Z}$ and morphisms

$$\varphi_j \in \mathrm{Hom}_{D^b(\mathfrak{A})}(\mathbb{F}^{j+1}K, \mathbb{F}^j K) \qquad \text{for } j = l, \dots, k,$$

where $\mathbb{F}^l K \simeq K$ and $\mathbb{F}^{k+1} K \simeq 0$. $\mathbb{F}^j K$ will be denoted by $\mathbb{F}^j$ when no confusion is likely. For convenience let $\mathbb{F}^i K = K$ for $i < l$ and $\mathbb{F}^i K = 0$ for $i > k$.

The *p-th associated graded complex* of a hyperfiltration $\mathbb{F}$ is

$$\mathbb{G}^p = \mathbb{G}r_{\mathbb{F}}^p K = M(\varphi_p),$$

the mapping cone[1] of the morphism φ_p.

Then one has the following standard result:

[1] Strictly speaking this is the class of the mapping cone of a morphism in $C(X)$ whose class in $D(X)$ is φ_p. The important point is that $\mathbb{F}^{p+1} \to \mathbb{F}^p \to \mathbb{G}^p \xrightarrow{+1}$ forms a distinguished triangle.

1.2.2 THEOREM. *There exists a spectral sequence E_r with $E_1^{p,q} = R^{p+q}\Gamma(\mathbb{G}^p)$ abutting to $R^{p+q}\Gamma(K)$.*

§2. The smooth case

Let $f : X \to S$ be a morphism of smooth algebraic varieties of dimension n and k respectively. Let Ψ, Φ, θ be defined as in (1.1) and for any $p \in \mathbb{Z}$ let $\underline{\Omega}^p_{X/S} = \underline{\mathfrak{Q}}^p_{\theta_X}$ as defined in [Kovács97a]. As follows we briefly review the construction.

Let $p \in \mathbb{N}$, and define $\mathfrak{F}^p_i = \mathfrak{F}^p_i(X/S) \in \mathrm{Ob}(C(X))$ and $\boxed{\mathfrak{F}^p_i} = \boxed{\mathfrak{F}^p_i(X/S)}$ a diagram in $C(X)$ for $i \in \mathbb{N}$ in the following way. First let $\mathfrak{F}^p_0 = \Omega^{n-p}_X \otimes f^*\omega_S^{-1}$.

2.1 DEFINITION. The $(p, 0)$-*filtration diagram of* X/S consists of $\mathfrak{F}^p_0 \otimes f^*\omega_S = \Omega^{n-p}_X$. It is denoted by $\boxed{\mathfrak{F}^p_0}$. A 0-*filtration morphism* for some p, q, consists of locally free sheaves $\mathcal{E}, \mathcal{F}$ and a morphism between $\Omega^{n-p}_X \otimes \mathcal{E}$ and $\Omega^{n-q}_X \otimes \mathcal{F}$.

For instance, $\curlywedge^f_p : \Omega^{n-p}_X \otimes f^*\omega_S \to \Omega^n_X \otimes f^*\Omega^{k-p}_S$ is a 0-filtration morphism. Let

$$\mathfrak{F}^p_1 = M(\curlywedge^f_p)[-1] \otimes f^*\omega_S^{-1}.$$

2.2 DEFINITION. The $(p, 1)$-*filtration diagram of* X/S consists of the predistinguished triangle,

$$\mathfrak{F}^p_1 \otimes f^*\omega_S \longrightarrow \Omega^{n-p}_X \otimes f^*\omega_S \longrightarrow \Omega^n_X \otimes f^*\Omega^{k-p}_S \xrightarrow{+1} .$$

It is denoted by $\boxed{\mathfrak{F}^p_1}$. A 1-*filtration morphism* for some p, q, consists of locally free sheaves $\mathcal{E}, \mathcal{F}$ and morphisms between the corresponding terms of $\boxed{\mathfrak{F}^p_1} \otimes \mathcal{E}$ and $\boxed{\mathfrak{F}^q_1} \otimes \mathcal{F}$ such that the resulting diagram is commutative.

