Du Bois pairs and vanishing theorems

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In memoriam, Professor Masayoshi Nagata

Abstract The main purpose of this article is to define the notion of Du Bois singularities for pairs and prove a vanishing theorem by using this new notion. The main vanishing theorem specializes to a new vanishing theorem for resolutions of log canonial singularities.

1. Introduction

The class of rational singularities is one of the most important classes of singularities. Their essence lies in the fact that their cohomological behavior is very similar to that of smooth points. For instance, vanishing theorems can be easily extended to varieties with rational singularities. Establishing that a certain class of singularities is rational opens the door to using very powerful tools on varieties with those singularities.

Du Bois (DB) singularities are probably somewhat harder to appreciate at first, but they are equally important. Their main importance comes from two facts. They are not too far from rational singularities, that is, they share many of their properties, but the class of DB singularities is more inclusive than that of rational singularities. For instance, log canonical singularities are DB, but not necessarily rational. The class of DB singularities is also more stable under degeneration.

Recently there has been an effort to extend the notion of rational singularities to pairs. There are at least two approaches: Schwede and Takagi [ST] are dealing with pairs (X,Δ), where $\lfloor\Delta\rfloor = 0$, while Kollár and Kovács [KK2] are studying pairs (X,Δ), where Δ is reduced.

The main goal of this article is to extend the definition of DB singularities to pairs in the spirit of the latter approach.

Here is a brief overview.

In Section 2 some basic properties of rational and DB singularities are reviewed, a few new ones are introduced, and the DB defect is defined. In

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Section 3 I recall the definition and some basic properties of pairs, generalized pairs, and rational pairs. I define the notion of a DB pair and the DB defect of a generalized pair and prove a few basic properties. In Section 4 I recall a relevant theorem from Deligne's Hodge theory and derive a corollary that is needed later. In Section 5 one of the main results is proven. A somewhat weaker version is the following. See Theorem 5.4 for the stronger statement.

THEOREM 1.1
Rational pairs are DB pairs.

This generalizes [Kov1, Theorem S] and [Sai, 5.4] to pairs. In Section 6 I prove a rather general vanishing theorem for DB pairs and use it to derive the following vanishing theorem for log canonical pairs.

THEOREM 1.2
Let (X,Δ) be a $\mathbb{Q}$-factorial log canonical pair, and let $\pi:\widetilde{X}\to X$ be a log resolution of (X,Δ). Let $\widetilde{\Delta}=(\pi_*^{-1}\lfloor\Delta\rfloor+\operatorname{Exc}_{\mathrm{nklt}}(\pi))_{\mathrm{red}}$. Then

$$\mathcal{R}^i\pi_*\mathscr{O}_{\widetilde{X}}(-\widetilde{\Delta})=0 \quad \textit{for } i>0.$$

A philosophical consequence one might draw from this theorem is that log canonical pairs are not too far from being rational. One may even view this as a vanishing theorem similar to the one in the definition of rational singularities (cf. (2.1), (3.4)) with a correction term as in vanishing theorems with multiplier ideals. Notice, however, that this is in a dual form compared with Nadel's vanishing and hence does not follow from that, especially since the target is not necessarily Cohen-Macaulay.

Theorem 1.2 is also closely related to Steenbrink's characterization of normal isolated DB singularities [Ste1, 3.6] (cf. [DB, Proposition 4.13], [KS, Theorem 6.1]).

A weaker version of this theorem was the cornerstone of a recent result on extending differential forms to a log resolution (see [GKKP]). For details on how this theorem may be applied, see the original article. It is possible that the current theorem will lead to a strengthening of that result.

DEFINITIONS AND NOTATION 1.3
Unless otherwise stated, all objects are assumed to be defined over $\mathbb{C}$, all schemes are assumed to be of finite type over $\mathbb{C}$, and a morphism means a morphism between schemes of finite type over $\mathbb{C}$.

If $\phi:Y\to Z$ is a birational morphism, then $\operatorname{Exc}(\phi)$ denotes the *exceptional set* of ϕ. For a closed subscheme $W\subseteq X$, the ideal sheaf of W is denoted by $\mathscr{I}_{W\subseteq X}$ or, if no confusion is likely, then simply by $\mathscr{I}_W$. For a point $x\in X$, $\kappa(x)$ denotes the residue field of $\mathscr{O}_{X,x}$.

For morphisms $\phi:X\to B$ and $\vartheta:T\to B$, the symbol X_T denotes $X\times_B T$ and $\phi_T:X_T\to T$ denotes the induced morphism. In particular, for $b\in B$ we write $X_b = \phi^{-1}(b)$. Of course, by symmetry, we also have the notation $\vartheta_X : T_X \simeq X_T \to X$, and if $\mathscr{F}$ is an $\mathscr{O}_X$-module, then $\mathscr{F}_T$ denotes the $\mathscr{O}_{X_T}$-module $\vartheta_X^* \mathscr{F}$.

Let X be a complex scheme (i.e., a scheme of finite type over $\mathbb{C}$) of dimension n. Let $D_{\mathrm{filt}}(X)$ denote the derived category of filtered complexes of $\mathscr{O}_X$-modules with differentials of order ≤ 1, and let $D_{\mathrm{filt,coh}}(X)$ denote the subcategory of $D_{\mathrm{filt}}(X)$ of complexes $K^\bullet$, such that for all i, the cohomology sheaves of $\mathrm{Gr}^i_{\mathrm{filt}} K^\bullet$ are coherent (see [DB], [GNPP]). Let $D(X)$ and $D_{\mathrm{coh}}(X)$ denote the derived categories with the same definition except that the complexes are assumed to have the trivial filtration. The superscripts $+, -, b$ carry the usual meaning (bounded below, bounded above, bounded). Isomorphism in these categories is denoted by $\simeq_{\mathrm{qis}}$. A sheaf $\mathscr{F}$ is also considered as a complex $\mathscr{F}^\bullet$ with $\mathscr{F}^0 = \mathscr{F}$ and $\mathscr{F}^i = 0$ for $i \neq 0$. If $K^\bullet$ is a complex in any of the above categories, then $h^i(K^\bullet)$ denotes the ith cohomology sheaf of $K^\bullet$.

The right derived functor of an additive functor F, if it exists, is denoted by $\mathcal{R}F$, and $\mathcal{R}^i F$ is short for $h^i \circ \mathcal{R}F$. Furthermore, $\mathbb{H}^i$, $\mathbb{H}^i_{\mathrm{c}}$, $\mathbb{H}^i_Z$, and $\mathscr{H}^i_Z$ denote $\mathcal{R}^i\Gamma$, $\mathcal{R}^i\Gamma_{\mathrm{c}}$, $\mathcal{R}^i\Gamma_Z$, and $\mathcal{R}^i\mathscr{H}_Z$, respectively, where Γ is the functor of global sections, Γ_{c} is the functor of global sections with proper support, Γ_Z is the functor of global sections with support in the closed subset Z, and $\mathscr{H}_Z$ is the functor of the sheaf of local sections with support in the closed subset Z. Note that according to this terminology, if $\phi : Y \to X$ is a morphism and $\mathscr{F}$ is a coherent sheaf on Y, then $\mathcal{R}\phi_*\mathscr{F}$ is the complex whose cohomology sheaves give rise to the usual higher direct images of $\mathscr{F}$.

We often use the notion that a morphism $f : \mathsf{A} \to \mathsf{B}$ in a derived category *has a left inverse*. This means that there exists a morphism $f^\ell : \mathsf{B} \to \mathsf{A}$ in the same derived category such that $f^\ell \circ f : \mathsf{A} \to \mathsf{A}$ is the identity morphism of A. That is, f^ℓ is a *left inverse* of f.

Finally, we also make the following simplification in notation. First, observe that if $\iota : \Sigma \hookrightarrow X$ is a closed embedding of schemes, then ι_* is exact and hence $\mathcal{R}\iota_* = \iota_*$. This allows one to make the following harmless abuse of notation: If $\mathsf{A} \in \mathrm{Ob}\, D(\Sigma)$, then, as usual for sheaves, we drop ι_* from the notation of the object $\iota_*\mathsf{A}$. In other words, without further warning, we consider A an object in $D(X)$.

2. Rational and DB singularities

DEFINITION 2.1

Let X be a normal variety, and let $\phi : Y \to X$ be a resolution of singularities. X is said to have *rational* singularities if $\mathcal{R}^i\phi_*\mathscr{O}_Y = 0$ for all $i > 0$ or, equivalently, if the natural map $\mathscr{O}_X \to \mathcal{R}\phi_*\mathscr{O}_Y$ is a quasi-isomorphism.

DB singularities are defined via Deligne's Hodge theory. We need a little preparation before we can define them.

