Unspecified Journal
Volume **000**, Number 0, 0000

ALGEBRAIC HYPERBOLICITY OF FINE MODULI SPACES

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Throughout the groundfield is $\mathbb{C}$, the field of complex numbers.

The typical example of a moduli space is $\mathfrak{M}_g$, the moduli space of smooth projective curves of genus g. Unfortunately, it is not a *fine* moduli space. However, if one only considers smooth projective curves of genus g that have no non-trivial automorphisms, then one obtains a fine moduli space, $\mathfrak{M}_g^\circ$.

More generally there exists a fine moduli space, $\mathfrak{M}$, for smooth projective canonically polarized varieties that have no non-trivial automorphisms [Viehweg95, 7.8]. Regarding these fine moduli spaces Shokurov made the following conjecture:

0.1 Algebraic Hyperbolicity Conjecture. [Shokurov97, 14.1] *There is no non-constant morphism* $\mathbb{A}^1 \to \mathfrak{M}$.

Remark. Shokurov's conjecture is somewhat more general. Please refer to [ibid.] for the exact statement.

The main result of this article is a proof of the above conjecture (in fact a little more is proved).

0.2 Theorem. *Let* $g^\circ : Y^\circ \to C^\circ$ *be a smooth family of canonically polarized varieties (i.e.,* ω_{Y_t} *is ample for all* $t \in C^\circ$*) such that* C° *is an open dense subset of* $\mathbb{P}^1$*. Let* $g : Y \to \mathbb{P}^1$ *be a projective family such that* $g^{-1}(C^\circ) = Y^\circ$ *and* $g\big|_{Y^\circ} = g^\circ$*.*

Then g *is either isotrivial or has at least* 3 *singular fibres. In particular there is no non-constant morphism from* $\mathbb{A}^1 \setminus \{0\}$ *to a fine moduli space of smooth canonically polarized varieties, hence, the latter space is "hyperbolic" as predicted by Shokurov's Conjecture.*

This also gives a partial answer to the following question of Catanese and Schneider.

0.3 Question. [Catanese-Schneider95, 4.1] *Let* Y *be a smooth variety of general type,* $g : Y \to \mathbb{P}^1$ *a fibration. Is it true that* g *has at least* 3 *singular fibres?*

The answer to this question is affirmative, when $\dim Y = 2$ by [Beauville81] and when $\dim Y = 3$ by [Migliorini95] and [Kovács97]. Here the question is answered in the affirmative in any dimension for canonically polarized varieties. This result was independently obtained by [Bedulev-Viehweg99].

Received by the editors March, 1998.
gSupported in part by an AMS Centennial Fellowship and by NSF Grant DMS-9818357.

0.4 Corollary. *Let Y be a smooth canonically polarized variety (i.e., ω_Y is ample), $g : Y \to \mathbb{P}^1$ a fibration. Then g has at least 3 singular fibres.*

The same methods give better estimates in low dimensions for semi-stable fibrations of canonically polarized varieties.

0.5 Theorem. *Let Y be a smooth canonically polarized variety of dimension $n \leq 3$, $g : Y \to \mathbb{P}^1$ a semi-stable fibration. Then g has at least 5 singular fibres if $n = 2$ and it has at least 4 singular fibres if $n = 3$.*

Remark. The case $n = 2$ is a special case of the main result of [Tan95], while the case $n = 3$ is new.

It would also be interesting to see whether the stronger assumption on the total space implies stronger restrictions:

0.6 Question. Let Y be a smooth variety of general type, $g : Y \to \mathbb{P}^1$ a fibration (not necessarily semi-stable). What is the minimal number of singular fibres of g?

Remark. Beauvillle gave examples of non-isotrivial families of curves of genus larger than 1 with only 3 singular fibers. However, the total space of those families does not satisfy the condition. For instance the total space of the example given in [Beauville81] is not of general type.

Tan gave an example of a semi-stable family of curves of genus 2 with 5 singular fibres [Tan95], however the total space of that example is still not of general type. On the other hand Tan's example might be altered to give an example with a total space of general type and only 5 singular fibres.

Definitions and Notation. Throughout the article the groundfield is $\mathbb{C}$, the field of complex numbers.

