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# Families over a base with a birationally nef tangent bundle

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It is a well-known consequence of the Torelli theorem that a smooth projective family of curves of genus at least 2 over a projective rational or elliptic curve is isotrivial, that is, the fibres of the family are isomorphic. Since the automorphism group of a curve of genus at least 2 is finite, this also implies that the family becomes trivial after a finite base change.

The above statement was generalized for smooth projective families of minimal surfaces of general type in [Migliorini95], and for smooth projective families of varieties (of arbitrary dimension) with ample canonical bundle in [Kovács96]. Both articles studied families over curves.

The aim of this article is to present a further generalization, namely let the base of the family have arbitrary dimension.

**0.1 Theorem = 4.2 Theorem.** *Let $f : X \to S$ be a smooth morphism of projective algebraic varieties such that the canonical bundle of every fibre of $f$ is ample. Assume that $S$ is birational to an $S'$ with $\Omega_{S'}$ semi-negative. Then $f$ is isotrivial.*

Smooth varieties with nef tangent bundle have been studied by several authors. Note that the results in this article are formulated for the cotangent bundle instead of the tangent bundle.

There are three essential parts of the proof. One is an Arakelov type result, proved in Sect. 2, which asserts that if $S$ is an abelian variety, then there is a reduction to another family which has an ample relative canonical bundle and the same variation in moduli. In particular if $S$ is an irreducible abelian variety, then either $\omega_{X/S}$ is ample or $f$ is isotrivial. The key ingredients are positivity results of [Kollár87] and [Esnault-Viehweg91].

The second essential part is a vanishing theorem, proved in Sect. 1, which in particular implies that $\omega_{X/S}$ cannot be ample, therefore proving (0.1) for an abelian base.

The third part, carried out in Sect. 3 and Sect. 4, consists of several reduction steps. The main theme of them is constructing a new family over a base for which the statement is already known, with the property that the set of isomorphism classes of the fibres is preserved.

Finally these yield (0.1) by the structure theorem of [DPS92] and because Fano varieties are rationally connected (cf. [KMM92b]).

*0.1.1 Remark.* The Arakelov type theorem and some of the reduction steps do not require $f$ to be smooth, and accordingly they are proved in a more general setting. Smoothness of $f$ is essential for the vanishing theorem.

One cannot expect (0.1) to be valid in the same generality as the other parts, since generic Lefschetz pencils provide counterexamples. However, it is expected that the ampleness assumption on the canonical bundle of the fibres can be weakened to that the fibres are varieties of general type with nef canonical bundle. This actually would follow if the vanishing theorem of Sect. 1 were extended to nef and big line bundles. The proof presented in Sect. 1, however, requires ampleness since the Kodaira-Akizuki-Nakano vanishing theorem is not valid for line bundles that are only nef and big.

*Definitions and Notation.* Throughout the article the groundfield will always be $\mathbb{C}$, the field of complex numbers.

A divisor $D$ on a scheme $X$ is called $\mathbb{Q}$-Cartier if $mD$ is Cartier for some $m > 0$.

A normal variety $X$ is said to have *canonical singularities* if $K_X$ is $\mathbb{Q}$-Cartier and for any resolution of singularities $\pi : \tilde{X} \to X$, with the collection of exceptional prime divisors $\{E_i\}$, there exist $a_i \in \mathbb{Q}$, $a_i \geqq 0$ such that $K_{\tilde{X}} = \pi^* K_X + \sum a_i E_i$ (cf. [CKM88]). $X$ is called a *canonical variety* if it has only canonical singularities and $K_X$ is ample.

A singularity is called *Gorenstein* if its local ring is a Gorenstein ring. A variety is *Gorenstein* if it admits only Gorenstein singularities. In particular, the dualizing sheaf of a Gorenstein variety is locally free.

A locally free sheaf $\mathscr{E}$ on a scheme $X$ is called *semi-positive* (or *nef*) if for every smooth complete curve $C$ and every map $\gamma : C \to X$, any quotient bundle of $\gamma^* \mathscr{E}$ has non-negative degree. $\mathscr{E}$ is called *semi-negative* if $\check{\mathscr{E}}$, the dual of $\mathscr{E}$, is semi-positive. $\mathrm{Sym}^l(\mathscr{E})$ denotes the $l$-th symmetric power of $\mathscr{E}$, and $\det \mathscr{E}$ the determinant bundle of $\mathscr{E}$, i.e., $\det \mathscr{E} = \bigwedge^r \mathscr{E}$ if $r = \mathrm{rk}\, \mathscr{E}$.

Let $f : X \to S$ be a morphism of schemes, then $X_s$ denotes the fibre of $f$ over the point $s \in S$ and $f_s$ denotes the restriction of $f$ to $X_s$. More generally, for a morphism $Z \to S$, let $f_Z : X_Z = X \times_S Z \to Z$. If $f$ is composed with another morphism $g : S \to T$, then $X_t$ denotes the fibre of $g \circ f$ over the point $t \in T$, i.e., $X_t = X_{S_t}$.