The starting point is Du Bois's construction, following Deligne's ideas, of the generalized de Rham complex, which we call the *Deligne–Du Bois complex.*

Recall that if X is a smooth complex algebraic variety of dimension n, then the sheaves of differential p-forms with the usual exterior differentiation give a resolution of the constant sheaf $\mathbb{C}_X$. That is, one has a filtered complex of sheaves,

$$\mathscr{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \Omega_X^3 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^n \simeq \omega_X,$$

which is quasi-isomorphic to the constant sheaf $\mathbb{C}_X$ via the natural map $\mathbb{C}_X \to \mathscr{O}_X$ given by considering constants as holomorphic functions on X. Recall that this complex *is not* a complex of quasi-coherent sheaves. The sheaves in the complex are quasi-coherent, but the maps between them are not $\mathscr{O}_X$-module morphisms. Notice, however, that this is actually not a shortcoming; as $\mathbb{C}_X$ is not a quasi-coherent sheaf, one cannot expect a resolution of it in the category of quasi-coherent sheaves.

The Deligne–Du Bois complex is a generalization of the de Rham complex to singular varieties. It is a complex of sheaves on X that is quasi-isomorphic to the constant sheaf $\mathbb{C}_X$. The terms of this complex are harder to describe, but its properties, especially cohomological properties, are very similar to the de Rham complex of smooth varieties. In fact, for a smooth variety the Deligne–Du Bois complex is quasi-isomorphic to the de Rham complex, so it is indeed a direct generalization.

The construction of this complex, $\underline{\Omega}^\bullet_X$, is based on simplicial resolutions. The reader interested in the details is referred to the original article [DB]. Note also that a simplified construction was later obtained in [Car] and [GNPP] via the general theory of polyhedral and cubic resolutions. An easily accessible introduction can be found in [Ste2]. Other useful references are the recent book [PS] and the survey [KS]. We actually do not use these resolutions here. They are needed for the construction, but if one is willing to believe the listed properties (which follow in a rather straightforward way from the construction), then one should be able follow the material presented here. The interested reader should note that recently Schwede found a simpler alternative construction of (part of) the Deligne–Du Bois complex that does not need a simplicial resolution (see [Sch1]). For applications of the Deligne–Du Bois complex and DB singularities other than the ones listed here, see [Ste1], [Kol, Chapter 12], [Kov1], [Kov3], [KSS], and [KK1].

The word "hyperresolution" refers to either a simplicial, polyhedral, or cubic resolution. Formally, the construction of $\underline{\Omega}^\bullet_X$ is the same regardless of the type of resolution used, and no specific aspects of either type are used.

The next theorem lists the basic properties of the Deligne–Du Bois complex.

THEOREM 2.2 ([DB, 3.2, COROLLAIRE 3.10, THÉORÈME 4.5, PROPOSITION 4.11])

Let X be a complex scheme of finite type. Then there exists a functorially defined object $\underline{\Omega}^\bullet_X \in \operatorname{Ob} D_{\text{filt}}(X)$ such that using the notation

$$\underline{\Omega}^p_X := \operatorname{Gr}^p_{\text{filt}} \underline{\Omega}^\bullet_X[p],$$

it satisfies the following properties.

(2.2.1) $$\underline{\Omega}^{\bullet}_X \simeq_{\mathrm{qis}} \mathbb{C}_X.$$

(2.2.2) $\underline{\Omega}^{\bullet}_{(\text{-})}$ *is functorial; that is, if* $\phi : Y \to X$ *is a morphism of complex schemes of finite type, then there exists a natural map* ϕ^* *of filtered complexes*

$$\phi^* : \underline{\Omega}^{\bullet}_X \to \mathcal{R}\phi_* \underline{\Omega}^{\bullet}_Y.$$

Furthermore, $\underline{\Omega}^{\bullet}_X \in \mathrm{Ob}(D^b_{\mathrm{filt,coh}}(X))$*, and if* ϕ *is proper, then* ϕ^* *is a morphism in* $D^b_{\mathrm{filt,coh}}(X)$.

(2.2.3) *Let* $U \subseteq X$ *be an open subscheme of* X. *Then*

$$\underline{\Omega}^{\bullet}_X|_U \simeq_{\mathrm{qis}} \underline{\Omega}^{\bullet}_U.$$

(2.2.4) *If* X *is proper, then there exists a spectral sequence degenerating at* E_1 *and abutting to the singular cohomology of* X*:*

$$E_1^{pq} = \mathbb{H}^q(X, \underline{\Omega}^p_X) \Rightarrow H^{p+q}(X, \mathbb{C}).$$

(2.2.5) *If* $\varepsilon_\bullet : X_\bullet \to X$ *is a hyperresolution, then*

$$\underline{\Omega}^{\bullet}_X \simeq_{\mathrm{qis}} \mathcal{R}\varepsilon_{\bullet *} \Omega^{\bullet}_{X_\bullet}.$$

In particular, $h^i(\underline{\Omega}^p_X) = 0$ *for* $i < 0$.

(2.2.6) *There exists a natural map,* $\mathscr{O}_X \to \underline{\Omega}^0_X$*, compatible with (2.2.2).*

(2.2.7) *If* X *is a normal crossing divisor in a smooth variety, then*

$$\underline{\Omega}^{\bullet}_X \simeq_{\mathrm{qis}} \Omega^{\bullet}_X.$$

In particular,

$$\underline{\Omega}^{p}_X \simeq_{\mathrm{qis}} \Omega^{p}_X.$$

(2.2.8) *If* $\phi : Y \to X$ *is a resolution of singularities, then*

$$\underline{\Omega}^{\dim X}_X \simeq_{\mathrm{qis}} \mathcal{R}\phi_* \omega_Y.$$

(2.2.9) *Let* $\pi : \widetilde{X} \to X$ *be a projective morphism, and let* $\Sigma \subseteq X$ *be a reduced closed subscheme such that* π *is an isomorphism outside of* Σ. *Let* E *denote the reduced subscheme of* $\widetilde{X}$ *with support equal to* $\pi^{-1}(X)$. *Then for each* p *one has an exact triangle of objects in the derived category,*

$$\underline{\Omega}^p_X \longrightarrow \underline{\Omega}^p_\Sigma \oplus \mathcal{R}\pi_* \underline{\Omega}^p_{\widetilde{X}} \xrightarrow{\ -\ } \mathcal{R}\pi_* \underline{\Omega}^p_E \xrightarrow{+1} .$$

(2.2.10) *Suppose that* $X = Y \cup Z$ *is the union of two closed subschemes and denote their intersection by* $W := Y \cap Z$. *Then for each* p *one has an exact triangle of objects in the derived category,*

$$\underline{\Omega}^p_X \longrightarrow \underline{\Omega}^p_Y \oplus \underline{\Omega}^p_Z \xrightarrow{\ -\ } \underline{\Omega}^p_W \xrightarrow{+1} .$$

It turns out that the Deligne–Du Bois complex behaves very much like the de Rham complex for smooth varieties. Observe that (2.2.4) says that the Hodge-to-de Rham (also known as Frölicher) spectral sequence works for singular varieties if one uses the Deligne–Du Bois complex in place of the de Rham complex. This has far-reaching consequences, and if the associated graded pieces $\underline{\Omega}^p_X$ turn out to be computable, then this single property leads to many applications.

Notice that (2.2.6) gives a natural map $\mathscr{O}_X \to \underline{\Omega}^0_X$; we are interested in situations when this map is a quasi-isomorphism. When X is proper over $\mathbb{C}$, such a quasi-isomorphism implies that the natural map

$$H^i(X^{\mathrm{an}}, \mathbb{C}) \to H^i(X, \mathscr{O}_X) = \mathbb{H}^i(X, \underline{\Omega}^0_X)$$

is surjective because of the degeneration at E_1 of the spectral sequence in (2.2.4). Notice that this is the condition that is crucial for Kodaira-type vanishing theorems (cf. [Kol, Section 9]).

Following Du Bois, Steenbrink was the first to study this condition, and he christened this property after Du Bois. It should be noted that many of the ideas that play important roles in this theory originated from Deligne. Unfortunately the now-standard terminology does not reflect this.

DEFINITION 2.3

A scheme X is said to have *DB singularities* if the natural map $\mathscr{O}_X \to \underline{\Omega}^0_X$ from (2.2.6) is a quasi-isomorphism.

REMARK 2.4

If $\varepsilon : X_\bullet \to X$ is a hyperresolution of X, then X has DB singularities if and only if the natural map $\mathscr{O}_X \to \mathcal{R}\varepsilon_{\bullet *}\mathscr{O}_{X_\bullet}$ is a quasi-isomorphism.

EXAMPLE 2.5

It is easy to see that smooth points are DB, and Deligne proved that normal crossing singularities are DB as well (cf. (2.2.7), [DJ, lemme 2(b)]).

In applications it is very useful to be able to take general hyperplane sections. The next statement helps with that.

