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The canonical sheaf of Du Bois singularities

Sándor J. Kovács [a,*,1], Karl Schwede [b,2], Karen E. Smith [b,3]

[a] *University of Washington, Department of Mathematics, Seattle, WA 98195, USA*
[b] *Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1109, USA*

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Abstract

We prove that a Cohen–Macaulay normal variety X has Du Bois singularities if and only if $\pi_*\omega_{X'}(G) \simeq \omega_X$ for a log resolution $\pi : X' \to X$, where G is the reduced exceptional divisor of π. Many basic theorems about Du Bois singularities become transparent using this characterization (including the fact that Cohen–Macaulay log canonical singularities are Du Bois). We also give a straightforward and self-contained proof that (generalizations of) semi-log-canonical singularities are Du Bois, in the Cohen–Macaulay case. It also follows that the Kodaira vanishing theorem holds for semi-log-canonical varieties and that Cohen–Macaulay semi-log-canonical singularities are cohomologically insignificant in the sense of Dolgachev.

1. Introduction

Consider a complex algebraic variety X. If X is smooth and projective, its De Rham complex plays a fundamental role in understanding the geometry of X. When X is singular, an analog of

* Corresponding author.
E-mail addresses: kovacs@math.washington.edu (S.J. Kovács), kschwede@umich.edu (K. Schwede), kesmith@umich.edu (K.E. Smith).

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the De Rham complex, the *Deligne–Du Bois complex* plays a similar role. Based on Deligne's theory of mixed Hodge structures, Du Bois defined a filtered complex of $\mathscr{O}_X$-modules, denoted by $\underline{\Omega}^{\cdot}_X$, that agrees with the algebraic De Rham complex in a neighborhood of each smooth point and, like the De Rham complex on smooth varieties, its analytization provides a resolution of the sheaf of locally constant functions on X [4].

Du Bois observed that an important class of singularities are those for which $\underline{\Omega}^0_X$, the zeroth graded piece of the filtered complex $\underline{\Omega}^{\cdot}_X$, takes a particularly simple form (see Discussion 2.2). He pointed out that singularities satisfying this condition enjoy some of the nice Hodge-theoretic properties of smooth varieties. Dubbed *Du Bois singularities* by Steenbrink, these singularities have been promoted by Kollár as a natural setting for vanishing theorems; see, for example, [17, Ch. 12]. Since the 1980's, Steenbrink, Kollár, Ishii, Saito and many others have investigated the relationship between Du Bois (or *DB*) singularities and better known singularities in algebraic geometry, such as rational singularities and log canonical singularities. Because of the difficulties in defining and understanding the Deligne–Du Bois complex $\underline{\Omega}^0_X$, many basic features of DB singularities have been slow to reveal themselves or have remained obscure. The purpose of this paper is to prove a simple characterization of DB singularities in the Cohen–Macaulay case, making many of their properties and their relationship to other singularities transparent.

Let $\pi : \widetilde{X} \to X$ be a log resolution of a normal complex variety X, and denote by G the reduced exceptional divisor of π. By Lemma 3.14 there exists a natural inclusion $\pi_*\omega_{\widetilde{X}}(G) \hookrightarrow \omega_X$. The main foundational result of this article is the following multiplier ideal-like criterion for DB singularities:

Theorem 1.1 *(= Theorem 3.1). Suppose that X is normal and Cohen–Macaulay. Let $\pi : X' \to X$ be any log resolution, and denote the reduced exceptional divisor of π by G. Then X has DB singularities if and only if $\pi_*\omega_{X'}(G) \simeq \omega_X$.*

Theorem 1.1 is analogous to the following well-known criterion for rational singularities due to Kempf: if X is normal and Cohen–Macaulay, then X has rational singularities if and only if the natural inclusion $\pi_*\omega_{\widetilde{X}} \hookrightarrow \omega_X$ is an isomorphism. In particular, Theorem 1.1 immediately implies that rational singularities are Du Bois, a statement that had been conjectured by Steenbrink in [31] and later proved by Kollár [17] in the projective case and finally by Kovács [20], and also independently by Saito [23] in general. Another immediate corollary is that normal quasi-Gorenstein DB singularities are log canonical; see Section 3 for a complete discussion.