$f$ is called *isotrivial* if $X_s \simeq X_t$ for every $s, t \in S$.

A smooth projective variety $X$ is called a *Fano variety* if $-K_X$ is ample. $X$ is a *Fano fibre space* over $S$ if the fibres of $f$ are connected Fano varieties.

A proper variety $X$ is called *rationally connected* if two arbitrary points of $X$ can be joined by an irreducible rational curve (cf. [KMM92a], [Campana91]).

$\Omega_X$ denotes the sheaf of differentials on $X$, $\Omega_{X/S}$ is the sheaf of relative differentials. $\Omega_X^p = \bigwedge^p \Omega_X$, $\Omega_{X/S}^p = \bigwedge^p \Omega_{X/S}$, and $\omega_{X/S} = \omega_X \otimes g^* \omega_S^{-1}$.

$k(s)$ denotes the residue field at $s \in S$. $h^i(S, \mathscr{F})$ is the dimension of $H^i(S, \mathscr{F})$. $\pi_1(X, \star)$ denotes the fundamental group of $X$ with an unspecified base point.

## 1 The vanishing theorem

The following well-known fact is included for ease of reference.

**1.1 Fact.** *Let $\mathscr{E}$ be a locally free sheaf of rank $r$. Assume that there is a filtration*

$$\mathscr{E} = F^0 \supset F^1 \supset \cdots \supset F^r = 0$$

*of $\mathscr{E}$ such that*

$$F^{i-1}/F^i = \mathscr{L}_i$$

*is a line bundle for all $i = 1, \ldots, r$. Then for every $1 \leqq t \leqq r$ there is a filtration*

$$\bigwedge^t \mathscr{E} = F_t^0 \supset F_t^1 \supset \cdots \supset F_t^{\binom{r}{t}} = 0$$

*of $\bigwedge^t \mathscr{E}$ such that*

$$F_t^{i-1}/F_t^i = \mathscr{L}_{i_1} \otimes \mathscr{L}_{i_2} \otimes \cdots \otimes \mathscr{L}_{i_t}$$

*for all $i = 1, \ldots, \binom{r}{t}$ and a suitable set of indices $1 \leqq i_1 < i_2 < \cdots < i_t \leqq r$.* □

**1.2 Definition.** *Let $\mathscr{E}$ be a locally free sheaf of rank $r$. $\mathscr{E}$ will be called* **semi-negative of splitting type** *if $\mathscr{E}$ has a filtration*

$$\mathscr{E} = F^0 \supset F^1 \supset \cdots \supset F^r = 0$$

*such that*

$$F^{i-1}/F^i = \mathscr{L}_i$$

*is a semi-negative line bundle, i.e., $\mathscr{L}_i^{-1}$ is nef.*

**1.3 Lemma.** *Let $f : X \to S$ be a smooth morphism of projective algebraic varieties of dimension $n$ and $k$ respectively. Assume that $f^*\Omega_S$ is semi-negative of splitting type. Let $i, j \in \mathbb{N}$ be natural numbers such that $i + j + k \geqq n$. Assume further that for every natural number $l > j$ and for every ample line bundle $\mathscr{L}$ on $X$,*

$$H^i(X, \Omega_{X/S}^l \otimes f^*\omega_S \otimes \mathscr{L}) = 0\ .$$

*Then*

$$H^{i+1}(X, \Omega_{X/S}^j \otimes f^*\omega_S \otimes \mathscr{L}) = 0$$

*for every ample line bundle $\mathscr{L}$.*

*Proof.* The standard exact sequence of $f$

$$0 \to f^*\Omega_S \to \Omega_X \to \Omega_{X/S} \to 0$$

yields a filtration on $\Omega_X^{j+k} \otimes \mathscr{L}$ which in turn induces a spectral sequence

$$E_1^{r,s} = H^{r+s}(X, \Omega_{X/S}^{j+k-r} \otimes f^*\Omega_S^r \otimes \mathscr{L}) \Rightarrow H^{r+s}(X, \Omega_X^{j+k} \otimes \mathscr{L}) .$$

Since $i+j+k \geqq n$, $H^{i+1}(X, \Omega_X^{j+k} \otimes \mathscr{L}) = 0$ by the Kodaira-Akizuki-Nakano vanishing theorem. Hence $E_\infty^{r,i+1-r} = 0$ for all $r$. In particular $E_\infty^{k,i+1-k} = 0$. Suppose now that

$$E_1^{k,i+1-k} = H^{i+1}(X, \Omega_{X/S}^j \otimes f^*\omega_S \otimes \mathscr{L}) \neq 0 .$$

