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A CHARACTERIZATION OF RATIONAL SINGULARITIES

SÁNDOR J. KOVÁCS

The main purpose of this note is to present a characterization of rational singularities in characteristic 0. The essence of the characterization is that it is enough to require less than the usual definition.

THEOREM 1. *Let $\phi : Y \to X$ be a morphism of varieties over $\mathbb{C}$, and let $\rho : \mathcal{O}_X \to R\phi_*\mathcal{O}_Y$ be the associated natural morphism. Assume that Y has rational singularities and there exists a morphism (in the derived category of $\mathcal{O}_X$-modules) $\rho' : R\phi_*\mathcal{O}_Y \to \mathcal{O}_X$ such that $\rho' \circ \rho$ is a quasi-isomorphism of $\mathcal{O}_X$ with itself. Then X has only rational singularities.*

If ρ' exists, it could be considered similar to a trace operator. In fact, for any finite morphism of normal varieties, ρ' exists because of the trace operator.

Note that for the first statement of Theorem 1, ϕ does not need to be birational. In particular, Theorem 1 implies that quotient singularities are rational, including quotients by reductive groups as in [B, Corollaire]. In the latter case, ρ' is given by the Reynolds operator.

A well-known and widely used theorem states that in characteristic 0, canonical singularities are Cohen-Macaulay and therefore rational (see [E] and [KMM]).

The original proofs are based on a very clever use of Grothendieck duality simultaneously for a resolution and its restriction onto the exceptional divisor and on a double loop induction. Kollár gave a simpler proof in [K2, §11] without using derived categories but still relying on a technically hard vanishing theorem. Recently Kollár and Mori found a simple proof allowing nonempty boundaries. They do not use derived categories either, but restrict to the projective case (see [KM, 5.18]). These proofs are ingenious, but one would like to have a simple natural proof (at least in the "classical" case, when the boundary is empty).

As an application of Theorem 1 a simple proof is given here in the "classical" case, but without the projective assumption. This proof seems even simpler than that of Kollár and Mori. Derived categories and Grothendieck duality are used, but in such a simple way that one is tempted to say that this proof is the most natural one. Note also that everything used here was already available when the question was raised for the first time.

A statement similar to Theorem 1 was given in [K2, 3.12]. Some ideas of the

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present proof already appear there. In addition, the results of [K1] and [K2] can be used to show that for projective varieties the existence of ρ' is actually equivalent to X having rational singularities. Note that the assumption of Theorem 1 implies that ϕ has to be surjective.

THEOREM 2. *Let $\phi : Y \to X$ be a surjective morphism of projective varieties over $\mathbb{C}$, and let $\rho : \mathcal{O}_X \to R\phi_*\mathcal{O}_Y$ be the associated natural morphism. Assume that both X and Y have rational singularities. Then there exists a morphism $\rho' : R\phi_*\mathcal{O}_Y \to \mathcal{O}_X$ such that $\rho' \circ \rho$ is a quasi-isomorphism of $\mathcal{O}_X$ with itself.*

Finally, as a byproduct of the proof, a partial generalization of Kempf's criterion for rational singularities (cf. [KKMS, p. 50]) is presented.

THEOREM 3. *Let $\phi : Y \to X$ be a surjective morphism of projective varieties over $\mathbb{C}$. Let $N = \dim Y$ and $n = \dim X$. Assume that Y has rational singularities. Then X has rational singularities if and only if X is Cohen-Macaulay and $R^{N-n}\phi_*\omega_Y \simeq \omega_X$.*

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Definitions and notation. Throughout the article, the ground field is always $\mathbb{C}$, the field of complex numbers. A variety means a separated variety of finite type over $\mathbb{C}$.