In addition, this criterion shows that CM DB singularities relate to rational singularities very much like log canonical singularities relate to (kawamata) log terminal singularities. This has been a general belief all along, but we feel that the criterion in Theorem 1.1 supports this belief more than anything else previously known.

A long-standing conjecture of Kollár's predicts that log canonical singularities are Du Bois. Using Theorem 1.1, it is easy to see that Kollár's conjecture holds in the Cohen–Macaulay case:

Theorem 1.2 *(= Theorem 3.16). Suppose that X is normal and Cohen Macaulay, and that Δ is an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. If (X, Δ) is log canonical, then X has Du Bois singularities.*

In fact, we prove the stronger result that Cohen–Macaulay semi-log canonical singularities are Du Bois, see Theorem 4.16 (even more, we prove that a generalization of semi-log canonical singularities are Du Bois). This is based on a technical generalization of certain aspects of

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Theorem 1.1 to the non-normal case, treated in Section 4. Many special cases of Kollár's conjecture had been known, including the isolated singularity case [12–14], the Cohen–Macaulay case when the singular set is not too big see also [20], and the local complete intersection case [25].

Very recently, Kollár's conjecture that log canonical singularities are Du Bois, has been verified by Kollár and the first named author, using recent advances in the Minimal Model Program [18]. In particular, there is now an independent and more general result proving Theorem 1.2. However, there are several reasons why Theorem 1.1 is still interesting (besides being the first general result of this kind). Kollár and Kovács also prove that the condition of being Cohen–Macaulay is constant in DB families. This means that a stable smoothable variety is necessarily Cohen–Macaulay, and hence the above condition is applicable.

Furthermore, Theorems 4.11 and 4.16, apply to non-normal singularities. These are the best results currently known. Notice that one of the main applications of DB singularities is to moduli theory and that is an arena where non-normal singularities may not be easily dismissed. Already for degenerations of curves one must deal with non-normal singularities. In particular, Kollár's conjecture is actually important in the non-normal case, that is, we want to know that semi-log canonical singularities are Du Bois.

Another immediate corollary, again predicted by Kollár, is that the Kodaira Vanishing Theorem holds for generalizations of (semi-)log canonical varieties. Fujino recently gave another proof of a closely related theorem using techniques of Ambro; see [7, Corollary 5.11].

Corollary 1.3 *(= Corollary 6.6). Kodaira vanishing holds for Cohen–Macaulay weakly semi-log canonical varieties. In particular, let (X, Δ) be a projective Cohen–Macaulay weakly semi-log canonical pair and $\mathscr{L}$ an ample line bundle on X. Then $H^i(X, \mathscr{L}^{-1}) = 0$ for $i < \dim X$.*

Of course, if X is not Cohen–Macaulay, Kodaira vanishing in the above form necessarily fails. But the Cohen–Macaulay condition is not sufficient for Kodaira vanishing. Examples show that some further restriction on the singularities is needed; see [1, Section 2]. In some sense, this is the most general form of the classical Kodaira vanishing theorem (that is, $H^i(X, \mathscr{L}^{-1}) = 0$ for $\mathscr{L}$ ample and $i < \dim X$) that could be hoped for.

We are also able to obtain some nice Hodge-theoretic properties for semi-log canonical singularities. In particular, we are able to show that Cohen–Macaulay semi-log canonical singularities are cohomologically insignificant in the sense of Dolgachev [3]; see Theorem 5.1. This fact has useful applications in the construction of compact moduli spaces of stable surfaces and higher dimensional varieties.