PROPOSITION 2.6

Let X be a quasi-projective variety, and let $H \subset X$ be a general member of a very ample linear system. Then $\underline{\Omega}^\bullet_H \simeq_{\mathrm{qis}} \underline{\Omega}^\bullet_X \otimes_L \mathscr{O}_H$.

Proof

Let $\varepsilon_\bullet : X_\bullet \to X$ be a hyperresolution. Since H is general, the fiber product $X_\bullet \times_X H \to H$ provides a hyperresolution of H. Then the statement follows from (2.2.5) applied to both X and H. $\square$

We saw in (2.2.5) that $h^i(\underline{\Omega}^0_X) = 0$ for $i < 0$. In fact, there is a corresponding upper bound by [GNPP, Section III.1.17], namely, that $h^i(\underline{\Omega}^0_X) = 0$ for $i > \dim X$. It turns out that one can make a slightly better estimate.

PROPOSITION 2.7 (CF. [GKKP, LEMMA 13.5], [KSS, LEMMA 4.9])

Let X be a positive-dimensional variety (i.e., reduced). Then the ith cohomology sheaf of $\underline{\Omega}^p_X$ vanishes for all $i \geq \dim X$; that is, $h^i(\underline{\Omega}^p_X) = 0$ for all p and for all $i \geq \dim X$.

Proof

For $i > \dim X$ or $p > 0$, the statement follows from [GNPP, Proposition III.1.17]. The case when $p = 0$ and $i = n := \dim X$ follows from either [GKKP, 13.5] or [KSS, Lemma 4.9]. $\square$

Another, much simpler fact that is used later is the following.

COROLLARY 2.8

If $\dim X = 1$, then $h^i(\underline{\Omega}^p_X) = 0$ for $i \neq 0$. In particular, X is DB if and only if it is seminormal.

Proof

The first statement is a direct consequence of (2.7). For the last statement, recall that the seminormalization of $\mathscr{O}_X$ is exactly $h^0(\underline{\Omega}^0_X)$, and so X is seminormal if and only if $\mathscr{O}_X \simeq h^0(\underline{\Omega}^0_X)$ (see [Sai, Proposition 5.2]; cf. [Sch3, 5.4.17], [Sch1, Remark 4.8], [Sch2, Lemma 5.6]). $\square$

DEFINITION 2.9

The *DB defect* of X is the mapping cone of the morphism $\mathscr{O}_X \to \underline{\Omega}^0_X$. It is denoted by $\underline{\Omega}^\times_X$. As a simple consequence of the definition, one has an exact triangle,

$$\mathscr{O}_X \longrightarrow \underline{\Omega}^0_X \longrightarrow \underline{\Omega}^\times_X \xrightarrow{+1} .$$

Notice that $h^0(\underline{\Omega}^\times_X) \simeq h^0(\underline{\Omega}^0_X)/\mathscr{O}_X$ and $h^i(\underline{\Omega}^\times_X) \simeq h^i(\underline{\Omega}^0_X)$ for $i > 0$.

PROPOSITION 2.10

Let X be a quasi-projective variety, and let $H \subset X$ be a general member of a very ample linear system. Then $\underline{\Omega}^\times_H \simeq_{\text{qis}} \underline{\Omega}^\times_X \otimes_L \mathscr{O}_H$.

Proof

This follows easily from the definition and Proposition 2.6. $\square$

The next simple observation explains the name of the DB defect.

LEMMA 2.11

A variety X is DB if and only if the DB defect of X is acyclic; that is, $\underline{\Omega}^{\times}_X \simeq_{\text{qis}} 0$.

Proof

This follows directly from the definition. $\square$

PROPOSITION 2.12

Let $X = Y \cup Z$ be a union of closed subschemes with intersection $W = Y \cap Z$. Then one has an exact triangle of the DB defects of X, Y, Z, and W:

$$\underline{\Omega}^{\times}_X \longrightarrow \underline{\Omega}^{\times}_Y \oplus \underline{\Omega}^{\times}_Z \xrightarrow{\ -\ } \underline{\Omega}^{\times}_W \xrightarrow{+1} .$$

Proof

Recall that there is an analogous exact triangle (also known as a short exact sequence) for the structure sheaves of X, Y, Z, and W, which forms a commutative diagram with the exact triangle of (2.2.10),

$$\begin{tikzcd}
\mathscr{O}_X \arrow[r] \arrow[d] & \mathscr{O}_Y \oplus \mathscr{O}_Z \arrow[r, "-"] \arrow[d] & \mathscr{O}_W \arrow[r, "+1"] \arrow[d] & {} \\
\underline{\Omega}^0_X \arrow[r] & \underline{\Omega}^0_Y \oplus \underline{\Omega}^0_Z \arrow[r, "-"] & \underline{\Omega}^0_W \arrow[r, "+1"] & {}
\end{tikzcd}$$

Then the statement follows by the (derived category version of the) 9-lemma. $\square$

3. Pairs and generalized pairs

3.A. Basic definitions

For an arbitrary proper birational morphism, $\phi : Y \to X$, $\operatorname{Exc}(\phi)$ stands for the exceptional locus of ϕ. A $\mathbb{Q}$-*divisor* is a $\mathbb{Q}$-linear combination of integral Weil divisors: $\Delta = \sum a_i \Delta_i$, $a_i \in \mathbb{Q}$, Δ_i (integral) a Weil divisor. For a $\mathbb{Q}$-divisor Δ, its *round-down* is defined by the formula $\lfloor \Delta \rfloor = \sum \lfloor a_i \rfloor \Delta_i$, where $\lfloor a_i \rfloor$ is the largest integer not larger than a_i.

A *log variety* or *pair* (X, Δ) consists of an equidimensional variety (i.e., a reduced scheme of finite type over a field k) X and an effective $\mathbb{Q}$-divisor $\Delta \subseteq X$. A morphism of pairs $\phi : (Y, B) \to (X, \Delta)$ is a morphism $\phi : Y \to X$ such that $\phi(\operatorname{supp} B) \subseteq \operatorname{supp} \Delta$.

Let (X, Δ) be a pair with Δ a reduced integral divisor. Then (X, Δ) is said to have *simple normal crossings* or to be an *snc pair at* $p \in X$ if X is smooth at p, and there are local coordinates $x_1, \dots, x_n$ on X in a neighborhood of p such that $\operatorname{supp} \Delta \subseteq (x_1 \cdots x_n = 0)$ near p. (X, Δ) is *snc* if it is snc at every $p \in X$.

A morphism of pairs $\phi : (Y, \Delta_Y) \to (X, \Delta)$ is a *log resolution of* (X, Δ) if $\phi : Y \to X$ is proper and birational, $\Delta_Y = \phi^{-1}_* \Delta$, and $(\Delta_Y)_{\text{red}} + \operatorname{Exc}(\phi)$ is an snc divisor on Y.

Note that we allow (X,Δ) to be snc and still call a morphism with these properties a log resolution. Also, note that the notion of a log resolution is not used consistently in the literature.

If (X,Δ) is a pair, then Δ is called a *boundary* if $\lfloor (1-\varepsilon)\Delta \rfloor = 0$ for all $0<\varepsilon<1$; that is, the coefficients of all irreducible components of Δ are in the interval $[0,1]$ (for the definition of *klt, dlt*, and *lc* pairs, see [KM]). Let (X,Δ) be a pair, and let $\mu : X^{\mathrm{m}} \to X$ be a proper birational morphism. Let $E=\sum a_i E_i$ be the discrepancy divisor, that is, a linear combination of exceptional divisors such that

$$K_{X^{\mathrm{m}}} + \mu_*^{-1}\Delta \sim_{\mathbb{Q}} \mu^*(K_X+\Delta)+E,$$

and let $\Delta^{\mathrm{m}} := \mu_*^{-1}\Delta + \sum_{a_i\le -1} E_i$. For an irreducible divisor F on a birational model of X, we define its discrepancy as its coefficient in E. Notice that as divisors correspond to valuations, this discrepancy is independent of the model chosen; it depends only on the divisor. A *non-klt place* of a pair (X,Δ) is an irreducible divisor F over X with discrepancy at most -1, and a *non-klt center* is the image of any non-klt place. $\mathrm{Exc}_{\mathrm{nklt}}(\mu)$ denotes the union of the loci of all non-klt places of ϕ.

Note that in the literature, non-klt places and centers are often called log canonical places and centers (for a more detailed and precise definition, see [HK, p. 37]).

Now if $(X^{\mathrm{m}},\Delta^{\mathrm{m}})$ is as above, then it is a *minimal dlt model* of (X,Δ) if it is a dlt pair and the discrepancy of every μ-exceptional divisor is at most -1 (see [KK1]). Note that if (X,Δ) is lc with a minimal dlt model $(X^{\mathrm{m}},\Delta^{\mathrm{m}})$, then $K_{X^{\mathrm{m}}}+\Delta^{\mathrm{m}} \sim_{\mathbb{Q}} \mu^*(K_X+\Delta)$.

