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On the minimal number of singular fibres in a family of surfaces of general type

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The following question was raised in Catanese-Schneider [2], 4.1.

0.1. Question. *Let Y be a smooth variety of general type, $g : Y \to \mathbb{P}^1$ a fibration. Is it true that g has at least 3 singular fibres?*

The answer is affirmative, when $\dim Y = 2$ by Beauville [1]. The purpose of this note is to prove the same when $\dim Y = 3$. In this case Migliorini [10] has already proved that g cannot be smooth, and in fact that article gave the initial inspiration to this work.

0.2. Theorem. *Let Y be a smooth threefold of general type, $g : Y \to \mathbb{P}^1$ a fibration. Then g has at least 3 singular fibres.*

The proof consists of two parts. In the first part a vanishing theorem is proved for the top cohomology group of certain line bundles (cf. 1.1). In the second part it is shown that if there exists a fibration of a smooth threefold of general type over $\mathbb{P}^1$ with at most two singular fibres, then one can construct another threefold admitting a fibration over $\mathbb{P}^1$ with at most two singular fibres such that the canonical bundle of the new threefold contains a line bundle, $\mathscr{L}$, such that the vanishing theorem of the first part can be applied to $\mathscr{L}$. Finally this leads to a contradiction, since the top cohomology group of the canonical bundle does not vanish.

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Definitions and notation. Throughout the article the groundfield is $\mathbb{C}$, the field of complex numbers.

A divisor D on a scheme X is called $\mathbb{Q}$-Cartier if mD is Cartier for some $m > 0$. X is said to have $\mathbb{Q}$-factorial singularities if every Weil divisor on X is $\mathbb{Q}$-Cartier.

A $\mathbb{Q}$-Cartier divisor D is called *ample* if mD is ample. It is called *nef* if $D \cdot C \geqq 0$ for every proper curve $C \subset X$. D is called *big* if X is proper and $|mD|$ gives a birational map for some $m > 0$. In particular ample implies nef and big.

Let $f: X \to S$ be a morphism of schemes. A $\mathbb{Q}$-Cartier divisor D on X is called *f-nef* if $D \cdot C \geqq 0$ for every proper curve $C \subset X$ such that $f(C)$ is a point.

A normal variety X is said to have *canonical* (resp. *terminal*) *singularities* if K_X is $\mathbb{Q}$-Cartier and for any resolution of singularities $\pi : \tilde{X} \to X$, with the collection of exceptional prime divisors $\{E_i\}$, there exist $a_i \in \mathbb{Q}$, $a_i \geqq 0$ (resp. $a_i > 0$) such that $K_{\tilde{X}} \equiv \pi^* K_X + \sum a_i E_i$ (cf. [5]). X is called a *canonical variety* if it has only canonical singularities and K_X is ample. X is called a *minimal variety* if it has only terminal singularities and K_X is nef.

Let $g : Y \to C$ be a morphism of normal varieties, then $K_{Y/C} = K_Y - g^* K_C$, similarly $\omega_{Y/C} = \omega_Y \otimes g^* \omega_C^{-1}$.

Let $f: X \to S$ be a morphism of schemes, then X_s denotes the fibre of f over the point $s \in S$ and f_s denotes the restriction of f to X_s. More generally, for a morphism $Z \to S$, let $f_Z : X_Z = X \times_S Z \to Z$. If f is composed with another morphism $g : S \to T$, then X_t denotes the fibre of $g \circ f$ over the point $t \in T$, i.e., $X_t = X_{S_t}$.

$Z_1(X)$ denotes the free abelian group generated by the irreducible reduced curves on X and $Z_1(X)_{\mathbb{R}} = Z_1(X) \otimes_{\mathbb{Z}} \mathbb{R}$. All cycles numerically equivalent to the zero cycle form a subgroup of $Z_1(X)_{\mathbb{R}}$ and the quotient is denoted by $N_1(X)_{\mathbb{R}}$.