Observe that $E_w^{u,v} = 0$ for every $u > k$ and arbitrary $v, w$, so in order to have $E_\infty^{k,i+1-k} = 0$, there must be a $t \geqq 1$ such that $E_t^{k-t,i-k+t} \neq 0$. Then $E_1^{k-t,i-k+t} \neq 0$ for the same $t$, so $l = j+t > j$ is such that

$$E_1^{k-t,i-k+t} = H^i(X, \Omega_{X/S}^l \otimes f^*\Omega_S^{k-t} \otimes \mathscr{L}) \neq 0 .$$

Now by assumption $f^*\Omega_S$ has a filtration

$$f^*\Omega_S = F^0 \supset F^1 \supset \cdots \supset F^k = 0$$

such that $F^{i-1}/F^i = \mathscr{L}_i$ and $\mathscr{L}_i^{-1}$ is a nef line bundle. Then by (1.1) there exist $i_1, \ldots, i_{k-t}$ such that

$$H^i(X, \Omega_{X/S}^l \otimes \mathscr{L}_{i_1} \otimes \cdots \otimes \mathscr{L}_{i_{k-t}} \otimes \mathscr{L}) \neq 0 .$$

Therefore

$$H^i(X, \Omega_{X/S}^l \otimes f^*\omega_S \otimes \mathscr{L} \otimes \mathscr{L}_{i_{k-t+1}}^{-1} \otimes \cdots \otimes \mathscr{L}_{i_k}^{-1}) \neq 0 ,$$

for $\{i_{k-t+1}, \ldots, i_k\} = \{1, \ldots, k\} \setminus \{i_1, \ldots, i_{k-t}\}$.

Since $\mathscr{L} \otimes \mathscr{L}_{i_{k-t+1}}^{-1} \otimes \cdots \otimes \mathscr{L}_{i_k}^{-1}$ is ample, this non-vanishing violates the assumption. Hence the statement follows. □

**1.4 Theorem.** *Let $f : X \to S$ be a smooth morphism of projective algebraic varieties of dimension $n$ and $k$ respectively. Let $\mathscr{L}$ be an ample line bundle and assume that $f^*\Omega_S$ is semi-negative of splitting type. Then*

$$H^i(X, \Omega_{X/S}^l \otimes f^*\omega_S \otimes \mathscr{L}) = 0 \quad \textit{for } i + l > n - k .$$

*Proof.* $\Omega_{X/S}$ is a locally free sheaf of rank $n-k$, so

$$H^i(X, \Omega_{X/S}^l \otimes f^*\omega_S \otimes \mathscr{L}) = 0$$

for $l > n-k$ and $i \geqq 0$. Then

$$H^{i+1}(X, \Omega_{X/S}^{n-k} \otimes f^*\omega_S \otimes \mathscr{L}) = 0$$

for $i \geqq 0$ by (1.3). Hence

$$H^i(X, \Omega^l_{X/S} \otimes f^*\omega_S \otimes \mathscr{L}) = 0$$

for $l > n-k-1$ and $i \geqq 1$. Then

$$H^{i+1}(X, \Omega^{n-k-1}_{X/S} \otimes f^*\omega_S \otimes \mathscr{L}) = 0$$

for $i \geqq 1$ by (1.3). Hence

$$H^i(X, \Omega^l_{X/S} \otimes f^*\omega_S \otimes \mathscr{L}) = 0$$

for $l > n-k-2$ and $i \geqq 2$.

Iterating this process one sees that

$$H^i(X, \Omega^l_{X/S} \otimes f^*\omega_S \otimes \mathscr{L}) = 0$$

for $i + l > n - k$. □

**1.5 Corollary.** *Let $f : X \to S$ be a smooth morphism of projective algebraic varieties of dimension n and k respectively. Let $\mathscr{L}$ be an ample line bundle and assume that $f^*\Omega_S$ is semi-negative of splitting type. Then*

$$H^i(X, f^*\omega_S \otimes \mathscr{L}) = 0 \quad \textit{for } i > n-k\ .$$

**1.6 Corollary.** *Let $f : X \to S$ be a smooth morphism of projective algebraic varieties of dimension n and k respectively, $k > 0$. Assume that $f^*\Omega_S$ is semi-negative of splitting type. Then $\omega_{X/S}$ is not ample.*

*Proof.* $H^n(X, \omega_X) \neq 0$. □

*1.6.1 Remark.* In fact (1.4) is true in a more general setting, but that generalization will not be used in this article. Using the Kodaira-Akizuki-Nakano vanishing theorem for $s$-ample line bundles (cf. [Shiffman-Sommese85, 3.36]) one can replace the condition $\mathscr{L}$ being ample by it being $s$-ample and change the condition on $i, l$ respectively. Another, trivial improvement can be made by observing that since $f^*\Omega^j_S \otimes f^*\omega_S^{-1}$ has a filtration whose successive quotients are nef line bundles, $f^*\omega_S$ can be replaced by $f^*\Omega^j_S$ for any $j$. Hence the proofs of (1.3) and (1.4) with the changes indicated above give:

**1.7 Theorem.** *Let $f : X \to S$ be a smooth morphism of projective algebraic varieties of dimension n and k respectively. Let $\mathscr{L}$ be an s-ample line bundle, $j \in \mathbb{N}$, and assume that $f^*\Omega_S$ is semi-negative of splitting type. Then*

$$H^i(X, \Omega^l_{X/S} \otimes f^*\Omega^j_S \otimes \mathscr{L}) = 0 \quad \textit{for } i + l > n - k + s\ .$$

**1.8 Corollary.** *Let $f : X \to S$ be a smooth morphism of projective algebraic varieties of dimension n and k respectively. Assume that $f^*\Omega_S$ is semi-negative of splitting type. Then $\omega_{X/S}$ is not $(k-1)$-ample.*

## 2 Families over abelian varieties

In the rest of the article the following condition will be used often:

**2.1 Condition.** Let $f:X \to S$ be a morphism of algebraic varieties. Assume the following:

(2.1.1) $f$ is flat, projective;
(2.1.2) $X$ is Gorenstein;
(2.1.3) $S$ is smooth;

and for all $s \in S$:

(2.1.4) $X_s$ is reduced, with only canonical singularities;
(2.1.5) $\omega_{X_s}$ is ample.

Note that $\omega_{X_s}$ is a line bundle by (2.1.2) and by [Stevens88, Prop. 7] these conditions also imply:

(2.1.6) $X$ has only canonical singularities.

*2.1.1 Remark.* If $f$ is smooth, then (2.1) reduces to:

(2.1.1′) $f$ is projective;
(2.1.5) $\omega_{X_s}$ is ample for all $s \in S$.

**2.2 Lemma.** *Let $f:X \to K$ be a morphism satisfying* (2.1)*. Assume that* $\det f_*\omega^m_{X/A} \in \mathrm{Pic}^\circ(K)$ *for some $m \gg 0$. Then $f$ is isotrivial.*

*Proof.* First let $\dim K = 1$. Suppose $f$ is not isotrivial. Then by the proofs of [Matsusaka-Mumford64, Theorems 1, 2] (cf. [Kollár87, 2.3(ii)] and [Kovács96, 2.16]), there is no open subset of $K$ over which $f$ is isotrivial. Then by [Kollár87, Theorem on p. 363] $\det f_*\omega^\eta_{X/A}$ is ample for some $\eta > 0$. However, this implies that $\det f_*\omega^m_{X/A}$ is ample for all $m \gg 0$ by [Esnault-Viehweg91, 0.1], leading to a contradiction. Therefore $f$ must be isotrivial.

For $\dim K > 1$ take a general hyperplane section of $K$ and use induction. □

Let $f:X \to A$ be a morphism satisfying (2.1) such that $A$ is an abelian variety. Fix an $m \gg 0$. Let $\mathscr{L} = \det f_*\omega^m_{X/A}$. Let $a \in A$ and $t_a$ the translation by $a$. Let $\hat{A}$ denote the dual variety of $A$ and

$$\begin{aligned} \phi_{\mathscr{L}}: A &\to \hat{A} \\ a &\mapsto t_a^*\mathscr{L} \otimes \mathscr{L}^{-1} . \end{aligned}$$

Next let $K$ be the connected component containing 0 of $\ker \phi_{\mathscr{L}}$ and finally let

$$\phi: A \to B = A/K .$$

Observe that $\mathscr{L}|_K \in \mathrm{Pic}^\circ(K)$, so (2.2) implies the following:

**2.3 Corollary.** *Let $a_1, a_2 \in A$ be such that $\phi(a_1) = \phi(a_2)$. Then $X_{a_1} \simeq X_{a_2}$.*

The next lemma provides an important reduction step for an abelian base. It allows one to regard only families with $\det f_*\omega^m_{X/A}$ ample.

**2.4 Lemma.** *There exists a commutative diagram*:

$$
\begin{array}{ccccc}
X & \longleftarrow & \tilde{X}_{\tilde{A}} & \longrightarrow & Y \\
{\scriptstyle f}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle g} \\
A & \underset{\eta}{\longleftarrow} & \tilde{A} & \underset{\psi}{\longrightarrow} & B
\end{array}
$$

*such that*

(2.4.1) $\eta$ *is étale*,

(2.4.2) $\psi = \phi \circ \eta$,

(2.4.3) $g$ *satisfies* (2.1),

(2.4.4) $Y_b \simeq X_a$ *for every* $b \in B$ *and* $a \in A_b$.

*Furthermore*, *if* $f$ *is smooth*, *then so is* $g$.