2. Preliminaries

In this section we will define the notion of log canonical, as well as DB singularities and state the forms of duality we will use. Throughout this paper, a *scheme* will always be assumed to be separated and noetherian of essentially finite type over $\mathbb{C}$. By a *variety*, we mean a reduced separated noetherian pure-dimensional scheme of finite type over $\mathbb{C}$. Note that a variety may have several irreducible components. All varieties and schemes will be assumed to be quasi-projective. The purpose of this assumption is to guarantee that these varieties are embedded in smooth schemes. Note that in the end this hypothesis is harmless because implications between various types of singularities are local questions, thus the varieties may assumed to be quasi-projective.

We will use the following notation: For a functor Φ, $\mathcal{R}\Phi$ denotes its derived functor on the (appropriate) derived category and $\mathcal{R}^i\Phi := h^i \circ \mathcal{R}\Phi$ where $h^i(C^\cdot)$ is the cohomology of the complex $C^\cdot$ at the ith term. Similarly, $\mathbb{H}^i_Z := h^i \circ \mathcal{R}\Gamma_Z$ where Γ_Z is the functor of cohomology with supports along a subscheme Z. Finally, $\mathcal{H}om$ stands for the sheaf-Hom functor.

Let $\alpha : Y \to Z$ be a birational morphism and $\Delta \subseteq Z$ a $\mathbb{Q}$-divisor. Then $\alpha_*^{-1}\Delta$ will denote the proper transform of Δ on Y.

2.1. *Log canonical singularities*

Let X be a normal irreducible variety of pure dimension d. The canonical sheaf ω_X of X is the unique reflexive $\mathcal{O}_X$-module agreeing with the sheaf of regular differential d-forms $\bigwedge^d \Omega_{X/\mathbb{C}}$ on the smooth locus of X. A canonical divisor is any member K_X of the (Weil) divisor class corresponding to ω_X. See Section 2.3 for the definition of the canonical sheaf on non-normal varieties.

A ($\mathbb{Q}$-)Weil divisor D is said to be $\mathbb{Q}$-Cartier if, for some non-zero integer r, the $\mathbb{Z}$-divisor rD is Cartier, meaning that it is given locally as the divisor of some rational function on X. For such a divisor, we can define the pullback π^*D, under any dominant morphism π, to be the $\mathbb{Q}$-divisor $\frac{1}{r}\pi^*(rD)$. We say that X is $\mathbb{Q}$-Gorenstein if K_X is $\mathbb{Q}$-Cartier.

Now consider a pair (X, Δ), where Δ is an effective $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. In this case, there exists a *log resolution* of the pair; that is, a proper birational morphism from a smooth variety

$$\pi : \widetilde{X} \to X$$

such that $\mathrm{Ex}(\pi)$ is a divisor and furthermore the set $\pi^{-1}(\Delta) \cup \mathrm{Ex}(\pi)$ is a divisor with simple normal crossing support. Here $\mathrm{Ex}(\pi)$ denotes the exceptional set of π. Let $\widetilde{\Delta}$ denote the birational (or proper, or strict) transform of Δ on $\widetilde{X}$, often denoted by $\pi_*^{-1}\Delta$. Then there is a numerical equivalence of divisors

$$K_{\widetilde{X}} + \widetilde{\Delta} - \pi^*(K_X + \Delta) \equiv \sum a_i E_i$$

where the E_i are the exceptional divisors of π and the a_i are some uniquely determined rational numbers. We can now define:

Definition 2.1. The pair (X, Δ) is called *log canonical* if $a_i \geqslant -1$ for all i.

Definition 2.1 is independent of the choice of log resolution. For this and other details about log resolutions, log pairs, and log canonical singularities see, for example, [19].

