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Number of automorphisms of principally polarized abelian varieties

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ABSTRACT. Explicit estimates on the size of the automorphism group of principally polarized abelian varieties are given adapting methods of Catanese-Schneider and Szabó for log varieties of general type.

There has been a considerable amount of work devoted to giving estimates on the size of the automorphism groups of varieties of general type. The recent articles [Catanese-Schneider95, Szabó96, Xiao94/95] and the references contained there provide ample material on this subject.

The purpose of this note is to point out that the methods of the above authors and their predecessors can be used to obtain estimates in more general situations, namely for log varieties of log general type. Note that among many others principally polarized abelian varieties belong to this class.

The first section contains the necessary modifications of the known arguments to the log case. The main result is (1.3.1). The general results are applied for principally polarized abelian varieties in the second section. The estimates are clearly far from being sharp, but the idea of viewing a principally polarized abelian variety as a log variety of general type may be worth mentioning.

ACKNOWLEDGEMENT. Most of the ideas behind the results in this note come from other people. My modest claim of novelty is simply the realization that these ideas apply under broader circumstences than previously used. In particular this article benefited greatly from the papers of Catanese and Schneider, Szabó, and in an indirect way from that of Huckleberry and Sauer. The results included in the first section are due to these authors as much as or perhaps more than to me.

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DEFINITIONS AND NOTATION. Every object is defined over an algebraically closed field of characteristic $p \geq 0$.

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A line bundle $\mathcal{L}$ on X is called *big* if X is proper and the global sections of $\mathcal{L}^m$ define a birational map for some $m > 0$. $\mathcal{L}$ is called *nef* if $\deg(\mathcal{L}|_C) \geq 0$ for every proper curve $C \subset X$. In particular every ample line bundle is nef and big.

In this paper a *log variety* (X, D) consists of a proper variety X and an effective divisor D, called the *boundary*. A *log morphism* takes boundary to boundary, in particular the image of the boundary is a divisor. (X, D) is of *log general type* if $K_X + D$ is big. It is *log canonically polarized* if $K_X + D$ is ample. Note that in this case X is necessarily projective.

For a log variety, (X, D), $\operatorname{Aut}(X, D)$ (resp. $\operatorname{Bir}(X, D)$) denotes the set of automorphisms (resp. proper birational automorphisms) of X that leave D fixed. If $X \subseteq \mathbb{P}^N$, then $\operatorname{Lin}(X, D) = \operatorname{Lin}(X) \cap \operatorname{Lin}(D)$ denotes the set of linear automorphisms that leave both X and D fixed. A smooth log variety will mean simply that X is smooth. Note that this is very different from the notion of log smooth.

Let G be a finite group and $g \in G$. Then $|G|$ (resp. $o(g)$) denotes the order of G (resp. the order g).

§1. Log varieties

§§1.1 Log canonical embeddings. Let (X, D) be a log canonically polarized smooth log variety of dimension n. Let $m \in \mathbb{N}$ be such that the complete linear system of $(m+1)K_X + mD$ gives a birational morphism $\phi : X \to \mathbb{P}^N$ for some $N > 0$ such that $\bar{D} = \phi(D)$ is still a divisor in $\bar{X} = \phi(X)$, i.e., $(\bar{X}, \bar{D})$ is a log variety and ϕ is a log morphism. This can be achieved for instance by requiring that ϕ separate points.

1.1.1 FACT. *In characteristic* 0, $|(m+1)K_X + mD|$ *separates the points of* X *as soon as* $m \geq \binom{n+2}{2}$ *by* [Angehrn-Siu95] (*cf.* [Kollár97, 5.8]).

Observe that any linear automorphism of $\mathbb{P}^N$ leaving the pair $(\bar{X}, \bar{D})$ fixed induces a proper birational automorphism of (X, D). In fact ϕ^* gives an isomorphism between $\operatorname{Lin}(\bar{X}, \bar{D})$ and $\operatorname{Bir}(X, D)$. Now $\operatorname{Bir}(X, D)$ is finite by [Iitaka82, 11.12], hence so is $\operatorname{Lin}(\bar{X}, \bar{D})$ and in order to estimate $|\operatorname{Aut}(X, D)|$ it is enough to estimate $|\operatorname{Lin}(\bar{X}, \bar{D})|$.

1.1.2 LEMMA. *Let* (X, D) *be a log canonically polarized smooth log variety of dimension* n *and* $\phi : X \to \mathbb{P}^N$ *a birational log morphism given by the complete linear system* $|(m+1)K_X + mD|$. *Let* $\bar{X} = \phi(X)$ *and* $\bar{D} = \phi(D)$.

