Du Bois singularities deform

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Abstract.

Let X be a variety and H a Cartier divisor on X. We prove that if H has Du Bois (or DB) singularities, then X has Du Bois singularities near H. As a consequence, if $X \longrightarrow S$ is a proper flat family over a smooth curve S whose special fiber has Du Bois singularities, then the nearby fibers also have Du Bois singularities. We prove this by obtaining an injectivity theorem for certain maps of canonical modules. As a consequence, we also obtain a restriction theorem for certain non-lc ideals.

1. Introduction

Du Bois singularities, or henceforth simply DB singularities, were introduced by Steenbrink in [Ste81]. They may be considered a generalization of the notion of rational singularities. The definition and its simple consequences makes DB singularities the natural class to consider in many situations including vanishing theorems and moduli theory. More precisely it is important and

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2010 *Mathematics Subject Classification.* 14B07, 14B05, 14F17, 14F18.

Key words and phrases. Du Bois singularities, DB singularities, deformation, log canonical singularities, non-lc ideal.

The first named author was supported in part by NSF Grants DMS #0856185 and #1301888, by a grant from the Simons Foundation #304043, and the Craig McKibben and Sarah Merner Endowed Professorship in Mathematics at the University of Washington.

The second named author was partially supported by the NSF grant DMS #1064485 and an NSF Postdoctoral Fellowship.

useful that the singularities considered in these situation are Du Bois. For instance, Steenbrink showed that families over smooth curves whose fibers have DB singularities possess particularly nice properties; this maxim and its consequences have been further explored in [KK10, Section 7]. These applications imply that the question of whether DB singularities are invariant under small deformations, that is whether the property of having DB singularities is open in flat families, is very important. In this paper we settle this question in the affirmative.

As both rational singularities [Kov99] and log canonical singularities [KK10] are DB, it is interesting to note that rational singularities are invariant under small deformations by [Elk78], while log canonical singularities are not unless the total space has a $\mathbb{Q}$-Cartier canonical divisor compatible with the canonical divisors of the family members. In this latter case the statement follows from inversion of adjunction [Kaw07].

Our main result is the following:

Main Theorem [Theorem 4.1]. *Let X be a scheme of finite type over $\mathbb{C}$ and H a reduced Cartier divisor on X. If H has DB singularities, then X has DB singularities near H.*

The openness of the Du Bois locus in proper flat families follows immediately, see Corollary 4.2.

In [Ish86], Ishii proved this result for isolated Gorenstein singularities, and it follows for normal Gorenstein singularities from a combination of [Kov99] and [Kaw07]. The first named author claimed a proof of the same statement in general in [Kov00]. That proof unfortunately is incomplete and only works under an additional condition. The problem lies in the first paragraph of the proof, namely that one may not always reduce to the case when the non-Du Bois locus of X is contained in $H = X_s$. For additional discussion of this issue see [KS11, Section 12].

In this paper we correct that proof by showing a more general injectivity theorem, Theorem 3.3, which should be viewed as playing the same role for Du Bois singularities that Grauert-Riemenschneider vanishing plays for rational singularities, see Corollary 3.5. Using this injectivity, we can follow the strategy of [Kov00] and mimic Elkik's proof [Elk78] that rational singularities deform in families to obtain the main result.

As another corollary of this injectivity theorem, we also prove a restriction theorem for the so-called *maximal non-lc ideals* defined in [FST11], at least in the case of a Gorenstein ambient variety, see Theorem 7.1.

Acknowledgements:

The authors would like to thank the referee, Kazuma Shimomoto and Burt Totaro for pointing out typos in previous drafts of this paper. We would also like to thank Florin Ambro for pointing out some references to us.

2. Preliminaries on DB singularities

Throughout this paper, all schemes are assumed to be separated and of finite type over $\mathbb{C}$, and all morphisms are defined over $\mathbb{C}$. A *variety* here means a reduced connected scheme.

We use $D^b_{\mathrm{coh}}(X)$ to denote the bounded derived category of $\mathscr{O}_X$-modules with coherent cohomology. Given an object $C^\bullet \in D^b_{\mathrm{coh}}(X)$, its ith cohomology is denoted by $h^i(C^\bullet)$. For any scheme X of finite type over $\mathbb{C}$, we use $\omega_X^\bullet$ to denote the dualizing complex of X which is defined as $\epsilon^!\mathbb{C}$ where $\epsilon : X \longrightarrow \mathbb{C}$ is the structure map of X. We will repeatedly use Grothendieck duality in the following form: For any proper map of schemes $f : Y \longrightarrow X$, and any $C^\bullet \in D^b_{\mathrm{coh}}(Y)$, there exists a functorial quasi-isomorphism:

$$\mathcal{R}f_*\mathcal{R}\mathcal{H}om^\bullet_Y(C^\bullet, \omega_Y^\bullet) \simeq \mathcal{R}\mathcal{H}om^\bullet_X(\mathcal{R}f_*C^\bullet, \omega_X^\bullet).$$

For an introduction to derived categories and Grothendieck duality in the context used in this paper, see [Har66].

Recall that given a variety X, a *resolution of singularities* $\pi : \widetilde{X} \longrightarrow X$ is a proper birational[1] map from a smooth variety $\widetilde{X}$. Given a closed subscheme $Z \subseteq X$

[1]birational = there exists a bijection of irreducible components with an induced isomorphism of fields of fractions.

with associated ideal sheaf $\mathscr{I}_Z$, we say that $\pi : \widetilde{X} \longrightarrow X$ is a *log resolution of* $Z \subseteq X$ if π is a resolution of singularities and if in addition $\pi^* \mathscr{I}_Z \simeq \mathscr{O}_{\widetilde{X}}(G)$ where G is a divisor, the exceptional set of π, $\mathrm{Exc}(\pi) \subseteq \widetilde{X}$, is also a divisor, and the divisor $\mathrm{Exc}(\pi) \cup \mathrm{supp}(G)$ has simple normal crossings. Note that resolutions of singularities, and log resolutions, exist by [Hir64].