A divisor D is called $\mathbb{Q}$-Cartier if mD is Cartier for some $m > 0$. A normal variety X is said to have log-terminal, (resp., canonical) singularities if K_X is $\mathbb{Q}$-Cartier. For any resolution of singularities, $f : Y \to X$, with the collection of exceptional prime divisors $\{E_i\}$, there exist $a_i \in \mathbb{Q}$, $a_i > -1$ (resp., $a_i \geq 0$) such that $K_Y \equiv f^*K_X + \sum a_i E_i$ (cf. [KMM] and [CKM]).

The index of a normal variety X with K_X $\mathbb{Q}$-Cartier is the smallest positive integer m such that mK_X is Cartier. For a normal variety X with K_X $\mathbb{Q}$-Cartier, there exists locally an index-1 cover: that is, a finite surjective morphism $\sigma : X' \to X$ such that X' has index 1. A log-terminal variety of index 1 is canonical, and an easy computation shows that finite covers of log-terminal (resp., canonical) singularities are log-terminal (resp., canonical). In particular, the index-1 cover of a log-terminal variety is canonical (see [R, 1.7, 1.9] and [CKM, 6.7, 6.8]).

Let X be a normal variety and $\phi : Y \to X$ a resolution of singularities. X is said to have rational singularities if $R^i\phi_*\mathcal{O}_Y = 0$ for all $i > 0$ or, equivalently, if the natural map $\mathcal{O}_X \to R\phi_*\mathcal{O}_Y$ is a quasi-isomorphism.

Let $\omega_X^{\cdot}$ denote the dualizing complex of X; that is, $\omega_X^{\cdot} = f^!\mathbb{C}$, where $f : X \to \operatorname{Spec}\mathbb{C}$ is the structure map (cf. [H]).

Main ingredients. Let $\phi : Y \to X$ be a proper morphism. Then one has the following:

- Grothendieck duality [H, VII]: For all $G^{\cdot}$-bounded complexes of $\mathcal{O}_Y$-modules,

$$R\phi_* R\mathcal{H}om_Y(G^{\cdot}, \omega_Y^{\cdot}) \simeq R\mathcal{H}om_X(R\phi_* G^{\cdot}, \omega_X^{\cdot}).$$

- Adjointness [H, II.5.10]: For all $F^{\cdot}$-bounded complexes of $\mathcal{O}_X$-modules and

$G^{\cdot}$-bounded complexes of $\mathcal{O}_Y$-modules,

$$R\phi_* R\mathcal{H}om_Y(L\phi^* F^{\cdot}, G^{\cdot}) \simeq R\mathcal{H}om_X(F^{\cdot}, R\phi_* G^{\cdot}).$$

If ϕ is a resolution of singularities, then one has

• Grauert-Riemenschneider vanishing [GR]: $R^i\phi_*\omega_Y = 0$ for $i > 0$. This is referred to as "GR vanishing."

LEMMA 1 [KKMS, p. 50], [K2, 11.9]. *Let X be a normal variety, and let $\phi : Y \to X$ be a resolution of singularities. If X is Cohen-Macaulay and $\omega_X \simeq \phi_*\omega_Y$, then X has rational singularities.*

Proof. By GR vanishing, $\omega_X^{\cdot} \simeq R\phi_*\omega_Y^{\cdot}$, and then

$$\begin{aligned}\mathcal{O}_X &\simeq R\mathcal{H}om_X(\omega_X^{\cdot}, \omega_X^{\cdot}) \simeq R\mathcal{H}om_X(R\phi_*\omega_Y^{\cdot}, \omega_X^{\cdot}) \\ &\simeq R\phi_* R\mathcal{H}om_Y(\omega_Y^{\cdot}, \omega_Y^{\cdot}) \simeq R\phi_*\mathcal{O}_Y.\end{aligned}$$ □

Proof of Theorem 1. Let $\pi : \tilde{X} \to X$ be a resolution of X and $\sigma : \tilde{Y} \to Y$ be a resolution of Y such that $\phi \circ \sigma$ factors through π: There exists $\psi : \tilde{Y} \to \tilde{X}$ such that $\phi \circ \sigma = \pi \circ \psi$. Then one has the following commutative diagram:

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "\rho"] \arrow[d, "\alpha"'] & R\phi_*\mathcal{O}_Y \arrow[d, "\beta"] \\ R\pi_*\mathcal{O}_{\tilde{X}} \arrow[r, "\gamma"'] & R\phi_* R\sigma_*\mathcal{O}_{\tilde{Y}}. \end{tikzcd}$$

Now ρ has a left inverse by assumption and β is a quasi-isomorphism since Y has rational singularities. Therefore $(\rho' \circ \beta^{-1} \circ \gamma) \circ \alpha$ is a quasi-isomorphism of $\mathcal{O}_X$ with itself, so one may assume that ϕ is a resolution of singularities.

Next apply $R\mathcal{H}om_X(\_\_, \omega_X^{\cdot})$ to the quasi-isomorphism

$$\mathcal{O}_X \xrightarrow{\rho} R\phi_*\mathcal{O}_Y \xrightarrow{\rho'} \mathcal{O}_X.$$

Then

$$\omega_X^{\cdot} \longrightarrow R\phi_*\omega_Y^{\cdot} \longrightarrow \omega_X^{\cdot}$$

is a quasi-isomorphism as well. Hence $h^i(\omega_X^{\cdot}) \subseteq R^i\phi_*\omega_Y^{\cdot} \simeq R^{i+d}\phi_*\omega_Y$. Now $R^{i+d}\phi_*\omega_Y = 0$ for $i > -d$ by GR vanishing. Therefore $h^i(\omega_X^{\cdot}) = 0$ for $i > -d$, so X is Cohen-Macaulay. The above proof also shows that

$$\omega_X \longrightarrow \phi_*\omega_Y \longrightarrow \omega_X$$

is an isomorphism, so $\omega_X \simeq \phi_*\omega_Y$. Therefore X has rational singularities by Lemma 1. □

THEOREM 4 [E]. *Log-terminal singularities are rational.*

Proof. Let X be a variety with log-terminal singularities. By Theorem 1 it is enough to prove that the index-1 cover of X has rational singularities. Thus one can assume that X has canonical singularities and ω_X is a line bundle.

Let $\phi : Y \to X$ be a resolution of singularities of X. By assumption there exists a nontrivial morphism $\iota : L\phi^*\omega_X \simeq \phi^*\omega_X \to \omega_Y$. Its adjoint morphism on X is $\omega_X \to R\phi_*\omega_Y$, which is a quasi-isomorphism by GR vanishing.

Applying $R\mathcal{H}om_Y(\_\_, \omega_Y)$ (not $R\mathcal{H}om_Y(\_\_, \omega_Y^{\cdot})$) to ι, one obtains

$$\begin{tikzcd}
R\phi_* R\mathcal{H}om_Y(\omega_Y, \omega_Y) \arrow[r] & R\phi_* R\mathcal{H}om_Y(L\phi^*\omega_X, \omega_Y) \arrow[r, "\simeq"] & R\mathcal{H}om_X(\omega_X, R\phi_*\omega_Y) \arrow[d, "\simeq"'] \\
R\phi_*\mathcal{O}_Y \arrow[u, "\simeq"'] \arrow[rr, "\rho'"] & & \mathcal{O}_X .
\end{tikzcd}$$

The last quasi-isomorphism uses the fact that $R\phi_*\omega_Y \simeq \omega_X$ and that ω_X is a line bundle. It is easy to see that $\rho' \circ \rho$ acts trivially on $\mathcal{O}_X$; hence Theorem 1 can be applied. □

The following is a simple consequence of [K1, 7.6].