3.B. Rational pairs

Recall the definition of a *log resolution* from Section 3.A: A morphism of pairs $\phi : (Y,\Delta_Y)\to(X,\Delta)$ is a *log resolution of* (X,Δ) if $\phi : Y\to X$ is proper and birational, $\Delta_Y=\phi_*^{-1}\Delta$, and $(\Delta_Y)_{\mathrm{red}}+\mathrm{Exc}(\phi)$ is an snc divisor on Y.

DEFINITION 3.1

Let (X,Δ) be a pair, and let Δ be an integral divisor. Then (X,Δ) is called a *normal pair* if there exists a log resolution $\phi : (Y,\Delta_Y)\to(X,\Delta)$ such that the natural morphism $\phi^{\#} : \mathscr{O}_X(-\Delta)\to\phi_*\mathscr{O}_Y(-\Delta_Y)$ is an isomorphism.

DEFINITION 3.2

A pair (X,Δ) with Δ an integral divisor is called a *weakly rational pair* if there is a log resolution $\phi : (Y,\Delta_Y)\to(X,\Delta)$ such that the natural morphism $\mathscr{O}_X(-\Delta)\to \mathcal{R}\phi_*\mathscr{O}_Y(-\Delta_Y)$ has a left inverse.

LEMMA 3.3

Let (X,Δ) be a weakly rational pair. Then it is a normal pair.

Proof
The 0th cohomology of the left inverse of $\mathscr{O}_X(-\Delta) \to \mathcal{R}\phi_*\mathscr{O}_Y(-\Delta_Y)$ gives a left inverse of $\phi^\# : \mathscr{O}_X(-\Delta) \to \phi_*\mathscr{O}_Y(-\Delta_Y)$. As the morphism ϕ is birational, the kernel of the left inverse of $\phi^\#$ is a torsion sheaf. However, since $\phi_*\mathscr{O}_Y(-\Delta_Y)$ is torsion free, this implies that $\phi^\#$ is an isomorphism. $\square$

DEFINITION 3.4 ([KK2])
Let (X,Δ) be a pair where Δ is an integral divisor. Then (X,Δ) is called a *rational pair* if there exists a log resolution $\phi : (Y,\Delta_Y) \to (X,\Delta)$ such that

(3.4.1) $\mathscr{O}_X(-\Delta) \simeq \phi_*\mathscr{O}_Y(-\Delta_Y)$; that is, (X,Δ) is normal,
(3.4.2) $\mathcal{R}^i\phi_*\mathscr{O}_Y(-\Delta_Y) = 0$ for $i > 0$, and
(3.4.3) $\mathcal{R}^i\phi_*\omega_Y(\Delta_Y) = 0$ for $i > 0$.

LEMMA 3.5
Let (X,Δ) be a pair where Δ is an integral divisor. Then it is a rational pair if and only if it is a weakly rational pair and $\mathcal{R}^i\phi_*\omega_Y(\Delta_Y) = 0$ for $i > 0$.

Proof
This follows directly from [KK2, 105]. $\square$

REMARK 3.6
Note that the notion of a *rational pair* describes the "singularity" of the relationship between X and Δ. From the definition it is not clear, for instance, whether (X,Δ) being rational implies that X has rational singularities.

REMARK 3.7
If $\Delta = \emptyset$, then (3.4.3) follows from the Grauert-Riemenschneider vanishing theorem, and X is weakly rational if and only if it is rational by [Kov2].

3.C. Generalized pairs

DEFINITION 3.8
A *generalized pair* (X,Σ) consists of an equidimensional variety (i.e., a reduced scheme of finite type over a field k) X and a subscheme $\Sigma \subseteq X$. A morphism of generalized pairs $\phi : (Y,\Gamma) \to (X,\Sigma)$ is a morphism $\phi : Y \to X$ such that $\phi(\Gamma) \subseteq \Sigma$. A *reduced generalized pair* is a generalized pair (X,Σ) such that Σ is reduced.

The *log resolution* of a generalized pair (X,W) is a proper birational morphism $\pi : \widetilde{X} \to X$ such that $\operatorname{Exc}(\pi)$ is a divisor and $\pi^{-1}W + \operatorname{Exc}(\pi)$ is an snc divisor.

Let X be a complex scheme, and let Σ be a closed subscheme whose complement in X is dense. Then $(X_\bullet,\Sigma_\bullet) \to (X,\Sigma)$ is a *good hyperresolution* if $X_\bullet \to X$ is a hyperresolution and if $U_\bullet = X_\bullet \times_X (X\setminus\Sigma)$ and $\Sigma_\bullet = X_\bullet \setminus U_\bullet$; then, for all α, either Σ_α is a divisor with normal crossings on X_α or $\Sigma_\alpha = X_\alpha$. Notice that it is possible that $X_\bullet$ has components that map into Σ. These component are contained in $\Sigma_\bullet$ (for more details and the existence of such hyperresolutions, see [DB, 6.2], [GNPP, Sections IV.1.21, IV.1.25, IV.2.1]; for a primer on hyperresolutions, see the appendix of [KS]).

Let (X, Σ) be a reduced generalized pair. Consider the Deligne–Du Bois complex of (X, Σ) defined by Steenbrink [Ste2, Section 3].

DEFINITION 3.9

The *Deligne–Du Bois complex* of the reduced generalized pair (X, Σ) is the mapping cone of the natural morphism $\varrho : \underline{\Omega}^\bullet_X \to \underline{\Omega}^\bullet_\Sigma$ twisted by (-1). In other words, it is an object $\underline{\Omega}^\bullet_{X,\Sigma}$ in $D_{\text{filt}}(X)$ such that it completes ϱ to an exact triangle:

$$\underline{\Omega}^\bullet_{X,\Sigma} \longrightarrow \underline{\Omega}^\bullet_X \longrightarrow \underline{\Omega}^\bullet_\Sigma \xrightarrow{+1} . \tag{3.9.1}$$

The associated graded quotients of $\underline{\Omega}^\bullet_{X,\Sigma}$ are denoted as usual:

$$\underline{\Omega}^p_{X,\Sigma} := \operatorname{Gr}^p_{\text{filt}} \underline{\Omega}^\bullet_{X,\Sigma}[p].$$

Notice that the above triangle is in $D_{\text{filt}}(X)$ and hence for all $p \in \mathbb{N}$ we obtain another exact triangle:

$$\underline{\Omega}^p_{X,\Sigma} \longrightarrow \underline{\Omega}^p_X \longrightarrow \underline{\Omega}^p_\Sigma \xrightarrow{+1} . \tag{3.9.2}$$

EXAMPLE 3.10

Let (X, Σ) be an snc pair. Then $\underline{\Omega}^\bullet_{X,\Sigma} \simeq_{\text{qis}} \Omega^\bullet_X(\log \Sigma)(-\Sigma)$.

The Deligne–Du Bois complex of a pair is functorial in the following sense.

PROPOSITION 3.11

Let $\phi : (Y, \Gamma) \to (X, \Delta)$ be a morphism of generalized pairs. Then there exists a filtered natural morphism $\underline{\Omega}^\bullet_{X,\Sigma} \to \mathcal{R}\phi_* \underline{\Omega}^\bullet_{Y,\Gamma}$.

Proof

There exist compatible filtered natural morphisms $\underline{\Omega}^\bullet_X \to \mathcal{R}\phi_* \underline{\Omega}^\bullet_Y$ and $\underline{\Omega}^\bullet_\Sigma \to \mathcal{R}\phi_* \underline{\Omega}^\bullet_\Gamma$ by (2.2.2). They induce the following morphism between exact triangles:

$$\begin{tikzcd}
\underline{\Omega}^0_{X,\Sigma} \arrow[r] \arrow[d, dashed] & \underline{\Omega}^0_X \arrow[r] \arrow[d] & \underline{\Omega}^0_\Sigma \arrow[r, "+1"] \arrow[d] & {} \\
\mathcal{R}\phi_* \underline{\Omega}^0_{Y,\Gamma} \arrow[r] & \mathcal{R}\phi_* \underline{\Omega}^0_Y \arrow[r] & \mathcal{R}\phi_* \underline{\Omega}^0_\Gamma \arrow[r, "+1"] & {}
\end{tikzcd}$$

and thus one obtains the desired natural morphism. $\square$

It follows easily from the definition and Proposition 2.7 that we have the following bounds on the nonzero cohomology sheaves of $\underline{\Omega}^p_{X,\Sigma}$.

PROPOSITION 3.12

Let X be a positive-dimensional variety. Then the ith cohomology sheaf of $\underline{\Omega}^p_{X,\Sigma}$ vanishes for all $i \geq \dim X$; that is, $h^i(\underline{\Omega}^p_{X,\Sigma}) = 0$ for all p and for all $i \geq \dim X$.