2.2. *Du Bois singularities*

As mentioned in the introduction, DB singularities are defined using a fairly complicated filtered complex $\underline{\Omega}^{\cdot}_X$, which plays the role of the De Rham complex for singular varieties. It follows from the construction that there is a natural map (in the derived category of $\mathcal{O}_X$-modules)

$$\mathcal{O}_X \to \underline{\Omega}^0_X,$$

where $\underline{\Omega}^0_X$ denotes the zeroth graded complex of the filtered complex $\underline{\Omega}^{\cdot}_X$. By definition, X has *Du Bois (or DB) singularities* if this map is a quasi-isomorphism. For a careful development of this point of view see [4,9] or [32]. However, a recent result of the second author in [25] provides an alternate definition of Du Bois singularities, which we now review here (also compare with [6]).

First, since the transverse union of two smooth varieties of different dimensions is an important example of a DB singularity, we must leave the world of irreducible (and even that of equidimensional) varieties, and instead consider reduced schemes of finite type over $\mathbb{C}$. Recall that a reduced subscheme X of a smooth ambient variety Y is said to have *normal crossings* if at every point of x there exists a regular system of parameters for $\mathscr{O}_{Y,x}$ such that X is defined by some monomials in these parameters.

Now suppose that X is a reduced closed subscheme of a smooth ambient Y. Let $\pi : \widetilde{Y} \to Y$ be a proper birational morphism such that:

(a) π is an isomorphism outside X,
(b) $\widetilde{Y}$ is smooth,
(c) $\pi^{-1}(X)$ has normal crossings (even though it may not be equidimensional).

Such morphisms always exist by Hironaka's theorem; for example, π could be an embedded resolution of singularities for $X \subset Y$, or a log resolution of the pair (Y, X). The conditions (a)–(c) can be relaxed somewhat; see [27, Proposition 2.20] for a way to relax condition (a) and [25] for further discussion. We set $\overline{X}$ to be the reduced preimage of X in $\widetilde{Y}$, and again emphasize that $\overline{X}$ may not be equidimensional. Under these conditions, Schwede shows that the object $\mathcal{R}\pi_*\mathscr{O}_{\overline{X}}$ is naturally quasi-isomorphic to the object $\underline{\Omega}^0_X$ defined in [4]. This leads to the following definition, equivalent to Steenbrink's original definition:

Definition 2.2. We say that X has *Du Bois* (*or DB*) *singularities* if the natural map $\mathscr{O}_X \to \mathcal{R}\pi_*\mathscr{O}_{\overline{X}}$ is a quasi-isomorphism.

The fact that Definition 2.2 is independent of the choice of embedding and of the choice of resolution simply follows from the fact that $\mathcal{R}\pi_*\mathscr{O}_{\overline{X}}$ is quasi-isomorphic to the well-defined object $\underline{\Omega}^0_X$. See [25] for details (and again, compare with [6]).

2.3. *Dualizing complexes and duality*

For the convenience of the reader, we briefly review the form of duality we will use.

Associated to every quasi-projective scheme X of dimension d and of finite type over $\mathbb{C}$ there exists a (normalized) dualizing complex $\omega^{\cdot}_X \in D^b_{\mathrm{coh}}(X)$, unique up to quasi-isomorphism. To construct $\omega^{\cdot}_X$ concretely, one may proceed as follows. For a smooth irreducible variety Y of dimension n, take $\omega^{\cdot}_Y = \omega_Y[n]$, the complex that has the canonical module $\det \Omega_{Y/\mathbb{C}}$ in degree $-n$ and the zero module in all the other spots. Now, whenever $X \subseteq Y$ is a closed embedding of schemes of finite type over a field, we have

$$\omega^{\cdot}_X = \mathcal{R}\mathcal{H}om^{\cdot}_Y(\mathscr{O}_X, \omega^{\cdot}_Y).$$

Because every quasi-projective scheme of finite type over $\mathbb{C}$ embeds in a smooth variety, this determines the dualizing complex on any such finite type scheme. Alternatively, one can also define $\omega_X^{\cdot}$ as $h^!\mathbb{C}$ where $h : X \to \mathbb{C}$ is the structural morphism. See [10] and [2] for details.