Consider the following commutative diagram (cf. (1.1.2)):

$$\begin{tikzcd}
\mathfrak{F}^p_1 \otimes f^*\omega_S \otimes f^*\omega_S \arrow[d] & \mathfrak{F}^{p-q}_1 \otimes f^*\Omega^{k-q}_S \otimes f^*\omega_S \arrow[d] \\
\Omega^{n-p}_X \otimes f^*\omega_S \otimes f^*\omega_S \arrow[r, "\curlywedge^f_q"] \arrow[d, "\curlywedge^f_p"'] & \Omega^{n-p+q}_X \otimes f^*\Omega^{k-q}_S \otimes f^*\omega_S \arrow[d, "\curlywedge^f_{p-q}"] \\
\Omega^n_X \otimes f^*\omega_S \otimes f^*\Omega^{k-p}_S \arrow[r, "\curlywedge^{\mathrm{id}}_q"'] \arrow[d, "+1"'] & \Omega^n_X \otimes f^*\Omega^{k-q}_S \otimes f^*\Omega^{k-p+q}_S \arrow[d, "+1"] \\
{} & {}
\end{tikzcd}$$

Then there exists a morphism,

$$\mathfrak{F}^p_1 \otimes f^*\omega_S \otimes f^*\omega_S \to \mathfrak{F}^{p-q}_1 \otimes f^*\Omega^{k-q}_S \otimes f^*\omega_S,$$

that makes the above diagram commutative, and therefore it gives a 1-filtration morphism $\boxed{\mathfrak{F}^p_1} \otimes f^*\omega_S \to \boxed{\mathfrak{F}^{p-q}_1} \otimes f^*\Omega^{k-q}_S$.

In general we have the following.

2.3 Proposition. [Kovács97a, 2.4] *The following assumptions hold for all $p, q \geq 0$ and $i \geq 0$.*

(2.3.1) *The (p, i)-filtration diagram of X/S, denoted by $\boxed{\mathfrak{F}_i^p}$, consists of the diagram*

$$\mathfrak{F}_i^p \otimes f^*\omega_S \to \boxed{\mathfrak{F}_{i-1}^p} \otimes f^*\omega_S \to \boxed{\mathfrak{F}_{i-1}^{i-1}} \otimes f^*\Omega_S^{k-p+i-1}.$$

(2.3.2) *An i-filtration morphism, by definition, consists of locally free sheaves $\mathcal{E}, \mathcal{F}$ and a morphism between the corresponding terms of $\boxed{\mathfrak{F}_i^p} \otimes \mathcal{E}$ and $\boxed{\mathfrak{F}_i^q} \otimes \mathcal{F}$ such that the resulting diagram is commutative.*

(2.3.3) $\mathfrak{F}_i^p = 0$ *for $p < i$.*

(2.3.4) *There exists an i-filtration morphism,*

$$\curlywedge_q^{f,i} : \boxed{\mathfrak{F}_i^p} \otimes f^*\omega_S \longrightarrow \boxed{\mathfrak{F}_i^{p-q}} \otimes f^*\Omega_S^{k-q}.$$

(2.3.5) *The diagram,*

$$\begin{tikzcd}
\boxed{\mathfrak{F}_i^p} \otimes f^*\omega_S \otimes f^*\omega_S \arrow[r, "\curlywedge_q^{f,i}"] \arrow[d, "\curlywedge_{q+r}^{f,i}"'] & \boxed{\mathfrak{F}_i^{p-q}} \otimes f^*\Omega_S^{k-q} \otimes f^*\omega_S \arrow[d, "\curlywedge_r^{f,i}"] \\
\boxed{\mathfrak{F}_i^{p-q-r}} \otimes f^*\omega_S \otimes f^*\Omega_S^{k-q-r} \arrow[r, "\curlywedge_q^{\mathrm{id}}"'] & \boxed{\mathfrak{F}_i^{p-q-r}} \otimes f^*\Omega_S^{k-q} \otimes f^*\Omega_S^{k-r}
\end{tikzcd}$$

is commutative.