Proof

This follows directly from Proposition 2.7 using the long exact cohomology sequence associated to (3.9.2). $\square$

3.D. DB pairs and the DB defect

DEFINITION 3.13

Recall the short exact sequence for the restriction of regular functions from X to Σ:

$$0 \longrightarrow \mathscr{I}_{\Sigma\subseteq X} \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_\Sigma \longrightarrow 0.$$

By (2.2.6), there exist compatible natural maps $\mathscr{O}_X \to \underline{\Omega}^0_X$ and $\mathscr{O}_\Sigma \to \underline{\Omega}^0_\Sigma$, and they induce a morphism between exact triangles,

$$\begin{tikzcd} \mathscr{I}_{\Sigma\subseteq X} \arrow[r] \arrow[d, dashed] & \mathscr{O}_X \arrow[r] \arrow[d] & \mathscr{O}_\Sigma \arrow[r, "+1"] \arrow[d] & {} \\ \underline{\Omega}^0_{X,\Sigma} \arrow[r] & \underline{\Omega}^0_X \arrow[r] & \underline{\Omega}^0_\Sigma \arrow[r, "+1"] & {} \end{tikzcd} \tag{3.13.1}$$

A reduced generalized pair (X,Σ) is called a *DB pair* if the natural morphism $\mathscr{I}_{\Sigma\subseteq X} \to \underline{\Omega}^0_{X,\Sigma}$ from (3.13.1) is a quasi-isomorphism.

REMARK 3.14

Note that just like the notion of a rational pair, the notion of a *DB pair* describes the “singularity” of the relationship between X and Σ. From the definition it is not clear, for instance, whether (X,Σ) being DB implies that X has DB singularities.

PROPOSITION 3.15

Let $\phi : (Y,\Gamma) \to (X,\Sigma)$ be a morphism of generalized pairs. Then there exists a natural morphism $\underline{\Omega}^0_{X,\Sigma} \to \mathcal{R}\phi_*\underline{\Omega}^0_{Y,\Gamma}$ and a commutative diagram,

$$\begin{tikzcd} \mathscr{I}_{\Sigma\subseteq X} \arrow[r] \arrow[d] & \underline{\Omega}^0_{X,\Sigma} \arrow[d] \\ \mathcal{R}\phi_*\mathscr{I}_{\Gamma\subseteq Y} \arrow[r] & \mathcal{R}\phi_*\underline{\Omega}^0_{Y,\Gamma} \end{tikzcd}$$

Proof
Similarly to (3.13.1), one obtains a commutative diagram for (Y, Γ):

$$\begin{tikzcd}
\mathscr{I}_{\Gamma \subseteq Y} \arrow[r] \arrow[d] & \mathscr{O}_Y \arrow[r] \arrow[d] & \mathscr{O}_\Gamma \arrow[r, "+1"] \arrow[d] & {} \\
\underline{\Omega}^0_{Y,\Gamma} \arrow[r] & \underline{\Omega}^0_Y \arrow[r] & \underline{\Omega}^0_\Gamma \arrow[r, "+1"] & {}
\end{tikzcd}$$

Then ϕ induces a morphism between these diagrams:

$$\begin{tikzcd}
\mathscr{I}_{\Sigma \subseteq X} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathscr{O}_X \arrow[rr] \arrow[dr] \arrow[dd] & & \mathscr{O}_\Sigma \arrow[r, "+1"] \arrow[dr] \arrow[dd] & {} & \\
& \underline{\Omega}^0_{X,\Sigma} \arrow[rr] \arrow[dd] & & \underline{\Omega}^0_X \arrow[rr] \arrow[dd] & & \underline{\Omega}^0_\Sigma \arrow[r, "+1"] \arrow[dd] & {} \\
\mathcal{R}\phi_* \mathscr{I}_{\Gamma \subseteq Y} \arrow[rr] \arrow[dr] & & \mathcal{R}\phi_* \mathscr{O}_\Gamma \arrow[rr] \arrow[dr] & & \mathcal{R}\phi_* \mathscr{O}_Y \arrow[r, "+1"] \arrow[dr] & {} & \\
& \mathcal{R}\phi_* \underline{\Omega}^0_{Y,\Gamma} \arrow[rr] & & \mathcal{R}\phi_* \underline{\Omega}^0_Y \arrow[rr] & & \mathcal{R}\phi_* \underline{\Omega}^0_\Gamma \arrow[r, "+1"] & {}
\end{tikzcd}$$

The front face of this diagram provides the one claimed in the statement. $\square$

Similarly to Definition 2.9, we introduce the DB defect of the pair (X, Σ).

DEFINITION 3.16
The *DB defect* of the pair (X, Σ) is the mapping cone of the morphism $\mathscr{I}_{\Sigma \subseteq X} \to \underline{\Omega}^0_{X,\Sigma}$. It is denoted by $\underline{\Omega}^\times_{X,\Sigma}$. Again, one has the exact triangles

(3.16.1) $$\mathscr{I}_{\Sigma \subseteq X} \longrightarrow \underline{\Omega}^0_{X,\Sigma} \longrightarrow \underline{\Omega}^\times_{X,\Sigma} \xrightarrow{+1}$$

and

(3.16.2) $$\underline{\Omega}^\times_{X,\Sigma} \longrightarrow \underline{\Omega}^\times_X \longrightarrow \underline{\Omega}^\times_\Sigma \xrightarrow{+1} .$$

And, again, one has

(3.16.3) $$\begin{aligned} h^0(\underline{\Omega}^\times_{X,\Sigma}) &\simeq h^0(\underline{\Omega}^0_{X,\Sigma})/\mathscr{I}_{\Sigma \subseteq X} \qquad \text{and} \\ h^i(\underline{\Omega}^\times_{X,\Sigma}) &\simeq h^i(\underline{\Omega}^0_{X,\Sigma}) \quad \text{for } i > 0. \end{aligned}$$

LEMMA 3.17
Let (X, Σ) be a reduced generalized pair. Then the following are equivalent.

(3.17.1) *The pair (X, Σ) is DB.*
(3.17.2) *The DB defect of (X, Σ) is acyclic; that is, $\underline{\Omega}^\times_{X,\Sigma} \simeq_{\text{qis}} 0$.*

(3.17.3) *The induced natural morphism* $\underline{\Omega}^{\times}_X \to \underline{\Omega}^{\times}_\Sigma$ *is a quasi-isomorphism.*

(3.17.4) *The induced natural morphism* $h^i(\underline{\Omega}^{\times}_X) \to h^i(\underline{\Omega}^{\times}_\Sigma)$ *is an isomorphism for all* $i \in \mathbb{Z}$.

(3.17.5) *The induced natural morphism* $h^i(\underline{\Omega}^{0}_X) \to h^i(\underline{\Omega}^{0}_\Sigma)$ *is an isomorphism for all* $i \neq 0$ *and a surjection with kernel isomorphic to* $\mathscr{I}_{\Sigma\subseteq X}$ *for* $i = 0$.

REMARK 3.17.1

This statement also applies in the case when $\Sigma = \emptyset$, so it implies Lemma 2.11.

Proof

The equivalence of (3.17.1) and (3.17.2) follows from (3.16.1), the equivalence of (3.17.2) and (3.17.3) follows from (3.16.2), the equivalence of (3.17.3) and (3.17.4) follows from the definition of quasi-isomorphism, and the equivalence of (3.17.4) and (3.17.5) follows from the definition of the DB defect $\underline{\Omega}^{\times}_{X,\Sigma}$, Definition 3.16, and (3.16.3). ☐

Cutting by hyperplanes works the same way as in the absolute case.

PROPOSITION 3.18

Let (X, Σ) *be a reduced general pair, where* X *is a quasi-projective variety and* $H \subset X$ *is a general member of a very ample linear system. Then* $\underline{\Omega}^{\bullet}_{H,H\cap\Sigma} \simeq_{\mathrm{qis}} \underline{\Omega}^{\bullet}_{X,\Sigma} \otimes_L \mathscr{O}_H$ *and* $\underline{\Omega}^{\times}_{H,H\cap\Sigma} \simeq_{\mathrm{qis}} \underline{\Omega}^{\times}_{X,\Sigma} \otimes_L \mathscr{O}_H$.

Proof

This follows directly from Proposition 2.6, (3.9.1), and Proposition 2.10. ☐

We also have the following adjunction-type statement.