We briefly recall some common objects used in the study of DB singularities. For a more extensive discussion of DB singularities, please see [KS11], [HK10, Section 3.I], or [PS08].

Lemma 2.1. *Given a variety X, one may associate to X an object $\underline{\Omega}^0_X \in D^b_{\mathrm{coh}}(X)$ defined as follows: let $\pi_\bullet : X_\bullet \longrightarrow X$ be a (cubic or simplicial) hyperresolution of X, see* [GNPP88, Car85, Del74]*, then*

$$\underline{\Omega}^0_X := \mathcal{R}\pi_{\bullet\,*}\mathscr{O}_{X_\bullet}.$$

This object has the following properties:

(i) *$\underline{\Omega}^0_X$ is functorial with respect to morphisms of varieties, i.e., given a morphism of varieties $f : Y \longrightarrow X$, there is an induced morphism $\underline{\Omega}^0_X \longrightarrow \mathcal{R}f_*\underline{\Omega}^0_Y$.*

(ii) *There is a natural morphism $\mathscr{O}_X \longrightarrow \underline{\Omega}^0_X$ compatible with (i) in the obvious way.*

(iii) *If in addition X is proper, then the composition*

$$H^i(X^{\mathrm{an}}, \mathbb{C}) \longrightarrow H^i(X, \mathscr{O}_X) \longrightarrow \mathbb{H}^i(X, \underline{\Omega}^0_X)$$

is surjective.

Proof. See [DB81] and [Ste81] for the original definitions and proofs and [KS11] for a survey on DB singularities. Property (iii) follows directly from the E_1-degeneration of the Deligne-Du Bois variant of the Hodge-to-De Rham spectral sequence. Q.E.D.

Definition 2.2. We say that X has *DB singularities* if the morphism $\mathscr{O}_X \longrightarrow \underline{\Omega}^0_X$ from (ii) above is a quasi-isomorphism.

We also recall the following fact about DB singularities.

Lemma 2.3 (*cf.* [Kol95, Proof of Theorem 12.8]). *If X has DB singularities and H is a general member of a base-point-free linear system δ on X, then H also has DB singularities.*

In this paper, we will repeatedly use the Grothendieck dual of $\underline{\Omega}^0_X$. To make that easier we introduce the notation

$$\underline{\omega}^\bullet_X := \mathcal{RH}om^\bullet_X(\underline{\Omega}^0_X, \omega^\bullet_X).$$

We will also use the fact that there exists a morphism $\Phi : \underline{\omega}^\bullet_X \longrightarrow \omega^\bullet_X$, which is dual to the natural morphism $\mathcal{O}_X \longrightarrow \underline{\Omega}^0_X$.

Remark 2.4. Note that X has DB singularities if and only if Φ is a quasi-isomorphism since applying the Grothendieck duality functor again yields a morphism $\mathcal{O}_X \longrightarrow \underline{\Omega}^0_X$ which can be identified with the morphism from Lemma 2.1(ii) up to quasi-isomorphism.

3. The key injectivity

In Theorem 3.3 below, we prove the following injectivity. For every integer $j \in \mathbb{Z}$,

$$\Phi^j : h^j(\underline{\omega}^\bullet_X) \hookrightarrow h^j(\omega^\bullet_X)$$

is injective. In the case that $x \in X$ is a closed point such that $X \setminus \{x\}$ is DB, the injectivity of this morphism played a key role in proving that rational, log canonical and F-injective singularities are DB, see [Kov99, KK10, Sch09].

Because of its potential usefulness it has been asked several times whether this injectivity holds. In particular, it was asked in [Sch09, Question 8.3] and [KS11, Question 5.2].

First we prove a lemma that is interesting on its own.

Lemma 3.1. *Let X be a variety and $\mathscr{L}$ a semi-ample line bundle. Choose $s \in \mathscr{L}^n$ a general global section for some $n \gg 0$ and take the n^{th}-root of this section as in* [KM98, 2.50]*:*

$$\eta : Y = \mathbf{Spec} \bigoplus_{i=0}^{n-1} \mathscr{L}^{-i} \longrightarrow X.$$

Then $\eta_* = \mathscr{R}\eta_*$,

$$\eta_* \underline{\Omega}^0_Y \simeq \underline{\Omega}^0_X \otimes \eta_* \mathscr{O}_Y \simeq \bigoplus_{i=0}^{n-1} (\underline{\Omega}^0_X \otimes \mathscr{L}^{-i}),$$

and this direct sum decomposition is compatible with the decomposition $\eta_* \mathscr{O}_Y = \bigoplus_{i=0}^{n-1} \mathscr{L}^{-i}$.

Proof. We fix $\pi_\bullet : X_\bullet \longrightarrow X$ a finite cubic (or simplicial) hyperresolution of X as in [GNPP88]. On each component X_i of $X_\bullet$, $\mathscr{L}$ pulls back to a semi-ample line bundle and further s is still a general member of the base-point free linear subsystem of $\pi_i^* \mathscr{L}^n$. Thus we obtain a cyclic cover $\eta_i : Y_i \longrightarrow X_i$ for each i as well. Furthermore, each Y_i is smooth since it is ramified over a general element of a base-point free linear system. Obviously, these Y_i's glue to give a diagram of smooth $\mathbb{C}$-schemes $Y_\bullet$ with an augmentation morphism $\rho_\bullet : Y_\bullet \longrightarrow Y$. From the construction of a cubic hyperresolution, it is easy to see that $Y_\bullet$ is also a cubic hyperresolution.

We briefly sketch the idea of this last claim: if $X' \longrightarrow X$ is a resolution of singularities, then the induced $Y' \longrightarrow Y$ is also a resolution of singularities. Furthermore, if $X' \longrightarrow X$ is an isomorphism outside of $\Sigma \subseteq X$, then $Y' \longrightarrow Y$ is also an isomorphism outside of $\eta^{-1}(\Sigma)$, which is itself the induced cyclic cover of Σ.