*Proof.* For every $b \in B$ let $F_b = X_a$ for any $a \in A_b$. By (2.3) this is well-defined and $X_b \to A_b$ is a locally trivial fibre bundle with fibre $F_b$. Hence there is a representation of the free abelian group $\pi_1(A_b, \star)$:

$$\rho_b : \pi_1(A_b, \star) \to \mathrm{Aut}(F_b)\ .$$

Let $A'_b \to A_b$ be the finite étale cover corresponding to the subgroup

$$\ker\ \rho_b \subset \pi_1(A_b, \star)\ .$$

Then $X_{A'_b} \to A'_b$ is a trivial fibre bundle, namely

$$X_{A'_b} = F_b \times A'_b\ .$$

By [Szabó96, Theorem 4] (cf. [Kobayashi72, III.2.1]) there exists an $N \in \mathbb{N}$, independent of $b$, such that

$$|\mathrm{Aut}(F_b)| \leqq N\ .$$

Set $M = N!$, then $\rho_b$ factors through

$$\rho_b : \pi_1(A_b, \star) \to \pi_1(A_b, \star)/M \cdot \pi_1(A_b, \star) \to \mathrm{Aut}(F_b)\ .$$

Therefore the finite étale cover $\eta''_b : A''_b \to A_b$ corresponding to the subgroup

$$M \cdot \pi_1(A_b, \star) \subset \pi_1(A_b, \star)$$

factors through $A'_b \to A_b$, so $X_{A''_b} \to A''_b$ is a trivial fibre bundle as well, i.e.,

$$X_{A''_b} = F_b \times A''_b\ .$$

Next consider the finite étale cover $\eta : \tilde{A} \to A$ corresponding to the subgroup

$$M \cdot \pi_1(A, \star) \subset \pi_1(A, \star)\ .$$

Clearly $\eta_{b*}\pi_1(\tilde{A}_b, \star) = M \cdot \pi_1(A_b, \star)$, so $\eta_b : \tilde{A}_b \to A_b$ is simply $\eta''_b : A''_b \to A_b$.

Hence for $\tilde{f}: \tilde{X} = X \times_A \tilde{A} \to B$, one has $\tilde{X}_b = F_b \times \tilde{A}_b$, so

$$H^0(\tilde{X}_b, \omega^k_{\tilde{X}/\tilde{A}}) = H^0(F_b, \omega^k_F) \ .$$

Now $h^0(F_b, \omega^k_F)$ is independent of $b$ for $k \geqq 2$ by Riemann-Roch and Kawamata-Viehweg vanishing, so $\tilde{f}_* \omega^k_{\tilde{X}/\tilde{A}}$ is a locally free sheaf on $B$ for $k \geqq 2$. Then

$$g: Y = \mathbf{Proj}_B \sum \tilde{f}_* \omega^k_{\tilde{X}/\tilde{A}} \to B$$

has the required properties. □

The next proposition is a generic comparison result between $\omega_{X/S}$ and its push-forward (note that the base is not required to be abelian here).

After reducing to the case when $\det f_* \omega^m_{X/A}$ is ample using (2.4), this will allow one to appeal to the vanishing result of the previous section to finally prove (0.1) for an abelian base.

**2.5 Proposition.** *Let* $f: X \to S$ *be a morphism satisfying* (2.1). *Assume that* $\det f_* \omega^\eta_{X/S}$ *is ample for some* $\eta > 0$. *Then* $\omega_{X/S}$ *is ample.*

*Proof.* Let $a, b \in S$ and $\mathscr{I}_{a,b}$ their ideal sheaf. $f_* \omega^k_{X/S}$ is ample for $k \gg 0$ by [Esnault-Viehweg91, 0.1]. Fix a $k \gg 0$. Then there exists an $l_0 \in \mathbb{N}$ such that for every $l \geqq l_0$ and $i > 0$,

$$H^i(S, \mathrm{Sym}^l(f_* \omega^k_{X/S}) \otimes \mathscr{I}_{a,b}) = 0 \ .$$

Hence

$$v: H^0(S, \mathrm{Sym}^l(f_* \omega^k_{X/S})) \to (\mathrm{Sym}^l(f_* \omega^k_{X/S}) \otimes k(a)) \oplus (\mathrm{Sym}^l(f_* \omega^k_{X/S}) \otimes k(b))$$

is surjective.

Since $\omega_{X/S}$ restricted to any fibre is ample,

$$\varepsilon: \mathrm{Sym}^l(f_* \omega^k_{X/S}) \to f_* \omega^{lk}_{X/S}$$

is also surjective. Thus one has the following commutative diagram:

$$\begin{array}{ccc} H^0(S, \mathrm{Sym}^l(f_* \omega^k_{X/S})) & \xrightarrow{\ v\ } & (\mathrm{Sym}^l(f_* \omega^k_{X/S}) \otimes k(a)) \oplus (\mathrm{Sym}^l(f_* \omega^k_{X/S}) \otimes k(b)) \\ \downarrow & & \downarrow \varepsilon \\ H^0(S, f_* \omega^{lk}_{X/S}) & \xrightarrow[\sigma]{} & (f_* \omega^{lk}_{X/S} \otimes k(a)) \oplus (f_* \omega^{lk}_{X/S} \otimes k(b)) \ , \end{array}$$

with $v$ and $\varepsilon$ surjective, so $\sigma$ is surjective as well. Therefore

$$H^0(S, \omega^{lk}_{X/S}) \to H^0(X_a, \omega^{lk}_{X_a}) \oplus H^0(X_b, \omega^{lk}_{X_b})$$

is surjective.