Now we are ready to define $\underline{\Omega}_{X/S}^p \in \mathrm{Ob}(D(X))$ for $p \in \mathbb{Z}, p \geq -k$. Let $\underline{\Omega}_{X/S}^p$ be the class of $\mathfrak{F}_{n-k-p}^{n-k-p} \otimes f^*\omega_S^{-(n-k-p)}$ in $\mathrm{Ob}(D(X))$ for $-k \leq p \leq n-k$, and let $\underline{\Omega}_{X/S}^p = 0$ for $p > n-k$. It is easy to see that

$$\underline{\Omega}_{X/S}^{n-k} = \Omega_X^n \otimes f^*\omega_S^{-1}$$

and that there is a distinguished triangle:

$$\underline{\Omega}_{X/S}^{n-k-1} \otimes f^*\omega_S \longrightarrow \Omega_X^{n-1} \longrightarrow \underline{\Omega}_{X/S}^{n-k} \otimes f^*\Omega_S^{k-1} \xrightarrow{+1}$$

In general for $j \geq p-n+k$ let $\mathbb{F}^j\Omega_X^p$ be the class of $\mathfrak{F}_{n-k-p+j}^{n-p} \otimes f^*\omega_S^{1-(n-k-p+j)}$ in $\mathrm{Ob}(D(X))$. The predistinguished triangle ($\star$) from [Kovács97a, 2.4],

$$\mathfrak{F}_{n-k-p+j+1}^{n-p} \otimes f^*\omega_S \longrightarrow \mathfrak{F}_{n-k-p+j}^{n-p} \otimes f^*\omega_S^2 \longrightarrow \mathfrak{F}_{n-k-p+j}^{n-k-p+j} \otimes f^*\omega_S \otimes f^*\Omega_S^j \xrightarrow{+1}$$

gives the distinguished triangle,

$$\mathbb{F}^{j+1}\Omega_X^p \longrightarrow \mathbb{F}^j\Omega_X^p \longrightarrow \underline{\Omega}_{X/S}^{p-j} \otimes f^*\Omega_S^j \xrightarrow{+1}$$

Now $\mathbb{F}^{k+1}\Omega_X^p = 0$ by (2.3.3) and by construction $\mathbb{F}^{p-n+k}\Omega_X^p = \Omega_X^p$. Observe that if $p-n+k < 0$, then $\mathbb{F}^0\Omega_X^p \simeq \mathbb{F}^{-1}\Omega_X^p \simeq \mathbb{F}^{p-n+k}\Omega_X^p = \Omega_X^p$, since $f^*\Omega_S^j = 0$ for $j < 0$. If $p-n+k \geq 0$, define $\mathbb{F}^j\Omega_X^p = \Omega_X^p$ for $j = 0, \ldots, p-n+k$.

Therefore [Kovács97a, 2.6] gives:

2.4 Theorem. *Let $f : X \to S$ be a morphism of smooth algebraic varieties of dimension n and k respectively. Then there exists an $\underline{\Omega}^r_{X/S} \in \mathrm{Ob}(D(X))$ for all $r \geq -k$ with the following property. For any $p \in \mathbb{N}$ there exists a hyperfiltration $\mathbb{F}^j\Omega^p_X$ of Ω^p_X with $j = 0, \ldots, k+1$, such that $\mathbb{F}^0\Omega^p_X \simeq \Omega^p_X$, $\mathbb{F}^{k+1}\Omega^p_X \simeq 0$ and*

$$\mathbb{G}^j\Omega^p_X \simeq \underline{\Omega}^{p-j}_{X/S} \otimes f^*\Omega^j_S.$$

Furthermore $\underline{\Omega}^r_{X/S} \simeq 0$ if $r > n-k$.