Therefore,

$$\begin{aligned}
\mathscr{R}\eta_* \underline{\Omega}^0_Y &\simeq \mathscr{R}\eta_* \mathscr{R}\rho_{\bullet\,*} \mathscr{O}_{Y_\bullet} \simeq \mathscr{R}\pi_{\bullet\,*} \mathscr{R}\eta_{\bullet\,*} \mathscr{O}_{Y_\bullet} \\
&\simeq \mathscr{R}\pi_{\bullet\,*} \left(\oplus_{i=0}^{n-1} (\mathscr{O}_{X_\bullet} \otimes \pi_\bullet^* \mathscr{L}^{-i}) \right) \\
&\simeq \oplus_{i=0}^{n-1} \left((\mathscr{R}\pi_{\bullet\,*} \mathscr{O}_{X_\bullet}) \otimes \mathscr{L}^{-i} \right) \\
&\simeq \oplus_{i=0}^{n-1} (\underline{\Omega}^0_X \otimes \mathscr{L}^{-i}) \\
&\simeq \underline{\Omega}^0_X \otimes (\oplus_{i=0}^{n-1} \mathscr{L}^{-i}) \\
&\simeq \underline{\Omega}^0_X \otimes \eta_* \mathscr{O}_Y .
\end{aligned}$$

and the result follows, the compatibility statement following by construction.

Alternatively, if one wishes to avoid hyperresolutions one may proceed as follows. By restricting to an open set, we may assume that X embeds as a closed subscheme in a smooth scheme U such that $\mathscr{L}$ is the restriction of a globally generated line-bundle $\mathscr{M}$ on U. Further set $\pi : U' \longrightarrow U$ to be a log resolution of $X \subseteq U$ where we use $\overline{X}$ to denote the reduced divisor $\pi^{-1}(X)_{\mathrm{red}}$. Then $\mathcal{R}\pi_* \mathscr{O}_{\overline{X}} \simeq \underline{\Omega}^0_X$. Choosing a general section s of the globally generated line bundle $\mathscr{M}^n$, we obtain a diagram of cyclic covers:

$$\begin{tikzcd} \overline{Y} \arrow[r, hook] \arrow[d] & W' \arrow[d] \\ Y \arrow[r, hook] & W \end{tikzcd}$$

where Y, W, W' and $\overline{Y}$ are the induced cyclic covers of X, U, U' and $\overline{X}$ respectively. It is clear that W and W' are smooth and that $\overline{Y}$ is the reduced-preimage of Y and has simple normal crossings. Thus the result follows again since $\mathcal{R}\pi_* \mathscr{O}_{\overline{Y}} \simeq \underline{\Omega}^0_Y$ by [Sch07], also see [Esn90]. Q.E.D.

Before proving our main injectivity, we need one more result.

Proposition 3.2. *Let X be a proper variety over $\mathbb{C}$ and $\mathscr{L}$ a semi-ample line bundle on X. Then the natural map*

$$H^j(X, \mathscr{L}^{-i}) \longrightarrow \mathbb{H}^j(X, \underline{\Omega}^0_X \otimes \mathscr{L}^{-i})$$

is surjective for all $j, i \geq 0$.

Proof. Choose $n > i$ such that $\mathscr{L}^n$ is base-point-free and choose a general section $s \in \Gamma(X, \mathscr{L}^n)$. Consider the induced cyclic cover $\eta : Y \longrightarrow X$ and note that Y is also proper. Now, we have the following factorization

$$H^i(Y^{\mathrm{an}}, \mathbb{C}) \longrightarrow H^i(Y, \mathscr{O}_Y) \longrightarrow \mathbb{H}^i(Y, \underline{\Omega}^0_Y).$$

This composition is surjective by Lemma 2.1(iii). Thus $H^i(Y, \mathscr{O}_Y) \longrightarrow \mathbb{H}^i(Y, \underline{\Omega}^0_Y)$ is also surjective. Then the statement follows by Lemma 3.1. Q.E.D.

Now we are ready to prove the main result of the section.

Theorem 3.3. *Let X be a variety over $\mathbb{C}$. Then the natural map*

$$\Phi^j : h^j(\underline{\omega}^\bullet_X) \hookrightarrow h^j(\omega^\bullet_X)$$

is injective for every $j \in \mathbb{Z}$.

Proof. The statement is local and compatible with restriction to an open subset. Therefore we may assume that X is projective. Let $j \in \mathbb{Z}$ and $\mathscr{L}$ an ample line bundle on X. It follows from Proposition 3.2 that $H^{-j}(X, \mathscr{L}^{-i}) \longrightarrow \mathbb{H}^{-j}(X, \underline{\Omega}^0_X \otimes \mathscr{L}^{-i})$ is surjective. Next, apply $\mathrm{Hom}_{\mathbb{C}}(\_\_, \mathbb{C})$ and observe that then

$$\mathbb{H}^{-j}(X, \underline{\Omega}^0_X \otimes \mathscr{L}^{-i})^\vee \hookrightarrow H^{-j}(X, \mathscr{L}^{-i})^\vee$$

is injective. However,

$$\begin{aligned} H^{-j}(X, \mathscr{L}^{-i})^\vee &\simeq h^j(\mathscr{R}\Gamma(X, \mathscr{R}\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}^{-i}, \omega^\bullet_X))) \\ &\simeq \mathbb{H}^j(X, \omega^\bullet_X \otimes \mathscr{L}^i) \end{aligned}$$

by Grothendieck duality applied to the structure map $\epsilon : X \longrightarrow \mathbb{C}$. Likewise,

$$\mathbb{H}^{-j}(X, \underline{\Omega}^0_X \otimes \mathscr{L}^{-i})^\vee \simeq \mathbb{H}^j(X, \underline{\omega}^\bullet_X \otimes \mathscr{L}^i).$$