Now choose $l \gg 0$ such that $\omega_{X_a}^{lk}$ is very ample for all $a \in S$. Then the global sections of $\omega_{X/S}^{lk}$ separate the fibres and induce an embedding on every fibre, hence $\omega_{X/S}$ is ample. $\square$

**2.6 Theorem.** *Let $f : X \to A$ be a morphism satisfying* (2.1) *such that $A$ is an abelian variety. Then there exists a morphism from $A$ to an abelian variety $A'$ and a morphism $f' : X' \to A'$ satisfying* (2.1), *such that $\omega_{X'/A'}$ is ample and $X'_{a'} \simeq X_a$ for every $a' \in A'$ and $a \in A_{a'}$. Furthermore, if $f$ is smooth, then so is $f'$. In particular, if $A$ is an irreducible abelian variety, then either $\omega_{X/A}$ is ample or $f$ is isotrivial.*

*Proof.* Assume $\omega_{X/A}$ is not ample. Then $\phi$ has a positive dimensional kernel by (2.5), so $\dim B < \dim A$. By (2.4) there exists a morphism $Y \to B$ satisfying (2.1) and (2.4.4), so the statement follows by induction on the dimension of $A$. $\square$

**2.7 Corollary.** *Let $f : X \to A$ be a smooth morphism satisfying* (2.1) *such that $A$ is an abelian variety. Then $f$ is isotrivial.*[1]

*Proof.* (2.6) provides a smooth morphism $Y \to B$ with $\omega_{Y/B}$ ample. Then $\dim B = 0$ by (1.6), so $f$ is isotrivial. $\square$

**2.8 Corollary.** [Kovács96] *Let $f : X \to A$ be a smooth morphism satisfying* (2.1) *such that $S$ is rationally connected. Then $f$ is isotrivial.*

*Proof.* It is enough to prove the statement in the case $S = \mathbb{P}^1$. Pick $s, t \in S$ and consider a cover of $S$ by an elliptic curve $\eta : E \to S$ such that $\eta$ is unramified over the chosen points. Then $X \times_S E \to E$ is isotrivial by (2.7), so $X_s \simeq X_t$, hence $f$ is isotrivial. $\square$

*2.8.1 Remark.* This is weaker than the result of [Kovács96], since the latter proves that smooth families of varieties of general type with nef canonical bundle have birationally equivalent fibres.

## 3 Families over Fano fibre spaces

**3.1 Proposition.** *Let $f : X \to S$ be a smooth morphism satisfying* (2.1). *Assume that there is a proper flat morphism $\alpha : S \to A$ such that the fibres of $\alpha$ are rationally connected varieties. Then $(\alpha \circ f)_* \omega_{X/S}^m$ is a locally free sheaf on $A$ for $m \geqq 2$ and there exists a commutative diagram*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ S & \xrightarrow{\ \alpha\ } & A \end{array}$$

*such that $g$ is smooth and satisfies* (2.1), *and $Y_a \simeq X_s$ for every $a \in A$ and $s \in S_a$.*

[1] Recently Qi Zhang also obtained this result with somewhat different methods

*Proof.* $S_a$ is rationally connected, hence $f_a : X_a \to S_a$ is isotrivial by (2.8), so $X_a \to S_a$ is a locally trivial fibre bundle. Rationally connected varieties are simply connected by [Campana91, 3.5], hence $X_a = X_s \times S_a$, and then

$$H^0(X_a, \omega^m_{X_a/S_a}) = H^0(X_s, \omega^m_{X_s}) .$$

On the other hand $h^0(X_s, \omega^m_{X_s})$ is independent of $s$ by Riemann-Roch and Kodaira vanishing for $m \geqq 2$, hence $h^0(X_a, \omega^m_{X_a/S_a})$ is independent of $a$ for $m \geqq 2$. Therefore $(\alpha \circ f)_* \omega^m_{X/S}$ is a locally free sheaf on $A$ for $m \geqq 2$. Take

$$g : Y = \mathbf{Proj}_A \sum (\alpha \circ f)_* \omega^m_{X/S} \to A . \qquad \square$$

**3.2 Theorem.** *Let $f : X \to S$ be a smooth morphism satisfying* (2.1)*. Assume that there is a proper flat morphism $\alpha : S \to A$ such that the fibres of $\alpha$ are rationally connected varieties and $A$ is an abelian variety. Then $f$ is isotrivial.*

*Proof.* By (3.1) one can reduce to a family over the abelian variety $A$ and then $f$ is isotrivial by (2.7). $\square$

**3.3 Corollary.** *Let $f : X \to S$ be a smooth morphism satisfying* (2.1)*. Assume that $\Omega_S$ is semi-negative. Then $f$ is isotrivial.*

*Proof.* By [DPS94, Main Theorem] a finite étale cover of $S$ is a smooth Fano fibre space over an abelian variety and Fano varieties are rationally connected by [KMM92b, 3.3] (cf. [Campana92]). $\square$

## 4 The birational case

The following easy technical lemma enables us to reduce (0.1) to (3.3).