A smooth projective variety, X, is called *canonically polarized* if ω_X is ample.

A morphism $\phi : X \to Y$ is said to have *fibre dimension* at most r if $\dim \phi^{-1}(y) \leq r$ for all $y \in Y$. A Cartier divisor D on X is called *r-ample* if for some $m > 0$, mD is base point free and the morphism given by mD has fibre dimension at most r. In particular D is ample if and only if it is 0-ample.

D is called *nef* if $D \cdot C \geq 0$ for every proper curve $C \subset X$.

D is called *big* if X is proper and $|mD|$ gives a birational map for some $m > 0$. In particular ample implies nef and big.

The same notions make sense for line bundles in the obvious way.

Let $g : Y \to C$ be a morphism of normal varieties, then $\omega_{Y/C} = \omega_Y \otimes g^*\omega_C^{-1}$.

Let $f : X \to S$ be a morphism of schemes, then X_s denotes the fibre of f over the point $s \in S$ and f_s denotes the restriction of f to X_s. More generally, for a morphism $\sigma : Z \to S$, let $f_Z : X_Z = X \times_S Z \to Z$. If f is composed with another morphism $g : S \to T$, then for a $t \in T$, X_t denotes the fibre of $g \circ f$ over the point t, i.e., $X_t = X_{S_t}$. f_Z and X_Z may also be denoted by f_σ and X_σ respectively.

$f : X \to S$ is called *isotrivial* if the smooth fibers of f are pairwise isomorphic.

Acknowledgement. I would like to thank Ingrid Bauer for pointing out an error in an earlier draft and Meng Chen for helpful comments. Commutative diagrams were drawn with the help of Paul Taylor's TEX macro package.

§1. Vanishing

The following lemma is almost identical to the one proved in [Kovács97]. It is a tiny bit more general and more importantly it contains an extra assumption that was implicitly assumed there but was not stated, namely that X has the same dimension as Y. The proof is also almost the same, it is mainly reproduced here for the reader's convenience.

1.1 Lemma. *Let $g : Y \to C$ be a morphism from a smooth n-dimensional projective variety Y to a smooth projective curve C. Let $\Delta \subset C$ be the set of points over which g is not smooth and assume that $D = g^*(\Delta)$ is a simple normal crossing divisor. Further let $\phi : Y \to X$ be a morphism to an n-dimensional projective variety X and $f : X \to C$ a morphism such that $g = f \circ \phi$ and assume that $\dim \phi^{-1}(x) \leq 1$ for all $x \in X \setminus f^{-1}(\Delta)$. Let $\mathcal{L}$ be a line bundle on Y such that there exists an ample line bundle $\mathcal{A}$ on X and a natural number $m \in \mathbb{N}$ such that $\mathcal{L}^m \simeq \phi^*\mathcal{A}$. Assume finally that $\mathcal{A} \otimes f^*\omega_C(\Delta)^{-m(n-1)}$ is also ample. Then for any line bundle, $\mathcal{K}$, such that $\mathcal{K} \supseteq \mathcal{L}$,*

$$H^n(Y, \mathcal{K} \otimes g^*\omega_C) = 0$$

1.1.1 Remark. If $\omega_C(\Delta)^{-1}$ is nef, then $\mathcal{A}$ ample implies that $\mathcal{A} \otimes f^*\omega_C(\Delta)^{-m(n-1)}$ is also ample, so in that case the last condition is vacuous.

Proof. Taking exterior powers of the sheaves of logarithmic differentials one has the following short exact sequences for all $p = 0, \dots, n-1$.

$$0 \longrightarrow \Omega^{p-1}_{Y/C}(\log D) \otimes g^*\omega_C(\Delta) \longrightarrow \Omega^p_Y(\log D) \longrightarrow \Omega^p_{Y/C}(\log D) \longrightarrow 0$$

Define $\mathcal{L}_p = \mathcal{L} \otimes g^*\omega_C(\Delta)^{p-(n-1)}$. Then the above short exact sequence yields:

$$0 \longrightarrow \Omega^{p-1}_{Y/C}(\log D) \otimes \mathcal{L}^{-1}_{p-1} \longrightarrow \Omega^p_Y(\log D) \otimes \mathcal{L}^{-1}_p \longrightarrow \Omega^p_{Y/C}(\log D) \otimes \mathcal{L}^{-1}_p \longrightarrow 0$$