Thus we get that

$$\mathbb{H}^j(X, \underline{\omega}^\bullet_X \otimes \mathscr{L}^i) \hookrightarrow \mathbb{H}^j(X, \omega^\bullet_X \otimes \mathscr{L}^i)$$

is injective. Notice that

$$\mathbb{H}^j(X, \omega^\bullet_X \otimes \mathscr{L}^i) \simeq H^0(X, h^j(\omega^\bullet_X) \otimes \mathscr{L}^i)$$

for $i \gg 0$ by Serre-vanishing and the associated Grothendieck spectral sequence. Likewise,

$$\mathbb{H}^j(X, \omega^\bullet_X \otimes \mathscr{L}^i) \simeq H^0(X, h^j(\underline{\omega}^\bullet_X) \otimes \mathscr{L}^i)$$

for $i \gg 0$. Therefore,

$$H^0(X, h^j(\underline{\omega}^\bullet_X) \otimes \mathscr{L}^i) \hookrightarrow H^0(X, h^j(\omega^\bullet_X) \otimes \mathscr{L}^i) \tag{3.3.1}$$

is injective for $i \gg 0$. Observe that since $\mathscr{L}$ is ample, both $h^j(\underline{\omega}^\bullet_X) \otimes \mathscr{L}^i$ and $h^j(\omega^\bullet_X) \otimes \mathscr{L}^i$ are generated by global sections for $i \gg 0$. Therefore the injectivity of equation (3.3.1) implies, that

$$\Phi^j : h^j(\underline{\omega}^{\bullet}_X) \longrightarrow h^j(\omega^{\bullet}_X)$$

is also injective for every j. This completes the proof. Q.E.D.

We also have the following local-dual version of Theorem 3.3.

Corollary 3.4 (*cf.* [Kov99, Lemma 2.2]). *Let X be a variety and $P \in X$ is a point (not necessarily closed). Then the natural map*

$$H^i_P(X, \mathscr{O}_{X,P}) \twoheadrightarrow \mathbb{H}^i_P(X, \underline{\Omega}^0_X \otimes \mathscr{O}_{X,P})$$

is surjective for all $i \geq 0$.

Proof. We have the injection $h^i(\underline{\omega}^{\bullet}_X)_P \longrightarrow h^i(\omega^{\bullet}_X)_P$ for all i. After shifting (in case P is not a closed point), we have that $h^i(\underline{\omega}^{\bullet}_{\mathscr{O}_{X,P}}) \longrightarrow h^i(\omega^{\bullet}_{\mathscr{O}_{X,P}})$ also injects for all i. Let E be the injective hull of the residue field $\mathscr{O}_{X,P}/\mathfrak{m}_{X,P}$ and apply the (faithful and exact) functor $\mathrm{Hom}_{\mathscr{O}_{X,P}}(\_\_, E)$. Local duality in the form of [Har66, IV, Theorem 6.2] then yields the corollary. Q.E.D.

With respect for deciding whether X has DB singularities, the complex $\underline{\Omega}^0_X$ plays the same role as the complex $\mathscr{R}\pi_*\mathscr{O}_{\widetilde{X}}$ does for detecting rational singularities, here $\pi : \widetilde{X} \longrightarrow X$ is a resolution of singularities.

However, in many applications what makes $\mathscr{R}\pi_*\mathscr{O}_{\widetilde{X}}$ a useful object is the Grauert-Riemenschneider vanishing theorem [GR70] applied to its Grothendieck dual, $\mathscr{R}\pi_*\omega^{\bullet}_{\widetilde{X}} \simeq \mathscr{R}\mathscr{H}om^{\bullet}_{\mathscr{O}_X}(\mathscr{R}\pi_*\mathscr{O}_{\widetilde{X}}, \omega^{\bullet}_X)$ implying that it is a complex with non-zero cohomology in only one spot:

$$\mathscr{R}\pi_*\omega^{\bullet}_{\widetilde{X}} \simeq \pi_*\omega_{\widetilde{X}}[\dim X].$$

For X Cohen-Macaulay, Theorem 3.3 yields an analogous vanishing for DB singularities.

Corollary 3.5. *Let X be a Cohen-Macaulay variety of dimension d. Then*

$$\underline{\omega}^{\bullet}_X \simeq h^{-d}(\underline{\omega}^{\bullet}_X)[d].$$

If additionally X is normal and $\pi : \widetilde{X} \longrightarrow X$ is a log resolution of singularities with reduced exceptional divisor E, then

$$\underline{\omega}^{\bullet}_X \simeq \pi_* \omega_{\widetilde{X}}(E)[d]$$

Proof. Since X is Cohen-Macaulay and connected, it is equidimensional. The first statement is immediate since a submodule of the zero-module is zero and because $h^i(\omega^{\bullet}_X) = 0$ for $i \neq -d$. For the second statement, use the fact that $h^{-d}(\underline{\omega}^{\bullet}_X) \simeq \pi_* \omega_{\widetilde{X}}(E)$ by [KSS10, Theorem 3.8]. Q.E.D.

Remark 3.6. Notice that if X is DB, then $\underline{\omega}^{\bullet}_X \simeq \omega^{\bullet}_X$ and hence the statement is equivalent to X being Cohen-Macaulay.

A slight reinterpretation of the previous result also gives us the following corollary.

Corollary 3.7. *Let Y be a smooth n-dimensional variety and $X \subseteq Y$ a Cohen-Macaulay subvariety of pure dimension d. Let $\pi : \widetilde{Y} \longrightarrow Y$ be a log resolution of $X \subseteq Y$. Set $E \subseteq Y$ to be the reduced pre-image of X in Y (which is a divisor since π is a log resolution). Then*

$$R^i \pi_* \omega_{\widetilde{Y}}(E) = 0$$

for all $i \neq 0, n - d - 1$.