§3. The general case

The results regarding $\underline{\Omega}^{\cdot}_X$ that are essential in the sequel are summarized in the following theorem.

3.1 Theorem. [DuBois81], [GNPP88, III.1.12, V.3.6, V.5.1] *For every complex scheme X of dimension n there exists an $\underline{\Omega}^{\cdot}_X \in \mathrm{Ob}(D_{filt}(X))$ with the following properties.*

(3.1.1) *It is functorial, i.e., if $\phi : Y \to X$ is a morphism of complex schemes, then there exists a natural map ϕ^* of filtered complexes*

$$\phi^* : \underline{\Omega}^{\cdot}_X \to R\phi_*\underline{\Omega}^{\cdot}_Y.$$

Furthermore, $\underline{\Omega}^{\cdot}_X \in \mathrm{Ob}(D^b_{filt,coh}(X))$ and if ϕ is proper, then ϕ^* is a morphism in $D^b_{filt,coh}(X)$.

(3.1.2) *Let $\Omega^{\cdot}_X$ be the usual De Rham complex of Kähler differentials considered with the "filtration bête". Then there exists a natural map of filtered complexes*

$$\Omega^{\cdot}_X \to \underline{\Omega}^{\cdot}_X$$

and if X is smooth, it is a quasi-isomorphism.

(3.1.3) *Let $\{\nu_i : U_i \hookrightarrow X\}$ be a finite open cover of X. Then*

$$\underline{\Omega}^{\cdot}_X \simeq R\nu_{\cdot *}\underline{\Omega}^{\cdot}_{U_\cdot}.$$

(3.1.4) *Let $\underline{\Omega}^p_X = Gr^p_F\,\underline{\Omega}^{\cdot}_X[p]$. If X is projective and $\mathcal{L}$ is an ample line bundle on X, then*

$$\mathbb{H}^q(X, \underline{\Omega}^p_X \otimes \mathcal{L}) = 0 \qquad \text{for } p+q > n.$$

(3.1.5) *If $\varepsilon_\cdot : X_\cdot \to X$ is any hyperresolution of X, then $\underline{\Omega}^{\cdot}_X \simeq R\varepsilon_{\cdot *}\Omega^{\cdot}_{X_\cdot}$ and $\underline{\Omega}^p_X \simeq R\varepsilon_{\cdot *}\Omega^p_{X_\cdot}$.*

Now we turn to the relative case. Let $f : X \to S$ be a morphism of algebraic varieties of dimension n and k respectively, such that S is smooth. Let $\varepsilon_\cdot : X_\cdot \to X$ be a hyperresolution of X. By (2.4) there are complexes $\underline{\Omega}^p_{X_\cdot/S}$ and hyperfiltrations $\mathbb{F}_\cdot$ of $\Omega^p_{X_\cdot}$, such that

$$\mathbb{G}r^q_{\mathbb{F}_\cdot}\Omega^p_{X_\cdot} \simeq \underline{\Omega}^{p-q}_{X_\cdot/S} \otimes f^*_\cdot\Omega^q_S.$$

3.2 Definition. Let $\underline{\Omega}^p_{X/S} = R\varepsilon_{\cdot *}\underline{\Omega}^p_{X_\cdot/S}$ for $p \geq 0$ and define the hyperfiltrations in the obvious way: $\mathbb{F}^q\underline{\Omega}^p_X = R\varepsilon_{\cdot *}\mathbb{F}^q_\cdot\Omega^p_{X_\cdot}$

3.3 Lemma. *$\underline{\Omega}^p_{X/S}$ is independent of the hyperresolution chosen.*

Proof. Let α be a morphism of hyperresolutions.