For an equidimensional X admitting a dualizing complex one may define the canonical sheaf to be the $\mathscr{O}_X$-module $h^{-\dim X}(\omega_X^{\cdot})$, denoted by ω_X. If X is Cohen–Macaulay, the dualizing complex turns out to be exact at all other spots (just as in the case of smooth varieties above). In particular, for a Cohen–Macaulay variety of pure dimension d, we have $\omega_X^{\cdot} = \omega_X[d]$. If X is normal and irreducible, $h^{-\dim X}(\omega_X^{\cdot})$ agrees with the canonical sheaf ω_X as defined in Section 2.1, so there is no ambiguity of terminology. More generally, if X is Gorenstein in codimension one and satisfies Serre's S_2 condition, the canonical module $h^{-\dim X}(\omega_X^{\cdot}) = \omega_X$ is a rank one reflexive sheaf, and so corresponds to a Weil divisor class.

A very important tool we need is Grothendieck duality:

Theorem 2.3. *(See [10, III.11.1, VII.3.4].) Let $f : X \to Y$ be a proper morphism between finite dimensional noetherian schemes. Suppose that both X and Y admit dualizing complexes and that $\mathscr{F}^{\cdot} \in D^-_{\mathrm{qcoh}}(X)$. Then the duality morphism*

$$\mathcal{R}f_*\mathcal{R}\mathcal{H}om_X^{\cdot}(\mathscr{F}^{\cdot}, \omega_X^{\cdot}) \to \mathcal{R}\mathcal{H}om_Y^{\cdot}(\mathcal{R}f_*\mathscr{F}^{\cdot}, \omega_Y^{\cdot})$$

is a quasi-isomorphism.

Remark 2.4. In the previous theorem, $\omega_X^{\cdot}$ should be thought of as $f^!\omega_Y^{\cdot}$.

3. Proof of the main theorem

In this section we prove our main result, a simple new characterization of DB singularities in the normal Cohen–Macaulay case.

Theorem 3.1. *Suppose that X is a normal Cohen–Macaulay variety. Let $\varrho : X' \to X$ be any log resolution and denote the reduced exceptional divisor of ϱ by G. Then X has DB singularities if and only if $\varrho_*\omega_{X'}(G) \simeq \omega_X$.*

The proof of this theorem will take most of the present section. We will first show that it is true for a special choice of log resolutions in Corollary 3.10. Then, in Lemma 3.12, we complete the proof by showing that the statement is independent of the choice of the log resolution.

The following notation will be fixed for the rest of this section:

Setup 3.2. Let X be a reduced equidimensional scheme of finite type over $\mathbb{C}$ embedded in a smooth variety Y, and let Σ denote a closed subscheme $\Sigma \subsetneq X$ that contains the singular locus of X. Assume that no irreducible component of X is a hypersurface, i.e., the codimension of every irreducible component of X in Y is at least two. Fix an embedded resolution $\pi : \widetilde{Y} \to Y$ of X in Y, and let $\widetilde{X}$ denote the strict transform of X on $\widetilde{Y}$. Further assume that

(i) π is an isomorphism over $X \setminus \Sigma$, and
(ii) $\pi^{-1}(\Sigma)$ is a simple normal crossing divisor of $\widetilde{Y}$ that intersects $\widetilde{X}$ in a simple normal crossing divisor of $\widetilde{X}$.

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Let E denote the reduced preimage of Σ in $\widetilde{Y}$, and $\overline{X}$ the reduced pre-image of X in $\widetilde{Y}$. We emphasize that $\overline{X}$ is *not* equidimensional; in fact, $\overline{X}$ is the transverse union of the smooth variety $\widetilde{X}$ and the normal crossing divisor E. We will frequently abuse notation and use π to denote $\pi|_{\overline{X}}$. Note that $\pi|_{\overline{X}}$ is a projective (respectively proper) morphism as long as π is. Finally, recall that by [25] $R\pi_*\mathscr{O}_{\overline{X}} \simeq_{\mathrm{qis}} \underline{\Omega}^0_X$ and $R\pi_*\mathscr{O}_E \simeq_{\mathrm{qis}} \underline{\Omega}^0_\Sigma$.