Proof. Consider the long exact sequence

$$R^i \pi_* \omega_{\widetilde{Y}} \longrightarrow R^i \pi_* \omega_{\widetilde{Y}}(E) \longrightarrow R^i \pi_* \omega_E \longrightarrow R^{i+1} \pi_* \omega_{\widetilde{Y}}$$

and notice first that $R^i \pi_* \omega_{\widetilde{Y}} = 0$ for all $i \neq 0$ by [GR70]. Since $\omega_E[n-1] \simeq \omega^{\bullet}_E$ we have $R^{j+n-1} \pi_* \omega_E \simeq h^j(\mathcal{R}\pi_* \omega^{\bullet}_E)$. However, $\mathcal{R}\pi_* \omega^{\bullet}_E \simeq \underline{\omega}^{\bullet}_X$ by [Sch07]. Therefore, since $h^j(\underline{\omega}^{\bullet}_X) = 0$ for $j \neq -d$ by Corollary 3.5, we see that $R^{j+n-1} \pi_* \omega_E = 0$ for $j \neq -d$. Thus $R^i \pi_* \omega_E = 0$ for $i \neq n - d - 1$ and the result follows. Q.E.D.

Remark 3.8. The previous two corollaries do not hold if X is not Cohen-Macaulay. In fact they automatically fail for any non-Cohen-Macaulay variety with Du Bois singularities. For example, they fail for the affine cone over an Abelian variety of dimension > 1.

Theorem 3.3 also provides slightly simpler proofs of existing results.

Corollary 3.9 ([Kov99], *cf.* [Kol95, Section 12]). *If the morphism $\mathscr{O}_X \longrightarrow \underline{\Omega}^0_X$ has a left-inverse in $D^b_{\mathrm{coh}}(X)$, then X has DB singularities.*

Proof. The hypothesis implies that $\Phi^i : h^i(\underline{\omega}^{\bullet}_X) \longrightarrow h^i(\omega^{\bullet}_X)$ is surjective for every i. Thus Φ^i is an isomorphism by Theorem 3.3 and hence $\Phi : \underline{\omega}^{\bullet}_X \longrightarrow \omega^{\bullet}_X$ is a quasi-isomorphism and so X has DB singularities by Remark 2.4. Q.E.D.

4. Deformation of DB singularities

We now prove the main result of the paper. In fact, simply using Corollary 3.4 fills in the gap in the first author's proof of this statement in [Kov00, Theorem 3.2]. For completeness, we provide a proof below. This proof (as well as the proof of [Kov00, Theorem 3.2]) was inspired by Elkik's proof of the fact that rational singularities deform [Elk78].

Theorem 4.1. *Let X be a scheme of finite type over $\mathbb{C}$ and H a reduced effective Cartier divisor (if X is not normal, by a Cartier divisor we mean a subscheme locally defined by a single non-zero-divisor at each stalk). If H has DB singularities, then X has DB singularities near H.*

Proof. Choose hyperresolutions $\pi_{\bullet} : X_{\bullet} \longrightarrow X$ and $\mu_{\bullet} : H_{\bullet} \longrightarrow H$ with a map $H_{\bullet} \longrightarrow X_{\bullet}$ factoring through the diagram of schemes $Z_{\bullet} := X_{\bullet} \times_X H$ as pictured below, *cf.* [GNPP88].

$$\begin{tikzcd} H_{\bullet} \arrow[r] \arrow[dr, "\mu_{\bullet}"'] & Z_{\bullet} \arrow[r] \arrow[d, "\varepsilon_{\bullet}"] & X_{\bullet} \arrow[d, "\pi_{\bullet}"] \\ & H \arrow[r, hook] & X \end{tikzcd}$$

Note that the components of $Z_{\bullet}$ need not be smooth or even reduced.

Choose a closed point $\mathfrak{q}$ of X contained within H. It is sufficient to prove that X is DB at $\mathfrak{q}$. Let R denote the stalk $\mathscr{O}_{X,\mathfrak{q}}$ and choose $f \in R$ to denote a defining equation of H in R. We also define $\underline{\Omega}^0_R := \underline{\Omega}^0_X \otimes R$ and $\underline{\omega}^{\bullet}_R := \mathscr{R}\mathrm{Hom}^{\bullet}_R(\underline{\Omega}^0_R, \omega^{\bullet}_R)$. Consider the following diagram whose rows are exact triangles in $D^b_{\mathrm{coh}}(X)$:

$$\begin{tikzcd}
R \arrow[r, "\times f"] \arrow[d] & R \arrow[r] \arrow[d] & R/(f) \arrow[r, "+1"] \arrow[d, "\rho"'] & {} \\
\underline{\Omega}^0_R \arrow[r, "\times f"'] & \underline{\Omega}^0_R \arrow[r] & (\mathscr{R}\varepsilon_{\bullet *}\mathscr{O}_{Z_\bullet}) \otimes R \arrow[r, "+1"] \arrow[d, "\tau"'] & {} \\
 & & \underline{\Omega}^0_H \otimes R &
\end{tikzcd}$$

where $\tau \circ \rho$ is a quasi-isomorphism by hypothesis. Next we apply the functor $\mathscr{R}\mathrm{Hom}^{\bullet}_R(\underline{}, \omega^{\bullet}_R)$. Using the notation $\widetilde{\omega}^{\bullet}_{Z_\bullet} = \mathscr{R}\mathrm{Hom}^{\bullet}_R((\mathscr{R}\varepsilon_{\bullet *}\mathscr{O}_{Z_\bullet}) \otimes R, \omega^{\bullet}_R)$ and taking cohomology we obtain the following diagram of long exact sequences:

$$\begin{tikzcd}
h^i(\omega^{\bullet}_R) & h^i(\omega^{\bullet}_R) \arrow[l, "\times f"'] & h^i(\omega^{\bullet}_{R/f}) \arrow[l, "\delta_i"'] & h^{i-1}(\omega^{\bullet}_R) \arrow[l, "\alpha_i"'] & h^{i-1}(\omega^{\bullet}_R) \arrow[l, "\times f"'] \\
h^i(\underline{\omega}^{\bullet}_R) \arrow[u, hook, "\Phi^i"'] & h^i(\underline{\omega}^{\bullet}_R) \arrow[l, "\times f"] \arrow[u, hook, "\Phi^i"'] & h^i(\widetilde{\omega}^{\bullet}_{Z_\bullet}) \arrow[l] \arrow[u, two heads, "\gamma_i"'] & h^{i-1}(\underline{\omega}^{\bullet}_R) \arrow[l, "\beta_i"] \arrow[u, hook, "\Phi^{i-1}"'] & h^{i-1}(\underline{\omega}^{\bullet}_R) \arrow[l, "\times f"] \arrow[u, hook, "\Phi^{i-1}"']
\end{tikzcd}$$

where the vertical Φ maps are injective because of Theorem 3.3 and the morphism γ_i is surjective because $\tau \circ \rho$ is an isomorphism.