$$\begin{array}{ccc} X'_{\cdot} & \xrightarrow{\alpha} & X''_{\cdot} \\ {\scriptstyle \varepsilon'_{\cdot}}\downarrow & & \downarrow{\scriptstyle \varepsilon''_{\cdot}} \\ X & \xrightarrow[\mathrm{id}_X]{} & X \end{array}$$

Then by [Kovács97a, 4.1] there exists a commutative diagram:

$$\begin{array}{ccccccc} R\varepsilon'_{\cdot}\mathbb{F}^{q+1}_{\cdot}\Omega^p_{X'_{\cdot}} & \longrightarrow & R\varepsilon'_{\cdot}\mathbb{F}^{q}_{\cdot}\Omega^p_{X'_{\cdot}} & \longrightarrow & R\varepsilon'_{\cdot}\mathbb{G}^{q}_{\cdot}\Omega^p_{X'_{\cdot}} & \xrightarrow{+1} & \\ \downarrow & & \downarrow & & \downarrow & & \\ R\varepsilon''_{\cdot}\mathbb{F}^{q+1}_{\cdot}\Omega^p_{X''_{\cdot}} & \longrightarrow & R\varepsilon''_{\cdot}\mathbb{F}^{q}_{\cdot}\Omega^p_{X''_{\cdot}} & \longrightarrow & R\varepsilon''_{\cdot}\mathbb{G}^{q}_{\cdot}\Omega^p_{X''_{\cdot}} & \xrightarrow{+1} & \end{array}$$

Now $R\varepsilon'_{\cdot}\Omega^p_{X'_{\cdot}} \simeq R\varepsilon''_{\cdot}\Omega^p_{X''_{\cdot}}$ by [DuBois81, 2.3] or [GNPP88, V.3.3] and the statement follows from [DuBois81, 2.1.4] and the construction of the hyperfiltration. □

3.4 Theorem. *Let $f : X \to S$ be a morphism of algebraic varieties of dimension n and k respectively, such that S is smooth. Then there exists an $\underline{\Omega}^r_{X/S} \in$* Ob$(D(X))$ *for all $r \geq -k$ with the following properties:*

(3.4.1) *$\underline{\Omega}^r_{X/S} \simeq 0$ for $r > n-k$, and if f is proper, then $\underline{\Omega}^r_{X/S} \in \mathrm{Ob}(D^b_{coh}(X))$ for every r.*

(3.4.2) *Let $p \in \mathbb{N}$, then there exists a hyperfiltration $\mathbb{F}^q\underline{\Omega}^p_X$ of $\underline{\Omega}^p_X$ for $q = 0, \dots, k+1$, such that $\mathbb{F}^0\underline{\Omega}^p_X \simeq \underline{\Omega}^p_X$, $\mathbb{F}^{k+1}\underline{\Omega}^p_X \simeq 0$ and*

$$\mathbb{G}^q\underline{\Omega}^p_X \simeq \underline{\Omega}^{p-q}_{X/S} \otimes f^*\Omega^q_S.$$

(3.4.3) *$\mathbb{F}^q\underline{\Omega}^p_X$ is functorial, i.e., if $\phi : Y \to X$ is an S-morphism, then there are natural maps in $D(X)$ forming the commutative diagram:*

$$\begin{array}{ccccccc} \mathbb{F}^{q+1}\underline{\Omega}^p_X & \longrightarrow & \mathbb{F}^{q}\underline{\Omega}^p_X & \longrightarrow & \mathbb{G}^{q}\underline{\Omega}^p_X & \xrightarrow{+1} & \\ \downarrow & & \downarrow & & \downarrow & & \\ R\phi_*\mathbb{F}^{q+1}\underline{\Omega}^p_Y & \longrightarrow & R\phi_*\mathbb{F}^{q}\underline{\Omega}^p_Y & \longrightarrow & R\phi_*\mathbb{G}^{q}\underline{\Omega}^p_Y & \xrightarrow{+1} & . \end{array}$$