The effective 1-cycles generate a subsemigroup $NE(X) \subset N_1(X)_{\mathbb{R}}$. It is called the *cone of curves of X*. The *closed cone of curves* of X, denoted by $\overline{NE}(X)$, is the closure of $NE(X)$ in the Euclidean topology of $Z_1(X)_{\mathbb{R}}$.

If D is an arbitrary $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, then $\overline{NE}(X)_{D \geqq 0}$ denotes the set of vectors $\xi \in \overline{NE}(X)$ such that $\xi \cdot D \geqq 0$.

The class of a curve C in $N_1(X)_{\mathbb{R}}$ is denoted by $[C]$.

§ 1. A vanishing lemma

1.1. Lemma. *Let $g : Y \to C$ be a fibration of a smooth n-dimensional projective variety Y over a smooth projective curve C. Let $\Delta \subset C$ be the set of points over which g is not smooth and assume that $D = g^* \Delta$ is a normal crossing divisor. Further let $\phi : Y \to X$ be a morphism to a projective variety X and $f: X \to C$ a morphism such that $g = f \circ \phi$ and $\dim \phi^{-1}(x) \leqq 1$ for all $x \in X \setminus f^{-1}(\Delta)$. Let $\mathscr{L}$ be a line bundle on Y such that there exists an ample line bundle $\mathscr{A}$ on X and a natural number $\nu \in \mathbb{N}$ such that $\mathscr{L}^\nu \simeq \phi^* \mathscr{A}$. Assume further that $\mathscr{A} \otimes f^* \omega_C(\Delta)^{-\nu(n-1)}$ is also ample. Then*

$$H^n(Y, \mathscr{L} \otimes g^* \omega_C) = 0 .$$

Proof. Taking exterior powers of the sheaves of logarithmic differentials one has the following short exact sequences for all $p = 0, \dots, n-1$:

$$0 \to \Omega_{Y/C}^{p-1}(\log D) \otimes g^* \omega_C(\Delta) \to \Omega_Y^p(\log D) \to \Omega_{Y/C}^p(\log D) \to 0 .$$

Define $\mathscr{L}_p = \mathscr{L} \otimes g^* \omega_C(\Delta)^{p-(n-1)}$. Then the above short exact sequence yields:

$$0 \to \Omega^{p-1}_{Y/C}(\log D) \otimes \mathscr{L}^{-1}_{p-1} \to \Omega^p_Y(\log D) \otimes \mathscr{L}^{-1}_p \to \Omega^p_{Y/C}(\log D) \otimes \mathscr{L}^{-1}_p \to 0\,.$$

Now

$$\mathscr{L}^\nu_p = \mathscr{L}^\nu \otimes g^*\omega_C(\Delta)^{\nu(p-(n-1))} \simeq \phi^*\big(\mathscr{A} \otimes f^*\omega_C(\Delta)^{\nu(p-(n-1))}\big),$$

where

$$\mathscr{A} \otimes f^*\omega_C(\Delta)^{\nu(p-(n-1))} \simeq \underbrace{\mathscr{A}}_{\text{ample}} \otimes \big(f^*\omega_C(\Delta)^{-1}\big)^{\nu(n-1-p)}$$

$$\simeq \underbrace{\big(\mathscr{A} \otimes f^*\omega_C(\Delta)^{-\nu(n-1)}\big)}_{\text{ample}} \otimes f^*\omega_C(\Delta)^{\nu p}$$

is ample, since either $\omega_C(\Delta)$ or $\omega_C(\Delta)^{-1}$ is nef. Then $H^{n-p-1}(Y, \Omega^p_Y(\log D) \otimes \mathscr{L}^{-1}_p) = 0$ by Esnault-Viehweg [3], 6.7, so the map,

$$H^{n-p-1}(Y, \Omega^p_{Y/C}(\log D) \otimes \mathscr{L}^{-1}_p) \to H^{n-(p-1)-1}(Y, \Omega^{p-1}_{Y/C}(\log D) \otimes \mathscr{L}^{-1}_{p-1})$$