(a) If D *is nef, then*

$$\deg \bar{X} \leq ((m+1)(K_X + D))^n$$
$$\deg \bar{D} \leq ((m+1)(K_X + D))^n.$$

(b) If K_X *is nef, then*

$$\deg \bar{D} \leq \deg \bar{X} = ((m+1)K_X + mD)^n.$$

PROOF. (a) In this case we actually prove that

$$\deg \bar{X} + \deg \bar{D} \leq ((m+1)(K_X + D))^n.$$

$(m+1)(K_X + D)$, D, and $(m+1)K_X + mD$ are nef, so for all $i = 1, \ldots, n-1$,

$$((m+1)K_X + mD)^i \cdot D \cdot ((m+1)K_X + mD)^{n-i-1} \geq 0,$$

and hence $\deg \bar{X} + \deg \bar{D} = ((m+1)K_X + mD)^n + D \cdot ((m+1)K_X + mD)^{n-1} = (m+1)(K_X + D) \cdot ((m+1)K_X + mD)^{n-1} = (m+1)(K_X + D) \cdot ((m+1)K_X + mD) \cdot ((m+1)K_X + mD)^{n-2} \leq ((m+1)(K_X + D))^2 \cdot ((m+1)K_X + mD)^{n-2} \leq \cdots \leq ((m+1)(K_X + D))^n$.

(b) If K_X is nef, then so is $(m+1)K_X + (m-1)D = 2K_X + (m-1)(K_X + D)$, hence $\deg \bar{D} = D \cdot ((m+1)K_X + mD)^{n-1} \leq ((m+1)K_X + mD)^n = \deg \bar{X}$. □

§§**1.2 Abelian groups of automorphisms.** Obtaining bounds for abelian groups of automorphisms requires much less than the general case. The following lemma is an adaptation of [Szabó96, Bézout Lemma] to the log case.

1.2.1 LEMMA. *Let $(\bar{X}, \bar{D})$ be a projective log variety of dimension n with a fixed embedding $\bar{X} \subseteq \mathbb{P}^N$ and T a closed reduced subgroup of the projective linear group. Assume that* $\operatorname{Lin}(\bar{X}, \bar{D}) \cap T$ *is finite and that there exists a point $x \in \bar{X}$ such that its stabilizer, T_x, is trivial and its orbit, Tx, is open and dense in $\mathbb{P}^N$.*

Then $\operatorname{Lin}(\bar{X}, \bar{D}) \cap T$ *has at most* $(\deg \bar{X})^l \cdot (\deg \bar{D})^{n+1-l}$ *elements for some* $1 \leq l \leq n+1$.

PROOF. Identify Tx with T. Let $U = X \cap T$ and $V = D \cap T$. U is open and dense in X and V is open in D. Note that V may be empty.

An arbitrary $t \in T$ is in $\operatorname{Lin}(X, D)$ if and only if $ta \in U$, i.e., $t \in Ua^{-1}$ for all $a \in U$ and $tb \in V$, i.e., $t \in Vb^{-1}$ for all $b \in V$. Equivalently,

$$\operatorname{Lin}(X, D) \cap T = \bigcap_{a \in U} Ua^{-1} \cap \bigcap_{b \in V} Vb^{-1}.$$

$\operatorname{Lin}(X, D) \cap T$ is assumed to be finite so one can find $a_0, \ldots, a_{l-1} \in U$ and $b_l, \ldots, b_n \in V$ for some $1 \leq l \leq n+1$ such that $\left(\cap_{i=0}^{l-1} Ua_i^{-1} \right) \cap \left(\cap_{j=l}^{n} Vb_j^{-1} \right)$ is already finite. By Bézout's theorem this proves the statement. □

1.2.2 THEOREM. *Let $(\bar{X}, \bar{D})$ be a projective log variety of dimension n with a fixed embedding $\bar{X} \subseteq \mathbb{P}^N$. Assume that* $\operatorname{Lin}(\bar{X}, \bar{D})$ *is finite. Let* $G \leq \operatorname{Lin}(\bar{X}, \bar{D})$ *be an abelian subgroup whose order is coprime to p, the characteristic of the ground field and* $t \in \operatorname{Lin}(\bar{X}, \bar{D})$ *an arbitrary element. Then both $|G|$ and $o(t)$ is at most* $(\deg \bar{X})^l \cdot (\deg \bar{D})^{n+1-l}$ *for some* $1 \leq l \leq n+1$.

PROOF. This follows easily by the arguments of [Szabó96, Theorem 3 and Abelian Lemma] using (1.2.1) in place of Szabó's Bézout Lemma. □

1.2.3 THEOREM. *Let (X, D) be a log canonically polarized smooth variety of dimension n and m the smallest integer such that $|(m+1)K_X + mD|$ separates the points of X. Let* $G \leq \operatorname{Aut}(X, D)$ *be an abelian subgroup and* $t \in \operatorname{Aut}(X, D)$ *an arbitrary element. Let π denote the p-part of $|G|$, i.e., $|G|/\pi$ is an integer coprime to p, the characteristic of the ground field.*

(a) If D is nef, then

$$|G| \leq \pi \left((m+1)^n (K_X + D)^n \right)^{n+1}$$
$$o(t) \leq \left((m+1)^n (K_X + D)^n \right)^{n+1}.$$

(b) If K_X is nef, then

$$|G| \leq \pi \left(((m+1) K_X + mD)^n \right)^{n+1}$$
$$o(t) \leq \left(((m+1) K_X + mD)^n \right)^{n+1}.$$

Furthermore in characteristic 0, $m \leq \binom{n+2}{2}$ *by* (1.1.1).