Now

$$\mathcal{L}^m_p = \mathcal{L}^m \otimes g^*\omega_C(\Delta)^{m(p-(n-1))} \simeq \phi^*\left(\mathcal{A} \otimes f^*\omega_C(\Delta)^{m(p-(n-1))}\right),$$

where

$$\begin{aligned} \mathcal{A} \otimes f^*\omega_C(\Delta)^{m(p-(n-1))} &\simeq \underbrace{\mathcal{A}}_{\text{ample}} \otimes \left(f^*\omega_C(\Delta)^{-1}\right)^{m(n-1-p)} \\ &\simeq \underbrace{\left(\mathcal{A} \otimes f^*\omega_C(\Delta)^{-m(n-1)}\right)}_{\text{ample}} \otimes f^*\omega_C(\Delta)^{mp} \end{aligned}$$

is ample, since either $\omega_C(\Delta)$ or $\omega_C(\Delta)^{-1}$ is nef. Then $H^{n-p-1}(Y, \Omega^p_Y(\log D) \otimes \mathcal{L}^{-1}_p) = 0$ by [Esnault-Viehweg92, 6.7], so the map,

$$H^{n-p-1}(Y, \Omega^p_{Y/C}(\log D) \otimes \mathcal{L}^{-1}_p) \longrightarrow H^{n-(p-1)-1}(Y, \Omega^{p-1}_{Y/C}(\log D) \otimes \mathcal{L}^{-1}_{p-1})$$

is injective for all p, so in fact

$$H^0(Y, \Omega^{n-1}_{Y/C}(\log D) \otimes \mathcal{L}^{-1}_{n-1}) \longrightarrow H^{n-1}(Y, \Omega^0_{Y/C}(\log D) \otimes \mathcal{L}^{-1}_0), \text{ and then}$$
$$H^0(Y, \omega_{Y/C} \otimes \mathcal{L}^{-1}) \longrightarrow H^{n-1}(Y, \mathcal{L}^{-1}_0)$$

is injective. $H^{n-1}(Y, \mathcal{L}^{-1}_0) = 0$ by [Esnault-Viehweg92, 6.7], so $H^0(Y, \omega_{Y/C} \otimes \mathcal{L}^{-1}) = 0$, and

$$H^n(Y, \mathcal{L} \otimes g^*\omega_C) = 0$$

by Serre duality.

Now let $\mathcal{L} \subseteq \mathcal{K}$ and $\mathcal{Q} = \mathcal{K}/\mathcal{L}$. Since $\mathcal{Q}$ is supported on a proper subvariety, $H^n(Y, \mathcal{Q} \otimes g^*\omega_C) = 0$, so

$$H^n(Y, \mathcal{L} \otimes g^*\omega_C) \longrightarrow H^n(Y, \mathcal{K} \otimes g^*\omega_C)$$

is surjective. Therefore $H^n(Y, \mathcal{K} \otimes g^*\omega_C) = 0$. □

1.2 Corollary. *Let $g : Y \to C$ be a morphism from a smooth n-dimensional projective variety Y to a smooth projective curve C. Let $\Delta \subset C$ be the set of points over which g is not smooth and assume that $D = g^*(\Delta)$ is a simple normal crossing divisor. Further let $\phi : Y \to X$ be a morphism to an n-dimensional projective variety X and $f : X \to C$ a morphism such that $g = f \circ \phi$ and assume that $\dim \phi^{-1}(x) \leq 1$ for all $x \in X \setminus f^{-1}(\Delta)$. Let $\mathcal{L}$ be a line bundle on Y such that there exists an ample line bundle $\mathcal{A}$ on X and a natural number $m \in \mathbb{N}$ such that $\mathcal{L}^m \simeq \phi^*\mathcal{A}$. Assume further that $\omega_C(\Delta)$ is nef. Then for any line bundle, $\mathcal{K}$, such that $\mathcal{K} \supseteq \mathcal{L}$,*

$$H^n(Y, \mathcal{K} \otimes g^*\omega_C(\Delta)^{n-1} \otimes g^*\omega_C) = 0.$$