is injective for all p, so in fact

$$H^0(Y, \Omega^{n-1}_{Y/C}(\log D) \otimes \mathscr{L}^{-1}_{n-1}) \to H^{n-1}(Y, \Omega^0_{Y/C}(\log D) \otimes \mathscr{L}^{-1}_0),$$

i.e.,

$$H^0(Y, \omega_{Y/C} \otimes \mathscr{L}^{-1}) \to H^{n-1}(Y, \mathscr{L}^{-1}_0)$$

is injective. $H^{n-1}(Y, \mathscr{L}^{-1}_0) = 0$ by the Kawamata-Viehweg vanishing theorem, and then $H^0(Y, \omega_{Y/C} \otimes \mathscr{L}^{-1}) = 0$. Now the statement follows by Serre duality. □

1.1.1. Remark. The role of the last condition in the lemma is to simplify the statement. In fact (1) if $\omega_C(\Delta)$ is nef, then replacing $\mathscr{L}$ with $\mathscr{L} \otimes g^*\omega_C(\Delta)^{n-1}$ one actually changes the ample line bundle $\mathscr{A}$ to $\mathscr{A} \otimes f^*\omega_C(\Delta)^{\nu(n-1)}$, so for this new line bundle the last condition is satisfied, i.e., $(\mathscr{A} \otimes f^*\omega_C(\Delta)^{\nu(n-1)}) \otimes f^*\omega_C(\Delta)^{-\nu(n-1)} = \mathscr{A}$ is ample. Therefore the lemma gives that

$$H^n(Y, \mathscr{L} \otimes g^*\omega_C(\Delta)^{n-1} \otimes g^*\omega_C) = 0\,.$$

(2) If $\omega_C(\Delta)^{-1}$ is nef, then $\mathscr{A}$ ample implies that $\mathscr{A} \otimes f^*\omega_C(\Delta)^{-\nu(n-1)}$ is also ample, so the last condition is vacuous.

§ 2. Proof of the theorem

Let Y be a smooth projective threefold of general type, $g: Y \to \mathbb{P}^1$ a fibration, and $\Delta \subset \mathbb{P}^1$ a subset such that g is smooth over $\mathbb{P}^1 \setminus \Delta$.

Suppose $\# \Delta \leqq 2$. By semi-stable reduction (cf. [6]) we may assume that $g^*\Delta$ is a reduced normal crossing divisor.

First run the relative Minimal Model Program on $g: Y \to \mathbb{P}^1$ (cf. [5], [11]). Let the first step of the program be the blowing down of (-1)-curves of the smooth fibres $\mu: Y \to Y_1$

(cf. [7]). Let $g_1 : Y_1 \to \mathbb{P}^1$ be the new fibration. Then K_{Y_1} is g_1-nef over $\mathbb{P}^1 \backslash \Delta$ and there exists an effective Cartier divisor E on Y such that

$$K_Y \equiv \mu^* K_{Y_1} + E.$$

Now if $\overline{NE}(Y_1/\mathbb{P}^1) \nsubseteq \overline{NE}(Y_1/\mathbb{P}^1)_{K_{Y_1} \geqq 0}$, then $\overline{NE}(Y_1/\mathbb{P}^1)$ contains a Mori-extremal ray, which leads to either a divisorial contraction or a flip, both centered over Δ. This way one obtains a new projective threefold with only $\mathbb{Q}$-factorial terminal singularities which is fibred over $\mathbb{P}^1$ and it only differs from Y_1 over a point of Δ. Therefore the above step can be repeated until one arrives to a projective threefold Y_2 with only $\mathbb{Q}$-factorial terminal singularities, a fibration $g_2 : Y_2 \to \mathbb{P}^1$ smooth over $\mathbb{P}^1 \backslash \Delta$, such that K_{Y_2} is g_2-nef.