The outline of the proof of Theorem 3.1 goes as follows. Using the Grothendieck dual form of Schwede's characterization of DB singularities stated in Lemma 3.3 below, it is clear that a reduced Cohen–Macaulay scheme X of dimension d has DB singularities if and only if

(i) $\mathcal{R}^i\pi_*\omega^{\boldsymbol{\cdot}}_{\overline{X}} = 0$ for $i \neq -d$, and
(ii) $\mathcal{R}^{-d}\pi_*\omega^{\boldsymbol{\cdot}}_{\overline{X}} = \omega_X$.

Our main technical statement is Theorem 3.8, which implies that for any normal variety of dimension d, the sheaf $\mathcal{R}^{-d}\pi_*\omega^{\boldsymbol{\cdot}}_{\overline{X}}$ can be identified with $\pi_*\omega_{\widetilde{X}}(G)$. This is proven by comparing the dualizing complexes for $\overline{X}$, $\widetilde{X}$ and E via the dual of the short exact sequence

$$0 \to \mathscr{O}_{\widetilde{X}}(-E|_{\widetilde{X}}) \to \mathscr{O}_{\overline{X}} \to \mathscr{O}_E \to 0.$$

On the other hand, the vanishing statement of (i) follows from a reinterpretation of a result of the first author [20] by the second author [26].

We first state the following dual form of Schwede's characterization of DB singularities.

Lemma 3.3. *X has DB singularities if and only if the natural map*

$$\mathcal{R}\pi_*\omega^{\boldsymbol{\cdot}}_{\overline{X}} \to \omega^{\boldsymbol{\cdot}}_X$$

is a quasi-isomorphism.

Proof. The result follows from Definition 2.2 via a standard application of Grothendieck duality. □

Before beginning the proof of Theorem 3.1, we would like to make the following suggestive observation. If X has pure dimension d and Y has dimension n, then

$$h^{-d}\big(\omega^{\boldsymbol{\cdot}}_{\overline{X}}\big) \cong \omega_{\widetilde{X}}(E|_{\widetilde{X}}),$$
$$h^{-n+1}\big(\omega^{\boldsymbol{\cdot}}_{\overline{X}}\big) \cong \omega_E, \quad \text{and}$$
$$h^{i}\big(\omega^{\boldsymbol{\cdot}}_{\overline{X}}\big) = 0 \quad \text{for } i \text{ not equal to } -n+1 \text{ or } -d.$$

To see this, note that $\widetilde{X}$ and E have normal crossings and that there exists a short exact sequence,

$$0 \to \mathscr{O}_{\widetilde{X}}(-E|_{\widetilde{X}}) \to \mathscr{O}_{\overline{X}} \to \mathscr{O}_E \to 0.$$

Next, dualize this sequence by applying $\mathcal{R}\mathcal{H}om^{\cdot}_{\widetilde{Y}}(\_, \omega^{\cdot}_{\widetilde{Y}})$ (and Grothendieck duality) to get an exact triangle,

$$\omega^{\cdot}_{E} \longrightarrow \omega^{\cdot}_{\overline{X}} \longrightarrow \omega^{\cdot}_{\widetilde{X}} \otimes \mathscr{O}_{\widetilde{Y}}(E) \xrightarrow{+1} .$$

Note that $\overline{X}$ is not equidimensional, but $\widetilde{X}$ and E are (in fact, they are Gorenstein and connected). Since E has dimension $n-1$ and $\widetilde{X}$ has dimension d, we see that $h^{-n+1}(\omega^{\cdot}_{\overline{X}}) \cong h^{-n+1}(\omega^{\cdot}_{E})$, proving the second statement. Taking the $-d$th cohomology proves the first statement since $\widetilde{X}$ and E have normal crossings. It is easy to see that the third statement is true as well by taking any other cohomology. These three facts will not be used directly, but they do suggest a way to analyze $\mathcal{R}\pi_*\omega^{\cdot}_{\overline{X}}$.