PROPOSITION 3.19

Let $X = (Y \cup Z)_{\mathrm{red}}$ *be a union of closed reduced subschemes with intersection* $W = (Y \cap Z)_{\mathrm{red}}$. *Then the DB defects of the pairs* (X, Y) *and* (Z, W) *are quasi-isomorphic. That is,*

$$\underline{\Omega}^{\times}_{X,Y} \simeq_{\mathrm{qis}} \underline{\Omega}^{\times}_{Z,W}.$$

Proof

Consider the following diagram of exact triangles,

$$\begin{tikzcd}
\underline{\Omega}^{\times}_{X,Y} \arrow[r] \arrow[d, "\alpha"] & \underline{\Omega}^{\times}_{X} \arrow[r] \arrow[d, "\beta"] & \underline{\Omega}^{\times}_{Y} \arrow[r, "+1"] \arrow[d, "\gamma"] & {} \\
\underline{\Omega}^{\times}_{Z,W} \arrow[r] & \underline{\Omega}^{\times}_{Z} \arrow[r] & \underline{\Omega}^{\times}_{W} \arrow[r, "+1"] & {}
\end{tikzcd}$$

where β and γ are the natural restriction morphisms and α is the morphism induced by β and γ on the mapping cones. Then by [KK1, Lemma 2.1], there exists an exact triangle

$$\mathsf{Q} \longrightarrow \underline{\Omega}_Y^\times \oplus \underline{\Omega}_Z^\times \longrightarrow \underline{\Omega}_W^\times \xrightarrow{+1}$$

and a map $\sigma : \underline{\Omega}_X^\times \to \mathsf{Q}$ compatible with the above diagram such that α is an isomorphism if and only if σ is one. On the other hand, σ is indeed an isomorphism by Proposition 2.12, and so the statement follows. $\square$

4. Cohomology with compact support

Let X be a complex scheme of finite type, and let $\iota : \Sigma \hookrightarrow X$ be a closed subscheme. Deligne's Hodge theory applied in this situation gives the following theorem.

THEOREM 4.1 ([Del])

Let X be a complex scheme of finite type, let $\iota : \Sigma \hookrightarrow X$ be a closed subscheme, and let $j : U := X \setminus \Sigma \hookrightarrow X$.

(4.1.1) *The natural composition map $j_!\mathbb{C}_U \to \mathscr{I}_{\Sigma\subseteq X} \to \underline{\Omega}^\bullet_{X,\Sigma}$ is a quasi-isomorphism; that is, $\underline{\Omega}^\bullet_{X,\Sigma}$ is a resolution of the sheaf $j_!\mathbb{C}_U$.*

(4.1.2) *The natural map $H^\bullet_{\mathrm{c}}(U,\mathbb{C}) \to \mathbb{H}^\bullet(X,\underline{\Omega}^\bullet_{X,\Sigma})$ is an isomorphism.*

(4.1.3) *If in addition X is proper, then the spectral sequence*

$$E_1^{p,q} = \mathbb{H}^q(X,\underline{\Omega}^p_{X,\Sigma}) \Rightarrow H^{p+q}_{\mathrm{c}}(U,\mathbb{C})$$

degenerates at E_1 and abuts to the Hodge filtration of Deligne's mixed Hodge structure.

Proof

Consider an embedded hyperresolution of $\Sigma \subseteq X$:

$$\begin{tikzcd} \Sigma_\bullet \arrow[r, "\varrho_\bullet"] \arrow[d, "\varepsilon_\bullet"'] & X_\bullet \arrow[d, "\varepsilon_\bullet"] \\ \Sigma \arrow[r, "\varrho"'] & X \end{tikzcd}$$

Then by (2.2.5) and by definition, $\underline{\Omega}^\bullet_{X,\Sigma} \simeq_{\mathrm{qis}} \mathcal{R}\varepsilon_{\bullet *}\Omega^\bullet_{X_\bullet,\Sigma_\bullet}$. The statements then follow from [Del, Sections 8.1, 8.2, 9.3] (see also [GNPP, Section IV.4]). $\square$

COROLLARY 4.2

Let X be a proper complex scheme of finite type, let $\iota : \Sigma \hookrightarrow X$ be a closed subscheme, and let $j : U := X \setminus \Sigma \hookrightarrow X$. Then the natural map

$$H^i(X,\mathscr{I}_{\Sigma\subseteq X}) \to \mathbb{H}^i(X,\underline{\Omega}^0_{X,\Sigma})$$

is surjective for all $i \in \mathbb{N}$.

Proof
By (4.1.3), the natural composition map

$$H^i_{\mathrm{c}}(U,\mathbb{C}) \to H^i(X,\mathscr{I}_{\Sigma\subseteq X}) \to \mathbb{H}^i(X,\underline{\Omega}^0_{X,\Sigma})$$

is surjective. This clearly implies the statement. □

5. DB pairs in nature

PROPOSITION 5.1
Let (X,Σ) *be a reduced generalized pair. If either* X *or* Σ *is DB, then the other one is DB if and only if* (X,Σ) *is a DB pair.*

Proof
Consider the exact triangle (3.16.2):

$$\underline{\Omega}^{\times}_{X,\Sigma} \longrightarrow \underline{\Omega}^{\times}_{X} \longrightarrow \underline{\Omega}^{\times}_{\Sigma} \xrightarrow{+1} .$$

Clearly, if one of the objects in this triangle is acyclic, then it is equivalent that the other two are acyclic. Then the statement follows by Lemmas 2.11 and 3.17. □

As one expects from a good notion of singularity, smooth points are DB. For pairs, being smooth is replaced by being snc.

COROLLARY 5.2
Let (X,Δ) *be an snc pair. Then it is also a DB pair.*

Proof
This follows directly from Proposition 5.1 (cf. (2.2.7), [Ste2, Example 3.2]). It also follows from Corollary 5.3. □

COROLLARY 5.3
Let (X,Δ) *be a log canonical pair, and let* $\Lambda\subset X$ *be an effective integral Weil divisor such that* $\operatorname{supp}\Lambda \subseteq \operatorname{supp}\lfloor\Delta\rfloor$. *Then* (X,Λ) *is a DB pair.*

Proof
By choice, Λ is a union of non-klt centers of the pair (X,Δ), and hence by [KK1, Theorem 1.4], both X and Λ are DB. Then (X,Λ) is a DB pair by Proposition 5.1. □

THEOREM 5.4
Let (X,Σ) *be a reduced generalized pair. Assume that the natural morphism* $\mathscr{I}_{\Sigma\subseteq X} \to \underline{\Omega}^0_{X,\Sigma}$ *has a left inverse. Then* (X,Σ) *is a DB pair.*

Proof
We mimic the proof of [Kov3, Corollary 1.5]. The statement is local, so we may assume that X is affine and hence quasi-projective.

LEMMA 5.5
Assume that there exists a finite subset $P \subseteq X$ such that $(X \setminus P, \Sigma \setminus P)$ is a DB pair. Then the induced morphism

$$H^i_P(X, \mathscr{I}_{\Sigma \subseteq X}) \to \mathbb{H}^i_P(X, \underline{\Omega}^0_{X,\Sigma})$$

is surjective for all $i \in \mathbb{N}$.

Proof
Let $\overline{X}$ be a projective closure of X, and let $\overline{\Sigma}$ be the closure of Σ in $\overline{X}$. Let $Q = \overline{X} \setminus X$, $Z = P \overset{\bullet}{\cup} Q$, and $U = \overline{X} \setminus Z = X \setminus P$. Consider the exact triangle of functors,

$$\mathbb{H}^0_Z(\overline{X}, \_\_) \longrightarrow \mathbb{H}^0(\overline{X}, \_\_) \longrightarrow \mathbb{H}^0(U, \_\_) \xrightarrow{+1}, \tag{5.5.1}$$

and apply it to the morphism $\mathscr{I}_{\overline{\Sigma} \subseteq \overline{X}} \to \underline{\Omega}^0_{\overline{X},\overline{\Sigma}}$. One obtains a morphism of two long exact sequences:

$$\begin{tikzcd}
\mathbb{H}^{i-1}(U, \mathscr{I}_{\overline{\Sigma} \subseteq \overline{X}}) \arrow[r] \arrow[d, "\alpha_{i-1}"] & \mathbb{H}^i_Z(\overline{X}, \mathscr{I}_{\overline{\Sigma} \subseteq \overline{X}}) \arrow[r] \arrow[d, "\beta_i"] & \mathbb{H}^i(\overline{X}, \mathscr{I}_{\overline{\Sigma} \subseteq \overline{X}}) \arrow[r] \arrow[d, "\gamma_i"] & \mathbb{H}^i(U, \mathscr{I}_{\overline{\Sigma} \subseteq \overline{X}}) \arrow[d, "\alpha_i"] \\
\mathbb{H}^{i-1}(U, \underline{\Omega}^0_{\overline{X},\overline{\Sigma}}) \arrow[r] & \mathbb{H}^i_Z(\overline{X}, \underline{\Omega}^0_{\overline{X},\overline{\Sigma}}) \arrow[r] & \mathbb{H}^i(\overline{X}, \underline{\Omega}^0_{\overline{X},\overline{\Sigma}}) \arrow[r] & \mathbb{H}^i(U, \underline{\Omega}^0_{\overline{X},\overline{\Sigma}})
\end{tikzcd}$$

By assumption, α_i is an isomorphism for all i. By Corollary 4.2, γ_i is surjective for all i. Then by the 5-lemma, β_i is also surjective for all i.