Fix $z \in h^{i-1}(\omega^{\bullet}_R)$. Pick $w \in h^i(\widetilde{\omega}^{\bullet}_{Z_\bullet})$ such that $\alpha_i(z) = \gamma_i(w)$. Since $\delta_i(\alpha_i(z)) = 0$ and Φ^i is injective, it follows that there exists a $u \in h^{i-1}(\underline{\omega}^{\bullet}_R)$ such that $\beta_i(u) = w$. Therefore, $\alpha_i(\Phi^{i-1}(u)) = \alpha_i(z)$ and so

$$z - \Phi^{i-1}(u) \in f \cdot h^{i-1}(\omega^{\bullet}_R). \tag{4.1.1}$$

Now, fix C_{i-1} to be the cokernel of Φ^{i-1} and set $\overline{z} \in C_{i-1}$ to be the image of z. Equation (4.1.1) then guarantees that $\overline{z} \in f \cdot C_{i-1}$. But z was arbitrary and so the multiplication map $C_{i-1} \xrightarrow{\times f} C_{i-1}$ is surjective. But this contradicts Nakayama's lemma unless $C_{i-1} = 0$. Therefore $C_{i-1} = 0$ and Φ^{i-1} is also surjective. This holds for all i and so the natural morphism $\underline{\omega}^{\bullet}_X \longrightarrow \omega^{\bullet}_X$

is a quasi-isomorphism. Thus X has DB singularities by Remark 2.4. Q.E.D.

Corollary 4.2. *Let $f : X \longrightarrow S$ be a proper flat family of varieties over a smooth curve S and $s \in S$ a closed point. If the fiber X_s has DB singularities, then so do the other fibers near s.*

Proof. By Theorem 4.1, X has DB singularities near X_s. Let Σ denote the non-Du Bois locus of X. Since f is proper, $f(\Sigma)$ is a closed subset of S not containing $s \in S$. Thus by restricting S to an open set, we may assume that X has DB singularities. By Lemma 2.3, all fibers over nearby points of $s \in S$ have DB singularities. Q.E.D.

5. DB pairs

In [Kov11], the first author defined a notion of Du Bois (or simply DB) pairs. Indeed, given a (possibly non-reduced) subscheme $Z \subseteq X$ one has an induced map in $D^b_{\mathrm{coh}}(X)$,

$$\underline{\Omega}^0_X \longrightarrow \underline{\Omega}^0_Z,$$

noting that by definition $\underline{\Omega}^0_Z = \underline{\Omega}^0_{Z_{\mathrm{red}}}$. Then $\underline{\Omega}^0_{X,Z}$ to be the object in the derived category making the following an exact triangle:

$$\underline{\Omega}^0_{X,Z} \longrightarrow \underline{\Omega}^0_X \longrightarrow \underline{\Omega}^0_Z \xrightarrow{+1} .$$

If $\mathscr{I}_Z$ is the ideal sheaf of Z, then it is easy to see that there is a natural map $\mathscr{I}_Z \longrightarrow \underline{\Omega}^0_{X,Z}$, [Kov11, Section 3.D].

Definition 5.1. [Kov11, Definition 3.13] The *Du Bois defect of* (X, Z), denoted $\underline{\Omega}^{\times}_{X,Z}$, is the mapping cone of the morphism $\mathscr{I}_Z \longrightarrow \underline{\Omega}^0_{X,Z}$, so that there is an exact triangle

$$\mathscr{I}_Z \longrightarrow \underline{\Omega}^0_{X,Z} \longrightarrow \underline{\Omega}^{\times}_{X,Z} \xrightarrow{+1}$$

We say that (X, Z) has *Du Bois singularities* if $\underline{\Omega}^{\times}_{X,Z}$ is quasi-isomorphic to zero. In other words, if $\mathscr{I}_Z \longrightarrow \underline{\Omega}^0_{X,Z}$ is a quasi-isomorphism.

We now mimic our approach before:

Lemma 5.2 (*cf.* Lemma 3.1). *Let X be a variety and $\mathscr{L}$ a semi-ample line bundle. Choose $s \in \mathscr{L}^n$ a general global section for some $n \gg 0$ and take the n^{th}-root of this section as in* [KM98, 2.50]*:*

$$\eta : Y = \mathbf{Spec} \bigoplus_{i=0}^{n-1} \mathscr{L}^{-i} \longrightarrow X.$$

Set $W = \eta^{-1}(Z)$ (with the induced scheme structure). Note that we have $\eta|_W : W = \mathbf{Spec} \bigoplus_{i=0}^{n-1} \mathscr{L}^{-i}|_Z \longrightarrow Z$. Then as before $\eta_* = \mathscr{R}\eta_*$,

$$\eta_* \underline{\Omega}^0_{Y,W} \simeq \underline{\Omega}^0_{X,Z} \otimes \eta_* \mathscr{O}_Y \simeq \bigoplus_{i=0}^{n-1} (\underline{\Omega}^0_{X,Z} \otimes \mathscr{L}^{-i}),$$

and this direct sum decomposition is compatible with the decomposition $\eta_* \mathscr{O}_Y = \bigoplus_{i=0}^{n-1} \mathscr{L}^{-i}$.

Proof. This can be proven just as in Lemma 3.1 or alternately follows formally from Lemma 3.1 via the functoriality of the construction. Q.E.D.

