Sándor Kovács

Moduli Theory and Singularities

In this talk I will discuss recent advances in the moduli theory of higher-dimensional algebraic varieties. Of course, half of the words in that sentence merit explanation, so that will be included in the talk as well. Some of these advances concern the singularities that appear on stable varieties and their influence on the geometry of the corresponding moduli spaces…yet more words to explain.

The roots of moduli theory can be traced to a short remark of Bernhard Riemann in his 1857 treatise on abelian functions (nowadays we would say compact Riemann surfaces). This remark suggests that the space of equivalence classes of compact complex Riemann surfaces of genus $g > 1$ can be parametrized by $3g - 3$ complex parameters, which Riemann called *moduli*.

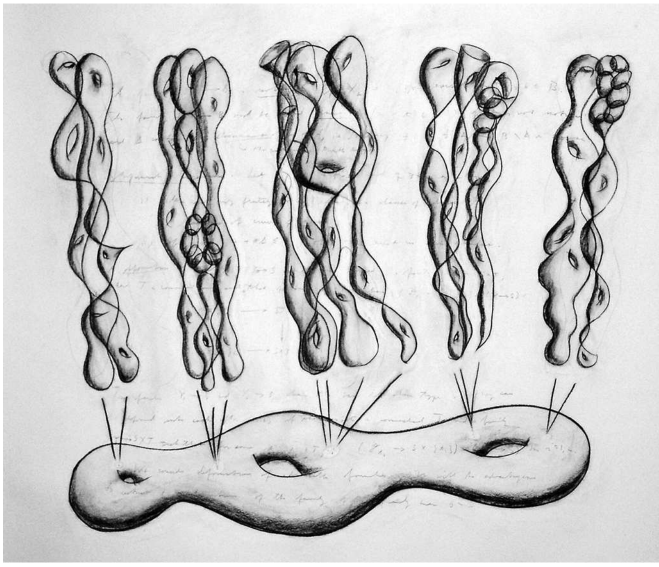

Figure 1. Riemann's $3g - 3$ dimensional moduli space of compact Riemann surfaces of genus g.

In other words, fix a compact connected orientable (topological) surface of genus g and consider the various ways one can equip it with a complex structure. According to this remark of Riemann, the possible complex structures can be described by $3g - 3$ parameters. Or, in modern language, the space of those complex structures is $3g - 3$-dimensional. After Riemann, these spaces are called *moduli spaces*.

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Lars Ahlfors, one of the 1936 Fields medalists and one of the first to embark on the quest to make Riemann's remark rigorous, said in his 1962 ICM address: "Riemann's classical problem of moduli is not a problem with a single aim, but rather a program to obtain maximum information about a whole complex of questions which can be viewed from several different angles."

An algebraic approach to moduli spaces was pioneered by David Mumford, a 1974 Fields Medalist. When viewed algebraically over the complex numbers, a Riemann surface is one-dimensional. Hence it is called an *algebraic curve*. Taking advantage of the algebraic point of view, Mumford extended the moduli problem to include degenerations of these algebraic curves, i.e., Riemann surfaces with singularities (as in Figure 2). This is a delicate matter, as allowing arbitrary singularities would lead to an intractable problem. In contrast, it is possible to improve the singularities of the degenerate fibers without changing the smooth fibers of a given family of curves. For example, consider the family of degree 5 plane curves with equations $x^5 - y^2 + t(5x^3 - 4x - 4) = 0$ (as in Figure 3), parametrized by t. As long as $t \neq 0$, the above equation defines a smooth algebraic curve, but for $t = 0$ the defined curve, $x^5 - y^2 = 0$ (denoted by red in Figure 3), is singular. However, this degeneration can be improved considerably by making a change of variables given by $y = x^2 z$. This will not affect the smooth members of the family, i.e., the ones with $t \neq 0$, but will replace the singular member by the curve defined by the equation $x^4(x - z^2) = 0$. Making another change of variables given by $x = zw$ leads to the equation $z^5 w^4 (z - w)$. The curve defined by this equation is the union of three lines, much simpler than the original singular degeneration.

Stable singularities in higher dimensions are much more diverse and complicated.

Mumford realized that a similar process can always be used to improve the singularities of the degenerate fibers. This led to the definition of *stable curves*, which are algebraic curves whose only allowable singularities are transversal intersections of smooth branches, such as in the curve in Figure 4 defined by the equation $y^2 = x^2(x+1)$. Note that this picture only shows the real points of this curve. All complex points and the true topology of this curve are shown by the green complex curve on the right-hand side in Figure 2.

In addition to giving an algebraic construction for the moduli space of smooth algebraic curves, Mumford also succeeded in constructing a moduli space for stable curves and proving that this moduli space can be equipped with the structure of a projective variety with the moduli space of smooth algebraic curves as a dense open set.

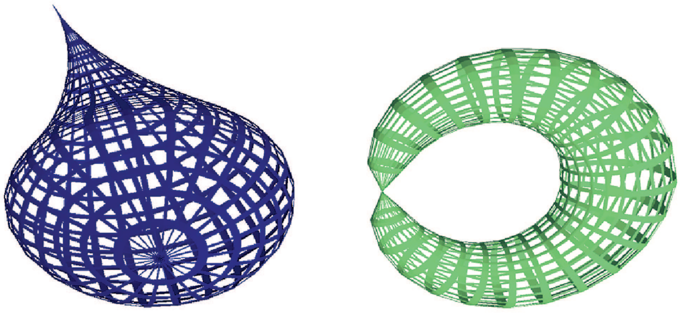

Figure 2. Mumford extended the moduli problem to include Riemann surfaces with singularities.