PROOF. (a) Using the birational morphism given by $|(m+1)K_X + mD|$ one obtains a projective log variety $(\bar{X}, \bar{D})$ that satisfies the conditions of (1.2.2).

Then by (1.1.2),

$$(\deg \bar{X})^l \cdot (\deg \bar{D})^{n+1-l} \leq ((m+1)^n (K_X + D)^n)^{n+1}$$

for any $1 \leq l \leq n+1$.

(b) If K_X is nef, then $(\deg \bar{X})^l \cdot (\deg \bar{D})^{n+1-l} \leq (\deg \bar{X})^{n+1}$ by (1.1.2). □

1.2.4 COROLLARY. *Under the assumptions of* (1.2.3), $|\operatorname{Aut}(X, D)|$ *is coprime to* p *as soon as* $p > \left((m+1)^n (K_X + D)^n\right)^{n+1}$.

§§**1.3 The general case.** The general case, that is when $\operatorname{Aut}(X, D)$ is not necessarily abelian, relies on group theoretical considerations. In particular the results cited below use the classification of finite simple groups. I have used [Szabó96] as a reference, but many of the ideas behind his results already appear in [Huckleberry-Sauer90] and possibly in the works of others.

1.3.1 THEOREM. *Let* (X, D) *be a log canonically polarized smooth variety of dimension* n *over an algebraically closed field of characteristic* $p \geq 0$, *and* m *the smallest integer such that* $|(m+1)K_X + mD|$ *separates the points of* X. *Let* π *denote the* p*-part of* $|\operatorname{Aut}(X, D)|$, *i.e.,* $|\operatorname{Aut}(X, D)|/\pi$ *is an integer coprime to* p, *the characteristic of the ground field.*

(a) If D *is nef, then*

$$|\operatorname{Aut}(X, D)| \leq \pi^{3^n} \left((m+1)^n (K_X + D)^n\right)^{16n3^n}.$$

(b) If K_X *is nef, then*

$$|\operatorname{Aut}(X, D)| \leq \pi^{3^n} \left(((m+1)K_X + mD)^n\right)^{16n3^n}.$$

Furthermore in characteristic 0, $m \leq \binom{n+2}{2}$ *by* (1.1.1).

PROOF. Follows from (1.2.3) and [Szabó96, Main bound]. □

REMARK. It is natural to allow some mild singularities when studying log varieties. In fact the arguments of this section stay valid if (X, D) is only assumed to be a klt pair. The estimates however become somewhat worse as one have to include the index of X. (For the definition of klt see [Kollár97]).

One may also weaken the assumptions requiring $K_X + D$ be only semi-ample and big. In that case however the Angehrn-Siu bound have to be replaced by something larger, e.g., Kollár's effective base point freeness bound.

§2. Principally polarized abelian varieties

In this section the general results are applied for principally polarized abelian varieties. It is very likely that by direct methods one can improve the actual estimates. According to an argument of J.-P. Serre [Previato] a recent result of Feit can be used to obtain significantly better estimates in characteristic 0. That argument however does not work in positive characteristic.

2.1 THEOREM. *Let* (A, Θ) *be a principally polarized abelian variety of dimension* g *over an algebraically closed field of characteristic* $p \geq 0$, $G \leq \operatorname{Aut}(A, \Theta)$ *an abelian subgroup and* π *the* p*-part of* $|\operatorname{Aut}(A, \Theta)|$. *Then*

$$|G| \leq \pi(3^g g!)^{g+1}$$

and

$$|\operatorname{Aut}(A, \Theta)| \leq \pi^{3^g}(3^g g!)^{16g3^g}$$

PROOF. 3Θ is very ample by [Mumford74, 17]. Then by (1.3.1) $|\operatorname{Aut}(A, \Theta)| \leq \pi^{3^g}((3\Theta)^g)^{16g3^g}$. Finally $\Theta^g = g!$ by Riemann-Roch. The estimates now follow from (1.2.3) and (1.3.1). $\square$

2.2 COROLLARY. *Under the conditions of* (2.1) *if* $p = 0$ *or* $p > (3^g g!)^{g+1}$, *then* $\pi = 1$, *i.e.,*

$$|\operatorname{Aut}(A, \Theta)| \leq (3^g g!)^{16g3^g}$$

PROOF. The order of any $t \in \operatorname{Aut}(A, \Theta)$ is at most $(3^g g!)^{g+1}$ by (1.2.3). $\square$

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