Just as in Proposition 3.2, we also obtain that

$$H^j(X, \mathscr{I}_Z \otimes \mathscr{L}^{-i}) \longrightarrow \mathbb{H}^j(X, \underline{\Omega}^0_{X,Z} \otimes \mathscr{L}^{-i})$$

simply by using [Kov11, Theorem 4.1] in place of Lemma 2.1(iii).

If we set $\underline{\omega}^\bullet{}_{X,Z} = \mathscr{R}\mathscr{H}om^\bullet_{\mathscr{O}_X}(X, \omega^\bullet_X)$, then we easily obtain.

Theorem 5.3. *Let X be a variety over $\mathbb{C}$. Then the natural map*

$$\Phi^j : h^j(\underline{\omega}^\bullet{}_{X,Z}) \hookrightarrow h^j(\mathscr{R}\mathscr{H}om_{\mathscr{O}_X}(\mathscr{I}_Z, \omega^\bullet_X))$$

is injective for every $j \in \mathbb{Z}$.

Proof. The proof is the same as in Theorem 3.3 Q.E.D.

6. Tranversality

Lemma 6.1. *Let X be a reduced scheme and $\Sigma \subseteq X$ a reduced subscheme with ideal sheaf $\mathscr{I}_\Sigma$. Further let $H \subseteq X$ be a Cartier divisor with ideal sheaf $\mathscr{I}_H$ such that H does not contain any irreducible components of either X or Σ. Then*

$$\mathscr{I}_H \cap \mathscr{I}_\Sigma = \mathscr{I}_H \cdot \mathscr{I}_\Sigma.$$

Proof. The statement is local, so we may assume that $X = \operatorname{Spec} A$. Let $I \subseteq A$ be the ideal of Σ, i.e., $\mathscr{I}_\Sigma = \widetilde{I}$. Since Σ is reduced, $I = \sqrt{I}$ and hence $I = \cap_{i=1}^r \mathfrak{p}_i$ with prime ideals $\mathfrak{p}_i \subset A$. Assume that this is an economic decomposition, i.e., none of the $\mathfrak{p}_i$ are redundant. Further let $f \in A$ a local equation for H, i.e., $\mathscr{I}_H = \widetilde{(f)}$. The assumption that H does not contain any irreducible components of either X or Σ imply that

(6.1.1) f is not contained in any minimal primes of A, and

(6.1.2) f is not contained in any of the $\mathfrak{p}_i$.

Claim 6.2. For any prime ideal $\mathfrak{p} \subseteq A$ such that $f \notin \mathfrak{p}$,

$$(f) \cap \mathfrak{p} = f\mathfrak{p}.$$

Proof. $(f) \cap \mathfrak{p} \supseteq f\mathfrak{p}$ trivially, so we only need to prove the opposite containment. Let $fg \in (f) \cap \mathfrak{p}$. Since $f \notin \mathfrak{p}$, it follows that $g \in \mathfrak{p}$, so $fg \in f\mathfrak{p}$ as desired. Q.E.D.

Applying this to the $\mathfrak{p}_i$ we obtain that

(6.2.1)
$$(f) \cap I = (f) \bigcap (\cap_{i=1}^r \mathfrak{p}_i) = \bigcap_{i=1}^r ((f) \cap \mathfrak{p}_i) = \bigcap_{i=1}^r f\mathfrak{p}_i$$

Claim 6.3. Assume that f is not contained in any minimal primes of A. Then for any set of prime ideals $\{\mathfrak{p}_i \subseteq A\}$,

$$\bigcap_{i=1}^r f\mathfrak{p}_i = f\left(\cap_{i=1}^r \mathfrak{p}_i\right). \tag{6.3.1}$$

Proof. Let $x \in \bigcap_{i=1}^r f\mathfrak{p}_i$ and let $g_i \in \mathfrak{p}_i$ such that $x = fg_i$ for all i. We claim that $g_i = g_j$ for any i, j. Indeed, $fg_i = x = fg_j$ so

$$f(g_i - g_j) = 0 \in \bigcap_{\substack{\mathfrak{p} \subseteq A \\ \text{is a minimal prime}}} \mathfrak{p}$$

By assumption $f \notin \mathfrak{p}$ for any of the $\mathfrak{p}$, so we must have $g_i - g_j \in \mathfrak{p}$ for all $\mathfrak{p}$. However, since X is reduced,

$$\bigcap_{\substack{\mathfrak{p} \subseteq A \\ \text{is a minimal prime}}} \mathfrak{p} = 0,$$

so it follows that $g_i = g_j =: g$. Finally this implies that $x = fg \in f\left(\cap_{i=1}^r \mathfrak{p}_i\right)$. Q.E.D.

Combining (6.2.1) and (6.3.1) implies that $(f) \cap I = f \cdot I$. Q.E.D.

7. Application to restriction theorems for maximal non-LC ideals

In this section we assume the reader is familiar with log canonical singularities; see [KM98] for an introduction. Let X be a normal variety, Δ an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $\pi : \widetilde{X} \longrightarrow X$ is a log resolution for (X, Δ). Write $K_{\widetilde{X}} - \pi^*(K_X + \Delta) = \sum a_i E_i$ and set $E^{=-1} = \sum_{a_i=-1} E_i$. The following ideal

$$\mathcal{J}_{\mathrm{NLC}}(X, \Delta) := \pi_* \mathscr{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta) + E^{=-1} \rceil)$$

is defined to be the *non-log canonical ideal of* X.