Ever since Mumford's seminal work, many properties of these moduli spaces have been studied. The various applications discovered are so numerous and broad ranging that it would be impossible to list them in a concise manner. In fact, several disciplines grew out of the study of moduli problems, as shown by the number of MSC classification categories devoted to such disciplines.

Naturally, the question arises whether something similar is possible for higher-dimensional varieties as in Figure 5. Interestingly, this question was not answered satisfactorily until recently, and to some extent it is still not answered completely. There are several reasons for this. While the singularities of stable curves are the most simple curve singularities—the transversal intersection of two smooth branches—stable singularities in higher dimensions are much more diverse and complicated. The families considered in the moduli problem are also more complicated. A stable family of curves is simply a family of stable curves, but this is no longer true in higher dimensions. A stable family, beyond being a family of stable objects, has further properties which reflect conditions on the family, not only on its fibers. In this talk I will discuss these intriguing issues along with the most recent related results.

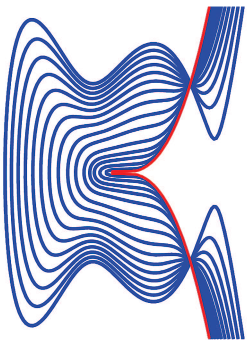

Figure 3. This family of plane curves can be reparametrized to make the components of the singular red one smooth.

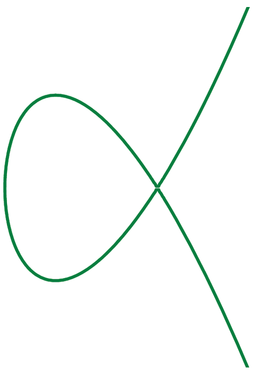

Figure 4. A stable curve, such as $y^2 = x^2(x+1)$, is one with no singularities except for transversal self-intersections.

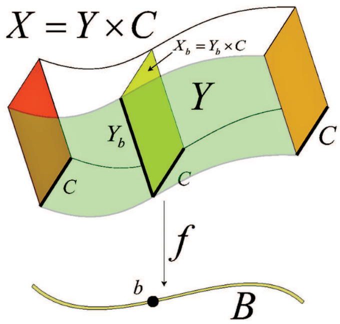

Figure 5. Constructing moduli spaces of higher dimensional varieties leads to new challenges.

Image Credits

Figure 1, courtesy of Lun-Yi Tsai, shows Lun-Yi Tsai's *Shafarevich's Conjecture*, `lunyitsai.com/demonstrations/collaborations.htm`.

Figures 2–5, and author photo courtesy of Sándor Kovács.

Sándor Kovács

ABOUT THE AUTHOR

When **Sándor Kovács** is not thinking about algebraic geometry, he enjoys swimming, biking, running, and hiking. One of his current goals is to improve his butterfly technique. He is also working toward a perfect headstand.

Dimitri Shlyakhtenko

A (Co)homology Theory for Subfactors and Planar Algebras

I am very grateful to be speaking on my joint work with S. Vaes and S. Popa [3].

I would like to start with a construction that a priori has nothing to do with subfactors or planar algebras. Let S^2_n denote the two-sphere with $n+1$ distinct points $p_1, \dots, p_{n+1}$ removed and let $\delta \geq 2$ be a fixed real number. Let V_n be the linear space whose basis consists of isotopy classes of zero or more closed curves drawn on S^2_n, subject to the relation that if a curve bounds a disk that does not include any other curves or the points p_j, then the curve can be removed up to a multiplicative factor δ. Figure 1 shows two equalities of diagrams on S^2_2. The equalities are due to the relation $\bigcirc = \delta$ and the fact that the drawing is on a sphere.

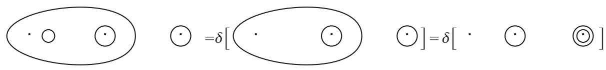

Figure 1. Removing a trivial curve is equivalent to multiplication by δ. The second equality holds because the drawing is on a sphere.

We define a differential complex structure on $(V_n)_{n\geq 0}$. Let $\alpha^{(n)}_j : S^2_n \to S^2_{n-1}$ be the map in which the point p_j is glued back into S^2_n and the remaining points are renumbered (in order) as $p_1, \dots, p_n$. Let $\epsilon^{(n)}_j : V_n \to V_{n-1}$ be the map in which a collection of curves on S^2_n is redrawn on S^2_{n-1} via the map $\alpha^{(n)}_j$. Then $\partial_n = \sum (-1)^k \epsilon^{(n)}_k$ satisfies $\partial_n \circ \partial_{n+1} = 0$, and so we can define a sequence of homology spaces $H_n = \ker \partial_n / \operatorname{im} \partial_{n+1}$.

The next challenge is to compute these spaces. One is tempted to do this "by hand"; indeed for $n = 0, 1, 2$ one can easily describe all arrangements of curves on S^2_n. Some amount of computation then shows that $H_0 = \mathbb{C}$ and $H_1 = H_2 = 0$. However, for $n \geq 3$ things get complicated, and I am actually not aware of an easy combinatorial computation, even of H_3.

To compute H_n we need to reinterpret them by redrawing and making apparent the connection with *Temperley-Lieb-Jones diagrams*; see Figure 2.

Leaving Temperley-Lieb-Jones diagrams aside for the moment, let us consider what happens if we permit other types of elements x as in Figure 2. Suppose that G is a group with a finite generating set X. For each element

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