**4.1 Lemma.** *Let $\varphi : T \to S$ be a morphism of algebraic varieties. Let $T_s$ be a fibre of $\varphi$ and $\mathscr{I}$ its ideal sheaf. Assume that $H^1(T_s, \mathscr{I}^n/\mathscr{I}^{n+1}) = 0$ for all $n \geqq 1$. Let $\mathscr{E}$ be a locally free sheaf on $T$ of rank $r$ such that $\mathscr{E}|_{T_s}$ is trivial. Then $\varphi_*\mathscr{E}$ is locally free near $s \in S$.*

*Proof.* Let $E_n$ be the subscheme of $T$ with ideal sheaf $\mathscr{I}^n$. In particular $E_1 = T_s$. By assumption $\mathscr{E} \otimes \mathscr{O}_{E_1} \simeq \mathscr{O}_{E_1}^{\oplus r}$ and then also $\mathscr{E} \otimes \mathscr{I}^n/\mathscr{I}^{n+1} \simeq (\mathscr{I}^n/\mathscr{I}^{n+1})^{\oplus r}$.

**Claim.** $\mathscr{E} \otimes \mathscr{O}_{E_n} \simeq \mathscr{O}_{E_n}^{\oplus r}$ for all $n \geqq 1$.

*Proof.* Consider

$$0 \to \mathscr{I}^n/\mathscr{I}^{n+1} \to \mathscr{O}_{E_{n+1}} \to \mathscr{O}_{E_n} \to 0 .$$

Tensoring this by $\mathscr{E}$ one has

$$0 \to \mathscr{E} \otimes \mathscr{I}^n/\mathscr{I}^{n+1} \to \mathscr{E} \otimes \mathscr{O}_{E_{n+1}} \to \mathscr{E} \otimes \mathscr{O}_{E_n} \to 0 .$$

By induction one can assume that $\mathscr{E} \otimes \mathscr{O}_{E_n} \simeq \mathscr{O}_{E_n}^{\oplus r}$. Thus the latter short exact sequence can be written as:

$$0 \to (\mathscr{I}^n/\mathscr{I}^{n+1})^{\oplus r} \to \mathscr{E} \otimes \mathscr{O}_{E_{n+1}} \to \mathscr{O}_{E_n}^{\oplus r} \to 0 .$$

Since $H^1(T_s, \mathscr{I}^n/\mathscr{I}^{n+1}) = 0$, the mapping

$$H^0(E_{n+1}, \mathscr{E} \otimes \mathcal{O}_{E_{n+1}}) \to H^0(E_n, \mathcal{O}_{E_n}^{\oplus r})$$

is surjective, so $\mathscr{E} \otimes \mathcal{O}_{E_{n+1}}$ has $r$ linearly independent global sections and there exists a morphism $\mathcal{O}_{E_{n+1}}^{\oplus r} \to \mathscr{E} \otimes \mathcal{O}_{E_{n+1}}$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathscr{I}^n/\mathscr{I}^{n+1})^{\oplus r} & \longrightarrow & \mathcal{O}_{E_{n+1}}^{\oplus r} & \longrightarrow & \mathcal{O}_{E_n}^{\oplus r} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{E} \otimes \mathscr{I}^n/\mathscr{I}^{n+1} & \longrightarrow & \mathscr{E} \otimes \mathcal{O}_{E_{n+1}} & \longrightarrow & \mathscr{E} \otimes \mathcal{O}_{E_n} & \longrightarrow & 0\,. \end{array}$$

Hence the claim follows by the 5-lemma. □

Therefore $\mathscr{E}$ is trivial in every infinitesimal neighborhood of $T_s$, so $\varphi_*\mathscr{E}$ is locally free near $s \in S$ by the Theorem on Formal Functions. □

**4.2 Theorem.** *Let $f: X \to T$ be a smooth morphism satisfying* (2.1)*. Assume that $T$ is birational to an $S$ with $\Omega_S$ semi-negative. Then $f$ is isotrivial.*

*Proof.* Resolving the indeterminacy locus of the birational map between $S$ and $T$ one has two birational morphisms $T' \to T$ and $\sigma : T' \to S$ such that $\sigma$ is obtained by a finite succession of blowing-ups along smooth subvarieties.

It is enough to prove that $f_{T'}$ is isotrivial, hence $f: X \to T$ can be replaced by $f_{T'}: X'_T \to T'$, so one may assume that there exists a morphism $\sigma: T \to S$, obtained by a finite succession of blowing-ups along smooth subvarieties.