With this in mind, we now prove that we really only need to understand the $-d$th cohomology of $\mathcal{R}\pi_*\omega^{\cdot}_{\overline{X}}$, at least in the Cohen–Macaulay case.

Proposition 3.4. *In addition to* (3.2) *assume further that X is Cohen–Macaulay. Then X has DB singularities if and only if the natural map*

$$\mathcal{R}^{-d}\pi_*\omega^{\cdot}_{\overline{X}} \to \omega_X$$

is surjective (if and only if it is an isomorphism).

Proof. If X has DB singularities, the statement (including the one in parentheses) follows trivially from Lemma 3.3.

Now assume that the natural map $\mathcal{R}^{-d}\pi_*\omega^{\cdot}_{\overline{X}} \to \omega_X$ is surjective, but X is not Du Bois. Let Σ_{DB} denote the non-Du Bois locus of X (cf. [20, 2.1]) and let $x \in \Sigma_{DB}$ a general point of (a component of) Σ_{DB}. By [26, 5.11], the natural map

$$\left(\mathcal{R}^{i}\pi_*\omega^{\cdot}_{\overline{X}}\right)_x \to h^i\left(\omega^{\cdot}_X\right)_x$$

is injective for every i. The right side of this equation is zero for $i \neq -d$ since X is Cohen–Macaulay, and thus the left side is zero as well. For $i = -d$, the map is surjective by assumption and, as we already noted, it is injective; hence it is an isomorphism. In particular, the localized map $(\mathcal{R}\pi_*\omega^{\cdot}_{\overline{X}})_x \to (\omega^{\cdot}_X)_x$ is a quasi-isomorphism, contradicting Lemma 3.3 and the fact that (X, x) is *not* Du Bois. □

Remark 3.5. Alternatively, one could use general hyperplane sections to reduce to the case of an isolated non-Du Bois point, and then apply local duality along with the key surjectivity of [21].

The following lemma will be important in the proof.

Lemma 3.6. *Let Z be a reduced closed subscheme of Y, a variety of finite type over $\mathbb{C}$. Then $h^i(\mathcal{R}\mathcal{H}om^{\cdot}_Y(\underline{\Omega}^0_Z, \omega^{\cdot}_Y)) = 0$ for $i < -\dim Z$.*

Proof. Without loss of generality, we may assume that Z and Y are affine. Let $z \in Z$ be an arbitrary closed point. By local duality (see [10, V, Theorem 6.2] or [22, 2.4]), it is sufficient to show that $\mathbb{H}^j_z(Y, \underline{\Omega}^0_Z) = \mathbb{H}^j_z(Z, \underline{\Omega}^0_Z) = 0$ for $j > \dim Z$. We consider the hypercohomology spectral sequence $H^p_z(Z, h^q(\underline{\Omega}^0_Z))$ that computes this cohomology. Note that $\dim(\mathrm{Supp}(h^q(\underline{\Omega}^0_Z))) \leqslant \dim Z - q$ by [9, V, 3.6], so that $H^p_z(Z, h^q(\underline{\Omega}^0_Z)) = 0$ for $p > \dim Z - q$ (i.e., for $p + q > \dim Z$). Therefore, we see that $\mathbb{H}^j_z(Z, \underline{\Omega}^0_Z)$ vanishes for $j > \dim Z$ because every term in the spectral sequence that might possibly contribute to $\mathbb{H}^j_z(Z, \underline{\Omega}^0_Z)$ is zero. □

Corollary 3.7. $\mathcal{R}^i \pi_* \omega^{\boldsymbol{\cdot}}_E = 0$ *for* $i < -\dim \Sigma$.