Proof. Replace $\mathcal{L}$ with $\mathcal{L} \otimes g^*\omega_C(\Delta)^{n-1}$ and $\mathcal{A}$ with $\mathcal{A} \otimes f^*\omega_C(\Delta)^{m(n-1)}$. This new line bundle satisfies the last condition of (1.1), i.e.,

$$(\mathcal{A} \otimes f^*\omega_C(\Delta)^{m(n-1)}) \otimes f^*\omega_C(\Delta)^{-m(n-1)} = \mathcal{A}$$

is ample. The lemma applied for $\mathcal{K} \otimes g^*\omega_C(\Delta)^{n-1}$ gives the statement. □

§2. General Families

2.1 Theorem. *Let $g^\circ : Y^\circ \to C^\circ$ be a smooth family of projective varieties such that ω_{Y_t} is big and 1-ample for all $t \in C^\circ$ where C° is an open dense subset of $\mathbb{P}^1$.*

Let $g : Y \to \mathbb{P}^1$ be a projective family such that $g^{-1}(C^\circ) = Y^\circ$ and $g\big|_{Y^\circ} = g^\circ$.

Then either g is isotrivial or has at least 3 singular fibres.

Proof. Let $\Delta \subset \mathbb{P}^1$ be the discriminant locus of g, i.e., g is smooth over the complement of Δ. One can assume that Y is smooth, g is non-isotrivial and that $\Delta = \mathbb{P}^1 \setminus C^\circ$.

- Suppose $\#\Delta = 2$.

Claim. One can construct two new families $\hat{g} : \hat{Y} \to \mathbb{P}^1$ and $\hat{f} : \hat{X} \to \mathbb{P}^1$ and a morphism $\hat{\phi} : \hat{Y} \to \hat{X}$ such that $\hat{g} = \hat{f} \circ \hat{\phi}$, $\hat{Y}$ is smooth, $\hat{g}\big|_{\hat{g}^{-1}(C^\circ)}$ is smooth, $\hat{D} = \hat{g}^*\Delta$ is a divisor with simple normal crossings, and there exists an ample line bundle $\hat{\mathcal{A}}$ on $\hat{X}$ and a line bundle $\hat{\mathcal{L}}$ on $\hat{Y}$ such that $\omega_{\hat{Y}/\mathbb{P}^1} \supseteq \hat{\mathcal{L}}$ and $\hat{\mathcal{L}}^m = \hat{\phi}^*\hat{\mathcal{A}}$ for some $m \in \mathbb{N}$.

By [Kollár87, p.363] $g_*\omega^r_{Y/\mathbb{P}^1}$ is ample for some $r > 0$, so

$$g_*\omega^r_{Y/\mathbb{P}^1} \simeq \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i) \qquad a_i \geq 1,$$

Let $t, s \in \mathbb{P}^1 \setminus \Delta$ and $\mathcal{I}_{t,s}$ their ideal sheaf. Then there exists an $l_0 \in \mathbb{N}$ such that for every $l \geq l_0$ and $i > 0$,

$$H^i(\mathbb{P}^1, \mathrm{Sym}^l(g_*\omega^r_{Y/\mathbb{P}^1}) \otimes \mathcal{I}_{t,s}) = 0.$$

Hence

$$\nu : H^0(\mathbb{P}^1, \mathrm{Sym}^l(g_*\omega^r_{Y/\mathbb{P}^1})) \longrightarrow \Big(\mathrm{Sym}^l(g_*\omega^r_{Y/\mathbb{P}^1}) \otimes k(t)\Big) \oplus \Big(\mathrm{Sym}^l(g_*\omega^r_{Y/\mathbb{P}^1}) \otimes k(s)\Big)$$

is surjective.