By construction we have a rational map $\varrho : Y_1 \dashrightarrow Y_2$ such that $g_1 = g_2 \circ \varrho$ and $\varrho|_{g_1^{-1}(\mathbb{P}^1 \backslash \Delta)}$ is an isomorphism. Let F (resp. F_1, F_2) denote the linear equivalence class of a fibre of g (resp. g_1, g_2). Observe that ϱ is defined on the complement of a curve contained in $g^{-1}(\Delta)$, so one can pull back $\mathbb{Q}$-Cartier divisors via ϱ. Since Y_2 has terminal singularities, there exist natural numbers r and a_i such that

$$K_{Y_1} \equiv \varrho^* K_{Y_2} + \sum \frac{a_i}{r} E_i,$$

where E_i are exceptional Cartier divisors of ϱ (contained in $g_1^{-1}(\Delta)$).

Next observe that by the proofs of Migliorini [10], 3.1 and 3.2, the complete linear system of $mK_{Y_2/\mathbb{P}^1}$ is base point free for every sufficiently large and divisible m, it induces the stable pluricanonical morphism on the smooth fibres, and separates the fibres of g_1 over $\mathbb{P}^1 \backslash \Delta$, so if $C \subset Y_2$ is a proper curve that is not contained in a fibre of g_2, then $K_{Y_2/\mathbb{P}^1} \cdot C \geqq 1$.

Now by [5], 4-2-4 (cf. [4], [8])

$$\overline{NE}(Y_2) = \overline{NE}(Y_2)_{K_{Y_2} \geqq 0} + \sum_{i=1}^{k} \mathbb{R}^+[C_i].$$

Note that these C_i curves are not contained in any fibre or else they could be included in the first term of this sum. Hence $K_{Y_2/\mathbb{P}^1} \cdot C_i \geqq 1$.

Let $C \subset Y_2$ be an arbitrary proper curve. Then

$$[C] = [C'] + \sum_{i=1}^{k} \alpha_i [C_i]$$

such that $K_{Y_2} \cdot C' \geqq 0$ and $\alpha_i \geqq 0$ for all $i = 1, \ldots, k$. Then for all $m \gg 0$

$$\left(K_{Y_2/\mathbb{P}^1} - \frac{2}{m} F_2\right) \cdot C = \left(K_{Y_2} + \left(2 - \frac{2}{m}\right) F_2\right) \cdot C' + \sum_{i=1}^{k} \alpha_i \left(K_{Y_2/\mathbb{P}^1} - \frac{2}{m} F_2\right) \cdot C_i \geqq 0$$

for all C, so $K_{Y_2/\mathbb{P}^1} - \frac{2}{m} F_2$ is nef. Choosing m large enough one can guarantee that $K_{Y_2/\mathbb{P}^1} - \frac{2}{m} F_2$ is also big. Then $\left| \nu \left(K_{Y_2/\mathbb{P}^1} - \frac{2}{m} F_2 \right) \right|$ is base point free for every sufficiently large and divisible ν.

Now let $C \subset Y_2$ be a proper curve that is not contained in a fibre of g_2,

$$[C] = [C'] + \sum_{i=1}^{k} \alpha_i [C_i]$$

as above. If there exists an i such that $\alpha_i \neq 0$, then

$$\left(K_{Y_2/\mathbb{P}^1} - \frac{2}{m} F_2 \right) \cdot C \geqq \left(K_{Y_2/\mathbb{P}^1} - \frac{2}{m} F_2 \right) \cdot \alpha_i C_i > 0\,,$$

and if $C \in \overline{NE}(Y_2)_{K_{Y_2} \geqq 0}$, then

$$\left(K_{Y_2} + \left(2 - \frac{2}{m} \right) F_2 \right) \cdot C > 0\,,$$

since $F_2 \cdot C > 0$.

Hence $K_{Y_2/\mathbb{P}^1} - \frac{2}{m} F_2$ is positive on every curve that is not contained in a fibre. In particular the morphism given by the complete linear system $\left| \nu \left(K_{Y_2/\mathbb{P}^1} - \frac{2}{m} F_2 \right) \right|$ separates the fibres.