By construction, $P \cap Q = \emptyset$, and hence

$$H^i_Z(\overline{X}, \mathscr{I}_{\overline{\Sigma} \subseteq \overline{X}}) \simeq H^i_P(\overline{X}, \mathscr{I}_{\overline{\Sigma} \subseteq \overline{X}}) \oplus H^i_Q(\overline{X}, \mathscr{I}_{\overline{\Sigma} \subseteq \overline{X}}),$$
$$\mathbb{H}^i_Z(\overline{X}, \underline{\Omega}^0_{\overline{X},\overline{\Sigma}}) \simeq \mathbb{H}^i_P(\overline{X}, \underline{\Omega}^0_{\overline{X},\overline{\Sigma}}) \oplus \mathbb{H}^i_Q(\overline{X}, \underline{\Omega}^0_{\overline{X},\overline{\Sigma}}).$$

It follows that the natural map (which is also the restriction of β_i),

$$H^i_P(\overline{X}, \mathscr{I}_{\overline{\Sigma} \subseteq \overline{X}}) \to \mathbb{H}^i_P(\overline{X}, \underline{\Omega}^0_{\overline{X},\overline{\Sigma}}),$$

is surjective for all i.

Now, by excision on local cohomology, one has

$$H^i_P(\overline{X}, \mathscr{I}_{\overline{\Sigma} \subseteq \overline{X}}) \simeq H^i_P(X, \mathscr{I}_{\Sigma \subseteq X}) \qquad \text{and} \qquad \mathbb{H}^i_P(\overline{X}, \underline{\Omega}^0_{\overline{X},\overline{\Sigma}}) \simeq \mathbb{H}^i_P(X, \underline{\Omega}^0_{X,\Sigma}),$$

and so Lemma 5.5 follows. □

It is now relatively straightforward to finish the proof of Theorem 5.4.

By taking repeated hyperplane sections and using Proposition 3.18, we may assume that there exists a finite subset $P \subseteq X$ such that $(X \setminus P, \Sigma \setminus P)$ is a DB pair. Therefore we may apply Lemma 5.5.

By assumption, the natural morphism $\mathscr{I}_{\Sigma\subseteq X} \to \underline{\Omega}^0_{X,\Sigma}$ has a left inverse. This implies that applying any cohomology operator on this map induces an injective map on cohomology. In particular, this implies that the natural morphism

$$H^i_P(X, \mathscr{I}_{\Sigma\subseteq X}) \to \mathbb{H}^i_P(X, \underline{\Omega}^0_{X,\Sigma})$$

is injective for all $i \in \mathbb{N}$. By Lemma 5.5, they are also surjective and hence an isomorphism. Thus the DB defect $\underline{\Omega}^\times_{X,\Sigma}$ is such that all of its local cohomology groups are zero:

$$\mathbb{H}^i_P(X, \underline{\Omega}^\times_{X,\Sigma}) = 0 \quad \text{for all } i.$$

On the other hand, by assumption, $\underline{\Omega}^\times_{X,\Sigma}$ is supported entirely on P, so $\mathbb{H}^i(X \setminus P, \underline{\Omega}^\times_{X,\Sigma}) = 0$ as well. However, then $\mathbb{H}^i(X, \underline{\Omega}^\times_{X,\Sigma}) = 0$ by the long exact sequence induced by (5.5.1). Now $\dim P \le 0$, so the spectral sequence that computes hypercohomology from the sheaf cohomology of the cohomology of the complex $\underline{\Omega}^\times_{X,\Sigma}$ degenerates and gives that for any $i \in \mathbb{N}$, $\mathbb{H}^i(X, \underline{\Omega}^\times_{X,\Sigma}) = H^0(X, h^i(\underline{\Omega}^\times_{X,\Sigma}))$, so, since we assumed that X is affine, it follows that $h^i(\underline{\Omega}^\times_{X,\Sigma}) = 0$ for all i. Therefore $\underline{\Omega}^\times_{X,\Sigma} \simeq_{\text{qis}} 0$, and thus the statement is proven. $\square$

COROLLARY 5.6

Let (X, Δ) be a weakly rational pair. Then it is a DB pair.

Proof

Let $\phi : (Y, \Delta_Y) \to (X, \Delta)$ be a log resolution such that γ admits a left inverse γ^ℓ. Then by Proposition 3.15, one has the commutative diagram

$$\begin{tikzcd}
\mathscr{O}_X(-\Delta) \arrow[r] \arrow[d, shift left, "\gamma"] & \underline{\Omega}^0_{X,\Delta} \arrow[d, "\alpha"] \\
\mathcal{R}\phi_*\mathscr{O}_Y(-\Delta_Y) \arrow[u, shift left, bend left, "\gamma^\ell"] \arrow[r, "\delta", "\simeq_{\text{qis}}"'] & \mathcal{R}\phi_*\underline{\Omega}^0_{Y,\Delta_Y}
\end{tikzcd}$$

Recall that as (Y, Δ_Y) is an snc pair, it is also DB by Corollary 5.2, and hence δ is a quasi-isomorphism. Then $\gamma^\ell \circ \delta^{-1} \circ \alpha$ is a left inverse to $\mathscr{O}_X(-\Delta) \to \underline{\Omega}^0_{X,\Delta}$, so the statement follows from Theorem 5.4. $\square$

COROLLARY 5.7

A rational pair is a DB pair.

Proof

As a rational pair is also a weakly rational pair, this is straightforward from Corollary 5.6. $\square$

COROLLARY 5.8

Let (X,Δ) be a dlt pair, and let $\Lambda\subset X$ be an effective integral Weil divisor such that $\operatorname{supp}\Lambda\subseteq\operatorname{supp}\lfloor\Delta\rfloor$. *Then (X,Λ) is a DB pair.*

Proof

A dlt pair is also an lc pair, so this follows from Corollary 5.3. $\square$

6. Vanishing Theorems

The following is the main vanishing result of this article. Note that a weaker version of it appeared in [GKKP, Corollary 13.4].

THEOREM 6.1

Let (X,Σ) be a DB pair, and let $\pi:\widetilde{X}\to X$ be a proper birational morphism with $E:=\operatorname{Exc}(\pi)$. Let $\widetilde{\Sigma}=E\cup\pi^{-1}\Sigma$ and $\Upsilon:=\overline{\pi(E)\setminus\Sigma}$, both considered with their induced reduced subscheme structure. Further, let $s\in\mathbb{N}$, $s>0$ be such that $h^i(\underline{\Omega}^\circ_{\Upsilon,\Upsilon\cap\Sigma})=0$ for $i\geq s$. Then

$$\mathcal{R}^i\pi_*\underline{\Omega}^0_{\widetilde{X},\widetilde{\Sigma}}=0 \quad \textit{for all } i\geq s.$$

Proof

Let $\Gamma=\Sigma\cup\Upsilon$, and consider the exact triangle (2.2.9),

$$\underline{\Omega}^0_X \longrightarrow \underline{\Omega}^0_\Gamma\oplus\mathcal{R}\pi_*\underline{\Omega}^0_{\widetilde{X}} \longrightarrow \mathcal{R}\pi_*\underline{\Omega}^0_{\widetilde{\Sigma}} \xrightarrow{+1}, \tag{6.1.1}$$

which induces the long exact sequence of sheaves

$$\longrightarrow h^i(\underline{\Omega}^0_X) \xrightarrow{(\alpha^i,\sigma^i)} h^i(\underline{\Omega}^0_\Gamma)\oplus\mathcal{R}^i\pi_*\underline{\Omega}^0_{\widetilde{X}} \longrightarrow \mathcal{R}^i\pi_*\underline{\Omega}^0_{\widetilde{\Sigma}} \longrightarrow h^{i+1}(\underline{\Omega}^0_X) \longrightarrow\cdot$$

By Remark 3.17, the natural morphism $\gamma^i:h^i(\underline{\Omega}^0_X)\to h^i(\underline{\Omega}^0_\Sigma)$ is an isomorphism for $i>0$. By Proposition 3.19 and the assumption, we obtain $h^i(\underline{\Omega}^\circ_{\Gamma,\Sigma})=0$ for $i\geq s>0$, and hence the natural morphism $\beta^i:h^i(\underline{\Omega}^0_\Gamma)\to h^i(\underline{\Omega}^0_\Sigma)$ is an isomorphism for $i\geq s$. Using the fact that $\gamma^i=\beta^i\circ\alpha^i$, we obtain that the morphism $\alpha^i:h^i(\underline{\Omega}^0_X)\to h^i(\underline{\Omega}^0_\Gamma)$ is an isomorphism for $i\geq s>0$, and hence the natural restriction map

$$\varrho^i:\mathcal{R}^i\pi_*\underline{\Omega}^0_{\widetilde{X}}\to\mathcal{R}^i\pi_*\underline{\Omega}^0_{\widetilde{\Sigma}}$$

is an isomorphism for $i\geq s$. This in turn implies that $\mathcal{R}^i\pi_*\underline{\Omega}^0_{\widetilde{X},\widetilde{\Sigma}}=0$ for $i\geq s$, as desired. $\square$

As a corollary, a slight generalization of [GKKP, Corollary 13.4] follows.