In particular, there are natural maps in $D(X)$ forming the commutative diagram:

$$\begin{array}{ccc} \underline{\Omega}^p_X & \longrightarrow & \underline{\Omega}^p_{X/S} \\ \downarrow & & \downarrow \\ R\phi_*\underline{\Omega}^p_Y & \longrightarrow & R\phi_*\underline{\Omega}^p_{Y/S}. \end{array}$$

(3.4.4) *If f is smooth, then $\underline{\Omega}^p_{X/S} \simeq \Omega^p_{X/S}$.*

(3.4.5) *Let $\{\nu_i : U_i \hookrightarrow X\}$ be a finite open cover of X. Then*

$$\underline{\Omega}^p_{X/S} \simeq R\nu_{\cdot *}\underline{\Omega}^p_{U_{\cdot}/C}.$$

PROOF. (3.4.1) follows by definition.

$$R\varepsilon_{\cdot *}\left(\underline{\Omega}^{p-q}_{X_{\cdot}/S}\otimes f_{\cdot}^{*}\Omega^{q}_{S}\right)\simeq R\varepsilon_{\cdot *}\underline{\Omega}^{p-q}_{X_{\cdot}/S}\otimes f^{*}\Omega^{q}_{S},$$

since S is smooth, so (3.4.2) follows by (2.4). (3.4.3) is a simple consequence of the functoriality of $\underline{\Omega}^{\cdot}_{X}$ and the construction of $\mathbb{F}^{q}\underline{\Omega}^{p}_{X}$ (cf. [Kovács97a, 4.1]). The rest follows from (3.1) and the construction of $\mathbb{F}^{q}\underline{\Omega}^{p}_{X}$. □

3.5 COROLLARY. *Let $f : X \to S$ be a morphism of algebraic varieties of dimension n and k respectively, such that S is smooth. Then for any $p \geq 0$ and any locally free sheaf $\mathcal{E}$ there exists a spectral sequence*

$$E_1^{r,s}=\mathbb{H}^{r+s}(X,\underline{\Omega}^{p-r}_{X/S}\otimes f^{*}\Omega^{r}_{S}\otimes\mathcal{E})\Rightarrow\mathbb{H}^{r+s}(X,\underline{\Omega}^{p}_{X}\otimes\mathcal{E}).$$

PROOF. (3.4.2), (1.2.2). □

§4. A vanishing theorem and property SP

4.1 DEFINITION. Let $\mathcal{E}$ be a locally free sheaf of rank r. $\mathcal{E}$ will be called *semi-negative of splitting type* if $\mathcal{E}$ has a filtration

$$\mathcal{E}=F^{0}\supset F^{1}\supset\cdots\supset F^{r}=0$$

such that

$$F^{i-1}/F^{i}=\mathcal{L}_i$$

is a semi-negative line bundle, i.e., $\mathcal{L}_i^{-1}$ is nef.

4.2 THEOREM. *Let $\mathcal{L}$ be an ample line bundle and assume that $f^{*}\Omega_S$ is semi-negative of splitting type. Then*

$$\mathbb{H}^{i}(X,\underline{\Omega}^{l}_{X/S}\otimes f^{*}\omega_S\otimes\mathcal{L})=0\qquad \text{for } i+l>n-k.$$

PROOF. $\mathbb{H}^{i+1}(X,\underline{\Omega}^{j+k}_{X}\otimes\mathcal{L})=0$ by (3.1.4) and then the statement follows by the proofs of [Kovács97a, 3.3 and 3.4]. □

By (3.1.2) there exists a natural map $\rho:\mathcal{O}_X\to\underline{\Omega}^{0}_{X}$. This map composed with the map $\underline{\Omega}^{0}_{X}\to\underline{\Omega}^{0}_{X/S}$ given by (3.4.3) gives a natural map $\mathcal{O}_X\to\underline{\Omega}^{0}_{X/S}$ and it is functorial in the sense of (3.4.3).