THEOREM 5. *Let $\phi : Y \to X$ be a surjective morphism of projective varieties over $\mathbb{C}$, and assume that both X and Y have rational singularities. Let $N = \dim Y$ and $n = \dim X$. Then*

$$R^{N-n}\phi_*\omega_Y \simeq \omega_X .$$

Proof. Let $\pi : \tilde{X} \to X$ be a resolution of X and $\sigma : \tilde{Y} \to Y$ a resolution of Y such that $\phi \circ \sigma$ factors through π; that is, there exists $\psi : \tilde{Y} \to \tilde{X}$ such that $\phi \circ \sigma = \pi \circ \psi$. Then one has the following commutative diagram:

$$\begin{tikzcd}
R(\phi \circ \sigma)_*\omega_{\tilde{Y}}[N] \arrow[r, "\delta"] \arrow[d, "\alpha"'] & R\pi_*\omega_{\tilde{X}}[n] \arrow[d, "\beta"] \\
R\phi_*\omega_Y[N] \arrow[r, "\gamma"'] & \omega_X[n].
\end{tikzcd}$$

Because both X and Y have rational singularities, α and β are quasi-isomorphisms by GR vanishing. Next take $-n$th cohomology of these complexes. By [K2, 3.4],

$$R^{-n}(\phi \circ \sigma)_*\omega_{\tilde{Y}}[N] \simeq R^{N-n}(\pi \circ \psi)_*\omega_{\tilde{Y}} \simeq \pi_* R^{N-n}\psi_*\omega_{\tilde{Y}},$$

so one has

$$\begin{tikzcd}
\pi_* R^{N-n}\psi_*\omega_{\tilde{Y}} \arrow[r, "h^{-n}(\delta)"] \arrow[d, "h^{-n}(\alpha)"', "\simeq"] & \pi_*\omega_{\tilde{X}} \arrow[d, "h^{-n}(\beta)", "\simeq"'] \\
R^{N-n}\phi_*\omega_Y \arrow[r, "h^{-n}(\gamma)"'] & \omega_X .
\end{tikzcd}$$

Now $h^{-n}(\delta)$ is an isomorphism by [K1, 7.6], so $h^{-n}(\gamma)$ is an isomorphism as well. □

LEMMA 2. *Let $\phi : Y \to X$ be a surjective morphism of projective varieties over $\mathbb{C}$ and $\rho : \mathcal{O}_X \to R\phi_*\mathcal{O}_Y$ the associated natural morphism. Let $N = \dim Y$ and $n = \dim X$. Assume that Y has rational singularities. If X is Cohen-Macaulay and $R^{N-n}\phi_*\omega_Y \simeq \omega_X$, then there exists a morphism $\rho' : R\phi_*\mathcal{O}_Y \to \mathcal{O}_X$ such that $\rho' \circ \rho$ is a quasi-isomorphism of $\mathcal{O}_X$ with itself.*

Proof. Let $\sigma : \tilde{Y} \to Y$ be a resolution of singularities. Since Y has rational singularities, $\mathcal{O}_Y \simeq R\sigma_*\mathcal{O}_{\tilde{Y}}$ and $\omega_Y \simeq R\sigma_*\omega_{\tilde{Y}}$. Thus one may assume that Y is smooth.

By [K2, 3.1] $\omega_X^{\cdot} \simeq \omega_X[n] \simeq R^{N-n}\phi_*\omega_Y[n]$ is a direct summand of $R\phi_*\omega_Y[N] \simeq R\phi_*\omega_Y^{\cdot}$. Therefore there exist morphisms whose composition is a quasi-isomorphism

$$\omega_X^{\cdot} \longrightarrow R\phi_*\omega_Y^{\cdot} \longrightarrow \omega_X^{\cdot}.$$

Applying $R\mathcal{H}om_X(\_, \omega_X^{\cdot})$ to this quasi-isomorphism, one concludes that

$$\mathcal{O}_X \xrightarrow{\rho} R\phi_*\mathcal{O}_Y \xrightarrow{\rho'} \mathcal{O}_X$$

is a quasi-isomorphism as well. □

COROLLARY. *Theorems 2 and 3 hold.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637 USA; kovacs@math.uchicago.edu; skovacs@member.ams.org