Since $\omega^r_{Y/\mathbb{P}^1}$ restricted to Y_t and Y_s is semi-ample,

$$\varepsilon : \mathrm{Sym}^l(g_*\omega^r_{Y/\mathbb{P}^1}) \to g_*\omega^{lr}_{Y/\mathbb{P}^1}$$

is also surjective over $\mathbb{P}^1 \setminus \Delta$ for $l \gg 0$. Thus one has the following commutative diagram:

$$\begin{tikzcd}
H^0(\mathbb{P}^1, \mathrm{Sym}^l(g_*\omega^r_{Y/\mathbb{P}^1})) \arrow[r, "\nu"] \arrow[d] & \Big(\mathrm{Sym}^l(g_*\omega^r_{Y/\mathbb{P}^1}) \otimes k(t)\Big) \oplus \Big(\mathrm{Sym}^l(g_*\omega^r_{Y/\mathbb{P}^1}) \otimes k(s)\Big) \arrow[d, "\varepsilon"] \\
H^0(\mathbb{P}^1, g_*\omega^{lr}_{Y/\mathbb{P}^1}) \arrow[r, "\sigma"'] & \Big(g_*\omega^{lr}_{Y/\mathbb{P}^1} \otimes k(t)\Big) \oplus \Big(g_*\omega^{lr}_{Y/\mathbb{P}^1} \otimes k(s)\Big),
\end{tikzcd}$$

with ν and ε surjective, so σ is surjective as well.

Therefore

$$H^0(Y, \omega^m_{Y/\mathbb{P}^1}) \to H^0(Y_t, \omega^m_{Y_t}) \oplus H^0(Y_s, \omega^m_{Y_s}) \tag{2.1.2}$$

is surjective for sufficiently large and divisible $m > 0$.

Now choose m such that (2.1.2) is surjective and $\omega^m_{Y_t}$ is generated by global sections and the morphism, $\phi_{\omega^m_{Y_t}}$, induced by $\omega^m_{Y_t}$ is birational and has fibre dimension at most 1 for all $t \in C^\circ$. Then $\omega^m_{Y/\mathbb{P}^1}$ is generated by global sections over C°, and it defines a rational map

$$\phi = \phi_{\omega^m_{Y/\mathbb{P}^1}} : Y \dashrightarrow X,$$

such that $\phi\big|_{Y^\circ}$ is a morphism, ϕ separates the fibres of g over C°, and $\phi\big|_{Y_t} = \phi_{\omega^m_{Y_t}}$. Choosing a larger m if necessary one can assume that X is normal.

Thus ϕ is a birational map, so $\omega_{Y/\mathbb{P}^1}$ is big. Let

$$\begin{tikzcd} & Z \arrow[dl, "\pi"'] \arrow[dr, "\psi"] & \\ Y \arrow[rr, dashed, "\phi"'] & & X \end{tikzcd}$$

be the resolution of indeterminacies of ϕ.

Now let $x \in X$ an arbitrary point. Then $\psi^{-1}(x)$ is connected by Zariski's Main Theorem. Suppose $(g \circ \pi)(\psi^{-1}(x))$ is not a single point. Then $(g \circ \pi)(\psi^{-1}(x)) = \mathbb{P}^1$, so there exist $t, s \in C^\circ \subset \mathbb{P}^1$, $s \neq t$ and $y_t \in Y_t$ and $y_s \in Y_s$ such that $\phi(y_t) = x = \phi(y_s)$. This contradicts the fact that ϕ separates the fibres over C°. Therefore $(g \circ \pi)(\psi^{-1}(x))$ is a single point, so X is also a family over $\mathbb{P}^1$, i.e., there exists a morphism $f : X \to \mathbb{P}^1$ such that the following diagram is commutative:

$$\begin{tikzcd} & Z \arrow[dl, "\pi"'] \arrow[dr, "\psi"] & \\ Y \arrow[rr, dashed, "\phi"'] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

By construction there exists an ample line bundle $\mathcal{A}$ on X such that

$$\omega^m_{Z/\mathbb{P}^1} \supseteq \pi^*\omega^m_{Y/\mathbb{P}^1} \supseteq \psi^*\mathcal{A}.$$

Replacing $g : Y \to \mathbb{P}^1$ by $g \circ \pi : Z \to \mathbb{P}^1$ we may assume that ϕ is an everywhere defined birational morphism and that there exists an ample line bundle $\mathcal{A}$ on X such that $\omega^m_{Y/\mathbb{P}^1} \supseteq \mathcal{L} = \phi^*\mathcal{A}$ and $\omega^m_{Y/\mathbb{P}^1}\big|_{Y^\circ} = \mathcal{L}\big|_{Y^\circ}$

$$\begin{tikzcd} Y \arrow[r, "\phi"] \arrow[d, "g"'] & X \arrow[dl, "f"] \\ \mathbb{P}^1 & \end{tikzcd}$$