4.3 DEFINITION. Let $f : X \to S$ be a morphism of algebraic varieties, such that S is smooth. f will be called an *SP-morphism* (or f will be said to have *property SP*) if the natural map $\rho:\mathcal{O}_X\to\underline{\Omega}^{0}_{X/S}$ has a left inverse, i.e., there exists a map, $\tilde{\rho}:\underline{\Omega}^{0}_{X/S}\to\mathcal{O}_X$ in $D(X)$ such that $\tilde{\rho}\circ\rho:\mathcal{O}_X\to\mathcal{O}_X$ is a quasi-isomorphism. In particular every smooth morphism is an SP-morphism by (3.4.4).

4.4 Proposition. *SP is a local property in the following sense: Let $f : X \to S$ be a morphism of algebraic varieties, such that S is smooth and $\{\nu_i : U_i \hookrightarrow X\}$ a finite open cover of X in the complex topology such that $f : U_i \to S$ is an SP-morphism for all i. Then f is an SP-morphism.*

Proof. [GNPP88, III.1.12(v)] (cf. [Kovács96, 1.1]), (3.4.5), and the assumption imply that

$$\mathcal{O}_X \simeq R\nu_{\cdot *}\mathcal{O}_{U.} \to R\nu_{\cdot *}\underline{\Omega}^0_{U./C} \simeq \underline{\Omega}^0_{X/S}$$

has a left inverse. □

4.5 Proposition. *Let $f : X \to S$ be a morphism of algebraic varieties, such that S is smooth, $\phi : Y \to X$ a morphism such that $f \circ \phi$ is an SP-morphism, and $\mathcal{O}_X \to R\phi_*\mathcal{O}_Y$ has a left inverse. Then f is an SP-morphism.*

Proof. By functoriality there is a commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_X & \longrightarrow & \underline{\Omega}^0_{X/S} \\ \big\downarrow & & \big\downarrow \\ R\phi_*\mathcal{O}_Y & \longrightarrow & R\phi_*\underline{\Omega}^0_{Y/S}. \end{array}$$

The bottom horizontal and the left vertical arrows have a left inverse by assumption, therefore the natural map $\mathcal{O}_X \to \underline{\Omega}^0_{X/S}$ has a left inverse as well. □

4.6 Corollary. *Assume that X has rational singularities and there exists a resolution of singularities of X, $\phi : Y \to X$, such that $f \circ \phi$ is smooth. Then f is an SP-morphism.* □

Now we can formulate two important corollaries of (4.2).

4.7 Corollary. *Let $f : X \to S$ be an SP-morphism of projective algebraic varieties of dimension n and k respectively, such that S is smooth. Let $\mathcal{L}$ be an ample line bundle and assume that $f^*\Omega_S$ is semi-negative of splitting type. Then*

$$H^i(X, f^*\omega_S \otimes \mathcal{L}) = 0 \qquad \textit{for } i > n-k.$$

Proof. $\mathbb{H}^i(X, \underline{\Omega}^0_{X/S} \otimes f^*\omega_S \otimes \mathcal{L}) = 0$ for $i > n-k$ by (4.2), and $H^i(X, f^*\omega_S \otimes \mathcal{L})$ is a direct summand of this group by the definition of property SP. □

4.8 Corollary. *Let $f : X \to S$ be an SP-morphism of projective algebraic varieties of dimension n and k respectively, such that S is smooth, $k > 0$. Assume that $f^*\Omega_S$ is semi-negative of splitting type. Then $\omega_{X/S}$ is not ample.*

Proof. $H^n(X, \omega_X) \neq 0$. □

§5. Families of varieties of general type

5.1 Condition. Let $f : X \to S$ be a morphism of proper algebraic varieties. Assume the following:
(5.1.1) f is flat and projective;
(5.1.2) X is Gorenstein;
(5.1.3) S is smooth;
for all $s \in S$:
(5.1.4) X_s is reduced, with only canonical singularities;
(5.1.5) ω_{X_s} is ample.