Next consider the m-th root cover $\sigma_0 : \mathbb{P}^1 \to \mathbb{P}^1$ whose branch locus contains Δ. Let $\tilde{Y} = Y \times_{\sigma_0} \mathbb{P}^1$. Then one has the following commutative diagram:

$$\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\ \sigma\ } & Y \\
 & & \downarrow{\scriptstyle \mu} \\
 & & Y_1 \\
{\scriptstyle \tilde{g}}\big\downarrow & & \downarrow{\scriptstyle \varrho} \\
 & & Y_2 \\
 & & \downarrow{\scriptstyle g_2} \\
\mathbb{P}^1 & \xrightarrow[\ \sigma_0\ ]{} & \mathbb{P}^1\,.
\end{array}$$

$\tilde{Y}$ is normal, since $g^* \Delta$ is a reduced normal crossing divisor and then $\tilde{Y}$ has only canonical Gorenstein singularities. Let $\tilde{F}$ denote the linear equivalence class of a fibre of $\tilde{g}$. Then $\sigma^* F = m\tilde{F}$. By the construction there exists a set of Cartier divisors $\tilde{E}_i$ such that $\sigma^* \mu^* E_i = m\tilde{E}_i$, where E_i are the exceptional divisors of ϱ (contained in $g^{-1}(\Delta)$).

By the Hurwitz formula

$$\begin{aligned}
K_{\tilde{Y}} &\equiv \sigma^* K_Y + 2(m-1)\tilde{F} \\
&\equiv \sigma^* K_Y + 2(m-1)\frac{1}{m}\sigma^* F \\
&\equiv \sigma^*\left(K_Y + \left(2 - \frac{2}{m}\right)F\right) \\
&\equiv \sigma^*\left(\mu^*\left(K_{Y_1} + \left(2 - \frac{2}{m}\right)F_1\right) + E\right) \\
&\equiv \sigma^*\left(\mu^*\left(\varrho^*\left(K_{Y_2} + \left(2 - \frac{2}{m}\right)F_2\right) + \sum \frac{a_i}{r} E_i\right) + E\right) \\
&\equiv \sigma^*\mu^*\varrho^*\left(K_{Y_2/\mathbb{P}^1} - \frac{2}{m}F_2\right) + \sum \frac{m a_i}{r}\tilde{E}_i + \sigma^* E .
\end{aligned}$$

Now if m is divisible by r, then $\sigma^*\mu^*\varrho^*\left(K_{Y_2/\mathbb{P}^1} - \frac{2}{m}F_2\right)$ is Cartier. Let $\pi : \hat{Y} \to \tilde{Y}$ be a resolution of singularities such that π is an isomorphism over $\mathbb{P}^1 \setminus \Delta$ and $\pi^*\tilde{g}^*\Delta$ is a normal crossing divisor. Let $\hat{g} = \tilde{g} \circ \pi$. Then $\omega_{\hat{Y}}$ contains a nef and big line bundle, $\mathscr{L}$, such that $\mathscr{L}^\nu$ is globally generated for some $\nu > 0$, and there exists a commutative diagram

$$\begin{array}{ccc}
\hat{Y} & \xrightarrow{\ \phi\ } & X \\
{\scriptstyle \hat{g}}\downarrow & & \downarrow{\scriptstyle f} \\
\mathbb{P}^1 & \xrightarrow{\ \mathrm{id}\ } & \mathbb{P}^1
\end{array}$$

such that $\mathscr{L}^\nu \simeq \phi^*\mathscr{A}$ for some ample line bundle $\mathscr{A}$ on X and ϕ induces the stable pluricanonical morphism on every $\hat{Y}_P$ for $P \in \mathbb{P}^1 \setminus \Delta$, so $\dim \phi^{-1}(x) \leqq 1$ for all

$$x \in X \setminus f^{-1}(\Delta).$$

Then $H^3(\hat{Y}, \mathscr{L} \otimes \hat{g}^*\omega_{\mathbb{P}^1}) = 0$ by (1.1). That would however imply $H^3(\hat{Y}, \omega_{\hat{Y}}) = 0$, a contradiction. □

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