Next let $\sigma : \mathbb{P}^1 \to \mathbb{P}^1$ be the m-th root cover that ramifies over Δ and $\pi : \hat{Y} \to Y_\sigma = \mathbb{P}^1 \times_\sigma Y$ a resolution of singularities of Y_σ such that $\hat{D} = \pi^* g^*_\sigma \Delta$ is a divisor with simple normal crossings. (Replace m with a suitable multiple of itself if necessary.)

$$
\begin{tikzcd}
\hat{Y} \arrow[r, "\pi"] & Y_\sigma \arrow[rr, "\sigma_Y"] \arrow[dr, "\phi_\sigma"] \arrow[dd, "g_\sigma"'] & & Y \arrow[dr, "\phi"] \arrow[dd, "g"'] & \\
& & X_\sigma \arrow[rr, "\sigma_X"'] \arrow[dl, "f_\sigma"] & & X \arrow[dl, "f"] \\
& \mathbb{P}^1 \arrow[rr, "\sigma"'] & & \mathbb{P}^1 &
\end{tikzcd}
$$

Let $E = \sum e_i E_i$ be an effective divisor supported on $g^{-1}(\Delta)$ such that

$$\omega^m_{Y/\mathbb{P}^1} \simeq \mathcal{L} \otimes \mathcal{O}_Y(E).$$

By construction there exists an effective divisor, E_σ, on Y_σ supported on $g_\sigma^{-1}(\Delta)$ such that $\sigma_Y^* E = mE_\sigma$. Then

$$\sigma_Y^* \omega^m_{Y/\mathbb{P}^1} \simeq \sigma_Y^* \mathcal{L} \otimes \mathcal{O}_{Y_\sigma}(\sigma_Y^* E) \simeq \sigma_Y^* \mathcal{L} \otimes \mathcal{O}_{Y_\sigma}(mE_\sigma).$$

Now let $\mathcal{L}_\sigma = \sigma_Y^* \omega_{Y/\mathbb{P}^1} \otimes \mathcal{O}_{Y_\sigma}(-E_\sigma)$. Then

$$\sigma_Y^* \mathcal{L} \simeq \mathcal{L}_\sigma^m$$

Also note that $\mathcal{L} = \phi^* \mathcal{A}$, so

$$\mathcal{L}_\sigma^m \simeq \sigma_Y^* \phi^* \mathcal{A} \simeq \phi_\sigma^* \sigma_X^* \mathcal{A}.$$

σ_X is finite, so $\hat{\mathcal{A}} = \sigma_X^* \mathcal{A}$ is ample and

$$\mathcal{L}_\sigma^m \simeq \phi_\sigma^* \hat{\mathcal{A}}.$$

Let $\hat{\mathcal{L}} = \pi^* \mathcal{L}_\sigma$ and $\hat{\phi} = \phi_\sigma \circ \pi$. Then

$$
\begin{aligned}
\omega_{\hat{Y}/\mathbb{P}^1} \supseteq (\sigma_Y \circ \pi)^* \omega_{Y/\mathbb{P}^1} \supseteq \hat{\mathcal{L}} \\
\omega^m_{\hat{Y}/\mathbb{P}^1}\big|_{\hat{Y} \setminus \hat{g}^{-1}(\Delta)} = \hat{\mathcal{L}}\big|_{\hat{Y} \setminus \hat{g}^{-1}(\Delta)} \\
\hat{\mathcal{L}}^m \simeq \hat{\phi}^* \hat{\mathcal{A}}.
\end{aligned}
$$

Finally let $\hat{X} = X_\sigma$. This proves the CLAIM.