5.2 Remark. Note that ω_{X_s} is a line bundle by (5.1.2) and by [Stevens88, Proposition 7] these conditions imply:

(5.1.6) X has only canonical singularities.

Note also that (5.1) is the same as [Kovács97, 2.1].

5.3 Theorem. *Let $g : Y \to S$ be a smooth family of projective varieties of general type with nef canonical bundle. Assume that $\alpha : S \to A$ is a smooth Fano fibre space over the abelian variety A. Then the fibres of g are birational.*

Proof. By Poincaré's Complete Reducibility Theorem [Birkenhake-Lange92, Ch. 5 (3.7)] there exists an isogeny $B \to A$ where B is the product of irreducible abelian varieties. Thus applying appropriate base changes one may assume that A itself is an irreducible abelian variety.

Let
$$f : X = \mathbf{Proj}_S \sum g_* \omega^m_{Y/S} \to S.$$

Then $f : X \to S$ satisfies (5.1), and $\phi : Y \to X$ is a resolution of singularities of X.

Fano varieties are rationally connected by [KMM92b, 3.3] (cf. [Campana92]), so S_a is rationally connected for alll $a \in A$. Let $\mathbb{P}^1 = S' \to S_a \subseteq S$ be a rational curve in S_a. Then $f_{S'} : X_{S'} \to S'$ satisfies (5.1), and $Y_{S'}$ is a resolution of singularities of $X_{S'}$, so by (5.2) and (4.6) $f_{S'} : X_{S'} \to S'$ is an SP-morphism, and then it is isotrivial by [Kovács96, Theorem 2]. Therefore $f_a : X_a \to S_a$ is isotrivial for all $a \in A$.

Rationally connected varieties are simply connected by [Campana91, 3.5], so $X_a = X_s \times S_a$, and then

$$h^0(X_a, \omega^m_{X_a/S_a}) = h^0(X_s, \omega^m_{X_s}).$$

On the other hand $h^0(X_s, \omega^m_{X_s})$ is independent of s by Riemann-Roch and Kawamata-Viehweg vanishing for $m \geq 2$, hence $h^0(X_a, \omega^m_{X_a/S_a})$ is independent of a. Therefore $(\alpha \circ f)_* \omega^m_{X/S}$ is a locally free sheaf on A for $m \geq 2$. Let

$$h : Z = \mathbf{Proj}_A \sum (\alpha \circ f)_* \omega^m_{X/S} \to A,$$

so one has the commutative diagram,

$$\begin{tikzcd} Y \arrow[r, "\phi"] \arrow[d, "g"'] & X \arrow[r, "\beta"] \arrow[d, "f"'] & Z \arrow[d, "h"'] \\ S \arrow[r, equal] & S \arrow[r, "\alpha"'] & A. \end{tikzcd}$$

The fibres of $\beta : X \to Z$ are smooth Fano varieties, so

$$\mathcal{O}_Z \simeq R\beta_* \mathcal{O}_X \simeq R(\beta \circ \phi)_* \mathcal{O}_Y.$$

Since $\alpha \circ g$ is smooth, h is an SP-morphism by (4.5). Then h is isotrivial by (4.8) and [Kovács97, 2.6]. Hence f is isotrivial, since $Z_a \simeq X_s$ for all $a \in A$, $s \in S_a$. Therefore the statement follows as X_s is the canonical model of Y_s for all $s \in S$. $\square$

5.4 Corollary. *Let $g : Y \to S$ be a smooth family of projective varieties of general type with nef canonical bundle. Assume that Ω_S is semi-negative. Then the fibres of g are birational.*

Proof. By [DPS94, Main Theorem] a finite étale cover of S is a smooth Fano fibre space over an abelian variety. $\square$

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