COROLLARY 6.2

Let (X,Σ) be a DB pair, and let $\pi:\widetilde{X}\to X$ be a log resolution of (X,Σ) with $E:=\operatorname{Exc}(\pi)$. Let $\widetilde{\Sigma}=E\cup\pi^{-1}\Sigma$ and $\Upsilon:=\overline{\pi(E)\setminus\Sigma}$, both considered with their *induced reduced subscheme structure. Then*

$$\mathcal{R}^i\pi_*\mathscr{I}_{\widetilde{\Sigma}\subseteq\widetilde{X}} = 0 \quad \textit{for all } i \geq \max(\dim\Upsilon, 1).$$

In particular, if X *is normal of dimension* $n\geq 2$*, then* $\mathcal{R}^{n-1}\pi_*\mathscr{I}_{\widetilde{\Sigma}\subseteq\widetilde{X}} = 0$.

Proof

Let $s = \max(\dim\Upsilon, 1)$. Then $h^i(\underline{\Omega}^{\circ}_{\Upsilon,\Upsilon\cap\Sigma}) = 0$ for $i \geq s$ by Proposition 3.12. As the pair $(\widetilde{X}, \widetilde{\Sigma})$ is snc, it is also DB, and hence $\underline{\Omega}^0_{\widetilde{X},\widetilde{\Sigma}} \simeq_{\mathrm{qis}} \mathscr{I}_{\widetilde{\Sigma}\subseteq\widetilde{X}}$. Therefore the statement follows from Theorem 6.1. □

We have a stronger result for log canonical pairs and for that we need the following definition.

DEFINITION 6.3

A log resolution of a dlt pair (Z,Θ), $g:(Y,\Gamma)\to(Z,\Theta)$ is called a *Szabó-resolution*, if there exist A, B effective $\mathbb{Q}$-divisors on Y without common irreducible components, such that $\mathrm{supp}(A+B)\subset \mathrm{exc}(g)$, $\lfloor A\rfloor = 0$, and

$$K_Y + \Gamma \sim_{\mathbb{Q}} g^*(K_Z+\Theta) - A + B.$$

REMARK 6.4

Every dlt pair admits a Szabó-resolution by [Sza] (cf. [KM, 2.44]).

COROLLARY 6.5

Let (X,Δ) *be a* $\mathbb{Q}$*-factorial log canonical pair, and let* $\pi:\widetilde{X}\to X$ *be a log resolution of* (X,Δ)*. Let* $\widetilde{\Delta} = (\pi_*^{-1}\lfloor\Delta\rfloor + \mathrm{Exc}_{\mathrm{nklt}}(\pi))_{\mathrm{red}}$*. Then*

$$\mathcal{R}^i\pi_*\mathscr{O}_{\widetilde{X}}(-\widetilde{\Delta}) = 0 \quad \textit{for } i > 0.$$

Proof

First note that the statement is true if (X,Δ) is an snc pair and π is the blowup of X along a smooth center. Indeed, if the center is a non-klt center, then the statement is a direct consequence of the Kawamata-Viehweg vanishing theorem and if the center is not a non-klt center, then this is a Szabó-resolution and the statement follows as in the proof of [KK2, 111]. This implies the following.

LEMMA 6.5.1

Let $\pi_i:(X_i,\Delta_i)\to(X,\Delta)$ *for* $i=1,2$ *be two log resolutions of* (X,Δ)*, and let* $\widetilde{\Delta}_i = ((\pi_i^{-1})_*\lfloor\Delta\rfloor + \mathrm{Exc}_{\mathrm{nklt}}(\pi_i))_{\mathrm{red}} \subset X_i$*. Then*

$$\mathcal{R}(\pi_1)_*\mathscr{O}_{X_1}(-\widetilde{\Delta}_1) \simeq \mathcal{R}(\pi_2)_*\mathscr{O}_{X_2}(-\widetilde{\Delta}_2).$$

Proof
By [AKMW, Theorem 0.3.1(6)] (cf. [BL, Theorem 3.8]) the induced birational map between X_1 and X_2 can be written as a sequence of blowing ups and blowing downs along smooth centers. Then the statement follows from the above observation and the definition of the $\widetilde{\Delta}$'s. □

Now we turn to proving the general case. Consider the minimal dlt model $\mu: (X^{\mathrm{m}}, \Delta^{\mathrm{m}}) \to (X, \Delta)$ [KK1, Theorem 3.1]. Let $\Sigma := \lfloor \Delta \rfloor \cup \mu(\operatorname{Exc}(\mu))$ considered with the induced reduced subscheme structure. From the definition of a minimal dlt model, it follows that Σ is a union of non-klt centers of (X, Δ). Then by [KK1, Theorem 1.4], both X and Σ are DB, and hence (X, Σ) is a DB pair by Proposition 5.1.

Since $(X^{\mathrm{m}}, \Delta^{\mathrm{m}})$ is dlt, $(X^{\mathrm{m}}, \lfloor \Delta^{\mathrm{m}} \rfloor)$ is a DB pair by Corollary 5.3. Therefore,

$$\underline{\Omega}^0_{X^{\mathrm{m}}, \lfloor \Delta^{\mathrm{m}} \rfloor} \simeq_{\mathrm{qis}} \mathscr{O}_{X^{\mathrm{m}}}(-\lfloor \Delta^{\mathrm{m}} \rfloor).$$

By the definition of a minimal dlt model, $\lfloor \Delta^{\mathrm{m}} \rfloor = (\pi^{-1}\Sigma)_{\mathrm{red}} \supseteq \operatorname{Exc}(\mu)$, and then it follows from Theorem 6.1 that $\mathcal{R}^i \mu_* \mathscr{O}_{X^{\mathrm{m}}}(-\lfloor \Delta^{\mathrm{m}} \rfloor) = 0$ for $i > 0$. Hence

$$\mathcal{R}\mu_* \mathscr{O}_{X^{\mathrm{m}}}(-\lfloor \Delta^{\mathrm{m}} \rfloor) \simeq_{\mathrm{qis}} \mathscr{O}_X(-\lfloor \Delta \rfloor). \tag{6.5.2}$$

Next let $\tau : \widehat{X} \to X^{\mathrm{m}}$ be the Szabó-resolution of $(X^{\mathrm{m}}, \Delta^{\mathrm{m}})$, $\sigma = \mu \circ \tau$, $\widehat{\Delta} = \tau_*^{-1} \lfloor \Delta^{\mathrm{m}} \rfloor = (\sigma_*^{-1} \lfloor \Delta \rfloor + \operatorname{Exc}_{\mathrm{nklt}}(\sigma))_{\mathrm{red}}$, and $\lambda = \pi^{-1} \circ \sigma$.

Then by [KK2, 111], we have that

$$\mathcal{R}\tau_* \mathscr{O}_{\widehat{X}}(-\widehat{\Delta}) \simeq_{\mathrm{qis}} \mathscr{O}_{X^{\mathrm{m}}}(-\lfloor \Delta^{\mathrm{m}} \rfloor),$$

and hence, by (6.5.2),

$$\begin{aligned} \mathcal{R}\sigma_* \mathscr{O}_{\widehat{X}}(-\widehat{\Delta}) &\simeq_{\mathrm{qis}} \mathcal{R}\mu_* \mathcal{R}\tau_* \mathscr{O}_{\widehat{X}}(-\widehat{\Delta}) \simeq_{\mathrm{qis}} \\ \mathcal{R}\mu_* \mathscr{O}_{X^{\mathrm{m}}}(-\lfloor \Delta^{\mathrm{m}} \rfloor) &\simeq_{\mathrm{qis}} \mathscr{O}_X(-\lfloor \Delta \rfloor). \end{aligned}$$

The proof is finished by applying (6.5.1) to $\widehat{X}$ and $\widetilde{X}$. □

Finally, observe that Corollary 6.5 implies that log canonical singularities are not too far from being rational.

COROLLARY 6.6
Let X be a variety with log canonical singularities, and let $\pi : \widetilde{X} \to X$ be a resolution of X with $E_{\mathrm{lc}} := \operatorname{Exc}_{\mathrm{nklt}}(\pi)$. Then

$$\mathcal{R}^i \pi_* \mathscr{O}_{\widetilde{X}}(-E_{\mathrm{lc}}) = 0 \quad \textit{for } i > 0.$$

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