Therefore (1.1) can be applied to $\omega_{\hat{Y}/\mathbb{P}^1}$, but then

$$H^n(\hat{Y}, \omega_{\hat{Y}}) = H^n(\hat{Y}, \omega_{\hat{Y}/\mathbb{P}^1} \otimes g^* \omega_{\mathbb{P}^1}) = 0$$

leading to contradiction. Therefore the assumption that $\#\Delta \leq 2$ was false. $\square$

2.2 Corollary. (0.2) *follows.*

§3. Families of Curves and Surfaces

3.1 Theorem. *Let Y be a smooth threefold and $g : Y \to \mathbb{P}^1$ a fibration such that the general fibre is of general type. Then either the smooth fibers of g are pairwise birational or g has at least 3 singular fibres.*

Proof. Start the relative Minimal Model Program for $g : Y \to \mathbb{P}^1$ (cf. [KMM87]). Let the first step of the program be the blowing down of (-1)-curves of the smooth fibres $Y \to Z$ (cf. [Kodaira63]). Let $h : Z \to \mathbb{P}^1$ be the new fibration. Then ω_{Z_t} is nef and big and hence 1-ample for all $t \in \mathbb{P}^1 \setminus \Delta$.

Then either h is isotrivial or has at least 3 singular fibres by (2.1). Since the smooth fibres of h are the minimal models of the smooth fibres of g this proves the statement. $\square$

3.2 Corollary. *If in addition the fibers are minimal surfaces then either g is isotrivial or it has at least 3 singular fibres.* $\square$

3.2.3 Remark. The following example of I. Bauer shows that in the above theorem one cannot replace "the smooth fibres are pairwise birational" with "isotrivial". In other words it is possible that the number of singular fibers is less then 3, all smooth fibers are irreducible and birational to each other, but they are not isomorphic.

Let S be a surface of general type that contains a smooth rational curve $\iota : C \hookrightarrow S$. Let $\sigma : C \to C$ be the double cover of $C \simeq \mathbb{P}^1$ ramified at two points. Embed C via $\sigma \times \iota$ into $\mathbb{P}^1 \times S$, i.e., the projection onto the first component is the double cover and the projection to the second component is the inclusion.

Now let X be the blow up of $\mathbb{P}^1 \times S$ along C. Then X is a threefold fibred over $\mathbb{P}^1$ with exactly two singular fibres (coming from the two ramification points) and the general fibre is a surface of general type. The smooth fibres are pairwise birational, but they are not isomorphic since then S would have an infinite automorphism group.

When the total space of the family is a canonically polarized variety, then there are even better lower bounds for the number of singular fibres in a semi-stable family.

3.3 Theorem. *Let X be a smooth canonically polarized variety of dimension $n \leq 3$, $f : X \to \mathbb{P}^1$ a semi-stable fibration. Then f has at least $7 - n$ singular fibres.*

Proof.

Let $\delta = \#\Delta$. By (0.4) $\delta \geq 3$. Then $\omega_{\mathbb{P}^1}(\Delta)$ is nef, so one can use (1.2). Hence

$$H^n(X, \omega_X \otimes g^*\omega_{\mathbb{P}^1}(\Delta)^{n-1} \otimes g^*\omega_{\mathbb{P}^1}) = 0.$$

Suppose that $(n-1)\delta - 2n \leq 0$. Then $g^*\omega_{\mathbb{P}^1}(\Delta)^{n-1} \otimes g^*\omega_{\mathbb{P}^1} = g^*\mathcal{O}_{\mathbb{P}^1}((n-1)\delta - 2n)$ is a subsheaf of $\mathcal{O}_X$. Let $\mathcal{Q} = \mathcal{O}_X / g^*\mathcal{O}_{\mathbb{P}^1}((n-1)\delta - 2n)$. Since $\mathcal{Q}$ is supported on a proper subvariety, $H^n(X, \mathcal{Q}) = 0$, so

$$0 = H^n(X, \omega_X \otimes g^*\mathcal{O}_{\mathbb{P}^1}((n-1)\delta - 2n)) \longrightarrow H^n(X, \omega_X)$$

is surjective. That however leads to contradiction since $H^n(X, \omega_X) \neq 0$.

This means that $g^*\mathcal{O}_{\mathbb{P}^1}((n-1)\delta - 2n)$ cannot be embedded to $\mathcal{O}_X$, so $(n-1)\delta - 2n > 0$. Therefore $\delta \geq 5$ if $n = 2$ and $\delta \geq 4$ if $n = 3$. $\square$

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