**4.3 Lemma.** *Assume $\sigma: T \to S$ is a single blowing-up of $S$ along a smooth subvariety. Then $\sigma_* f_* \omega_{X/T}^m$ is a locally free sheaf on $S$ for $m \geqq 2$.*

*Proof.* $f_*\omega_{X/T}^m$ is a locally free sheaf on $T$ by Riemann-Roch and Kodaira vanishing. Let $s \in S$ and $\mathscr{I}$ the ideal sheaf of $T_s$. Then $T_s \simeq \mathbb{P}^r$ for some $r$, and $\mathscr{I}^n/\mathscr{I}^{n+1}$ is a sum of nef line bundles. Hence $H^1(T_s, \mathscr{I}^n/\mathscr{I}^{n+1}) = 0$ for all $n \geqq 1$. $X_s = F_s \times T_s$ by (2.8), where $F_s$ denotes the isomorphic fibres of $f$ over $T_s$. Let $p : X_s \to F_s$ denote the projection. Then $\omega_{X_s/T_s}^m \simeq p^*\omega_{F_s}^m$, so $f_*\omega_{X_s/T_s}^m \simeq f_*\omega_{X/T}^m \otimes \mathcal{O}_{T_s}$ is trivial on $T_s$. Therefore by (4.1) $\sigma_* f_* \omega_{X/T}^m$ is locally free. □

Now let $X' = \mathbf{Proj}_S \sum \sigma_* f_* \omega_{X/T}^m$. Then one has the following commutative diagram:

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

such that for every $s \in S$ and $t \in T_s$, $X'_s \simeq X_t$.

Finally consider the general case, i.e., when $\sigma$ is obtained by a finite succession of blowing-ups along smooth subvarieties. By repeated use of the previous argument the family over $T$ descends to a family over $S$, i.e., there exist commuting morphisms:

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

such that for every $s \in S$ and $t \in T_s$, $X'_s \simeq X_t$. Hence $f$ is isotrivial by (3.3). □

*4.3.1 Remark.* The above proof also shows that in general the isotriviality of $f$ only depends on the birational class of $T$ in the following sense: If $T$ is birational to an $S$ such that every smooth projective family of varieties with ample canonical bundle over $S$ is isotrivial, then the same is true for $T$.

## References


[Campana91] Campana, F.: On twistor spaces of the class $\mathscr{C}$. J. Diff. Geom. **33** (1991), 541–549

[Campana92] Campana, F.: Connexité rationnelle des variétés de Fano. Ann. Sci. E. N. S. **25** (1992), 539–545

[CKM88] Clemens, H., Kollár, J., Mori, S.: Higher Dimensional Complex Geometry, vol. 166. Astérisque, 1988

[DPS94] Demailly, J.P., Peternell, T., Schneider, M.: Compact complex manifolds with numerically effective tangent bundles. J. Alg. Geom. **3** (1994), 295–345

[Esnault-Viehweg91] Esnault, H., Viehweg, E.: Ample sheaves on moduli schemes. Algebraic Geometry and Analytic Geometry. Berlin, Heidelberg: Springer 1991, pp. 53–80

[Kobayashi72] Kobayashi, S.: Transformation groups in differential geometry. Ergebuisse 70, Heidelberg: Springer 1972

[Kollár87] Kollár, J.: Subadditivity of the Kodaira dimension: Fibers of general type. Advanced Studies in Pure Math. 10, Algebraic Geometry Sendai 85. Kinokuniya-North Holland, 1987, pp. 361–398

[KMM92a] Kollár, J., Miyaoka, Y., Mori, S.: Rationally Connected Varieties. J. Alg. Geom. **1** (1992), 429–448

[KMM92b] Kollár, J., Miyaoka, Y., Mori, S.: Rational connectedness and boundedness of Fano manifolds. J. Diff. Geom. **36** (1992), 765–769

[Kovács96] Kovács, S.J.: Smooth families over rational and elliptic curves. J. Alg. Geom. **5** (1996), 369–385

[Matsusaka-Mumford64] Matsusaka, T., Mumford, D.: Two fundamental theorems on deformations of polarized varieties. Amer. J. Math. **86** (1964), 668–684

[Migliorini95] Migliorini, L.: A smooth family of minimal surfaces of general type over a curve of genus at most one is trivial. J. Alg. Geom. **4** (1995), 353–361

[Shiffman-Sommese85] Shiffman, B., Sommese, A.J.: Vanishing theorems on complex manifolds. Progress in Math., vol. 56. Basel: Birkhäuser, 1985

[Stevens88] Stevens, J.: On canonical singularities as total spaces of deformations. Abh. Math. Sem. Univ. Hamburg **58** (1988), 275–283

[Szabó96] Szabó, E.: Bounding Automorphism Groups. Math. Ann. **304** (1996), 801–811

[Zhang96] Zhang, Q.: Global holomorphic one-forms on projective manifolds with ample canonical bundles. Preprint, University of Missouri