This ideal was first defined by F. Ambro in [Amb03, Definition 4.1] where it was denoted by $\mathcal{I}_{X_{-\infty}}$. The study of this ideal as an object similar to the multiplier ideal, was recently initiated by O. Fujino in [Fuj10]. One of the main facts about this ideal is that the zero set of $\mathcal{J}_{\mathrm{NLC}}$ is exactly the locus where (X, Δ) does not have log canonical singularities. Fujino proved the following restriction theorem for $\mathcal{J}_{\mathrm{NLC}}(X, \Delta)$ (in fact, he proved a more general result):

Theorem. [Fuj10, Theorem 1.2] *If H is a normal Cartier divisor on a $\mathbb{Q}$-Gorenstein variety X, then $\mathcal{J}_{\mathrm{NLC}}(X,H)\otimes \mathscr{O}_H \simeq \mathcal{J}_{\mathrm{NLC}}(H,0)$.*

However, there are other natural ideals that define the non-lc locus. With notation as above, set $E=\sum E_i$ and set $E^{\mathbb{Z}}=\sum_{a_i\in\mathbb{Z}} E_i$. Then consider the ideal

$$\begin{aligned}\mathcal{J}'(X,\Delta) &:= \pi_*\mathscr{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}}-\pi^*(K_X+\Delta)+E^{\mathbb{Z}}\rceil)\\ &= \pi_*\mathscr{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}}-\pi^*(K_X+\Delta)+\varepsilon E\rceil)\end{aligned}$$

where we choose $1\gg\varepsilon>0$. This is the largest ideal which canonically defines the non-log canonical locus of (X,Δ) and as such is called the *maximal non-lc ideal*. In [FST11], the authors explored this ideal (and other non-lc-ideals). In particular, they obtained restriction theorems in special cases [FST11, Theorem 12.7, Theorem 13.13]. As an application of Theorem 3.3, we obtain the following restriction theorem for $\mathcal{J}'(X,H)$ in the case that X is Gorenstein.

Theorem 7.1. *If X is a normal d-dimensional Gorenstein variety and H is a normal Cartier divisor on X, then $\mathcal{J}'(X,H)|_H\simeq\mathcal{J}'(H,0)$.*

The proof strategy is the same as in [FST11, Section 13]

Proof. By working sufficiently locally, we may assume that $K_X\sim 0$ and $H=V(f)\sim 0$ for some $f\in\Gamma(X,\mathscr{O}_X)$. Shrinking X again if necessary, we embed $X\subseteq Y$ as a closed subscheme in a smooth scheme Y. Let $\pi:\widetilde{Y}\longrightarrow Y$ be a log resolution of $H\subseteq Y$ which is simultaneously an embedded resolution of $X\subseteq Y$. Let $\overline{X}=\pi^{-1}(X)_{\mathrm{red}}$, $\widetilde{X}$ the strict transform of X, and $\overline{H}=\pi^{-1}(H)_{\mathrm{red}}$. We may assume that π is an isomorphism outside of $\operatorname{Sing}X\cup H$ and write $\overline{X}=\widetilde{X}\cup E\cup\overline{H}$ where $E=\pi^{-1}(\operatorname{Sing}X)_{\mathrm{red}}$. Finally, we may also assume that $E\cup\overline{H}$ is a reduced simple normal crossings divisor which intersects $\widetilde{X}$ with normal crossings so that $(E\cup\overline{H})\cap\widetilde{X}$ is a reduced simple normal crossings divisor on $\widetilde{X}$. We have the following short exact sequence:

$$0 \longrightarrow \mathscr{O}_{\widetilde{X}}(-E \cup \overline{H}) \longrightarrow \mathscr{O}_{\overline{X}} \longrightarrow \mathscr{O}_{E \cup \overline{H}} \longrightarrow 0.$$

By pushing forward and using [Sch07], we obtain the exact triangle,

$$\mathscr{R}\pi_* \mathscr{O}_{\widetilde{X}}(-E \cup \overline{H}) \longrightarrow \underline{\Omega}^0_X \longrightarrow \underline{\Omega}^0_{H \cup \operatorname{Sing} X} \xrightarrow{+1} .$$

Applying $\mathscr{R}\mathscr{H}om^{\bullet}_{\mathscr{O}_X}(\_\_, \omega^{\bullet}_X)$ gives

$$\underline{\omega}^{\bullet}_{H \cup \operatorname{Sing} X} \longrightarrow \underline{\omega}^{\bullet}_X \longrightarrow \mathscr{R}\pi_* \mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}} + E \cup \overline{H})[d] \xrightarrow{+1} ,$$

and by taking cohomology, we arrive at the exact sequence

$$\begin{aligned} (7.1.1) \quad 0 \longrightarrow h^{-d}(\underline{\omega}^{\bullet}_X) \longrightarrow \pi_* \mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}} + E \cup \overline{H}) \longrightarrow \\ \longrightarrow h^{-d+1}(\underline{\omega}^{\bullet}_{H \cup \operatorname{Sing} X}) \longrightarrow h^{-d+1}(\underline{\omega}^{\bullet}_X) = 0. \end{aligned}$$

The vanishing on the right follows by Corollary 3.5 since X is Gorenstein and thus Cohen-Macaulay.

By [FST11, Lemma 13.11],

$$h^{-d+1}(\underline{\omega}^{\bullet}_{H \cup \operatorname{Sing} X}) \simeq h^{-d+1}(\underline{\omega}^{\bullet}_H).$$

Furthermore, by [KSS10, Theorem 3.8] we have that

$$\mathcal{J}'(X, 0) \cong h^{-d}(\underline{\omega}^{\bullet}_X) \otimes \mathscr{O}_X(-K_X)$$

and

$$\mathcal{J}'(H, 0) \cong h^{-d+1}(\underline{\omega}^{\bullet}_H) \otimes \mathscr{O}_X(-K_X - H).$$

Hence twisting (7.1.1) by $\mathscr{O}_X(-K_X - H)$ we obtain the following short exact sequence: *cf.* [FST11, Lemma 13.8] [KSS10, Lemma 4.14],

$$\begin{aligned} 0 \longrightarrow \mathcal{J}'(X, 0) \otimes \mathscr{O}_X(-H) \longrightarrow \\ \longrightarrow \pi_* \mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}} - \pi^*(K_X + H) + E \cup \overline{H}) \longrightarrow \\ \longrightarrow \mathcal{J}'(H, 0) \longrightarrow 0. \end{aligned}$$

This completes the proof. Q.E.D.

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