Proof. By [25] (cf. (3.2)) $\mathcal{R}\pi_* \mathscr{O}_E \simeq_{\mathrm{qis}} \underline{\Omega}^0_\Sigma$, so by Grothendieck duality

$$\mathcal{R}\pi_* \omega^{\boldsymbol{\cdot}}_E \simeq_{\mathrm{qis}} \mathcal{R}\mathcal{H}om^{\boldsymbol{\cdot}}_Y\big(\underline{\Omega}^0_\Sigma, \omega^{\boldsymbol{\cdot}}_Y\big).$$

Then the statement follows from Lemma 3.6. □

Theorem 3.8. *If* X *is equidimensional and* $\mathrm{codim}_X \Sigma \geqslant 2$, *then* $\mathcal{R}^{-d}\pi_*\omega^{\boldsymbol{\cdot}}_{\widetilde{X}} \cong \pi_*\omega_{\widetilde{X}}(E|_{\widetilde{X}})$.

Remark 3.9. The codimension condition of this theorem implies that X must be R_1. It is satisfied, for instance, if X is normal and the maximal dimensional components of Σ and $\mathrm{Sing}\, X$ coincide.

Proof. Applying $\mathcal{R}\pi_*$ to the exact triangle,

$$\omega^{\boldsymbol{\cdot}}_E \to \omega^{\boldsymbol{\cdot}}_{\overline{X}} \to \omega^{\boldsymbol{\cdot}}_{\widetilde{X}} \otimes \mathscr{O}_{\widetilde{Y}}(E) \to$$

and taking cohomology leads to the exact sequence

$$\mathcal{R}^{-d}\pi_*\omega^{\boldsymbol{\cdot}}_E \to \mathcal{R}^{-d}\pi_*\omega^{\boldsymbol{\cdot}}_{\overline{X}} \to \mathcal{R}^{-d}\pi_*\big(\omega^{\boldsymbol{\cdot}}_{\widetilde{X}} \otimes \mathscr{O}_{\widetilde{Y}}(E)\big) \to \mathcal{R}^{-d+1}\pi_*\omega^{\boldsymbol{\cdot}}_E.$$

The outside terms are zero by Corollary 3.7 which implies that the middle two terms are isomorphic. To complete the proof, simply observe that

$$\mathcal{R}^{-d}\pi_*\big(\omega^{\boldsymbol{\cdot}}_{\widetilde{X}} \otimes \mathscr{O}_{\widetilde{Y}}(E)\big) = \mathcal{R}^{-d}\pi_*\big(\omega_{\widetilde{X}}[d] \otimes \mathscr{O}_{\widetilde{Y}}(E)\big) = \pi_*\big(\omega_{\widetilde{X}} \otimes \mathscr{O}_{\widetilde{Y}}(E)\big) = \pi_*\omega_{\widetilde{X}}(E|_{\widetilde{X}}). \qquad \square$$

Corollary 3.10. *In addition to* (3.2) *assume further that* X *is normal and Cohen–Macaulay. Then* X *has DB singularities if and only if the natural map*

$$\pi_*\omega_{\widetilde{X}}(E|_{\widetilde{X}}) \to \omega_X$$

(coming from (3.4) *and* (3.8)*) is surjective (if and only if it is an isomorphism).*

Remark 3.11. Note that $\pi_*\omega_{\widetilde{X}}(E|_{\widetilde{X}}) \to \omega_X$ is an isomorphism on the smooth locus of X by construction, and hence it is always injective. For the same reason, this natural map is the same as the one coming from Lemma 3.14.

At this point, we have proven Theorem 3.1 for a log resolution as in (3.2). The general case follows from the next lemma which is well known to experts (for example, it also follows from [15, Lemma 1.6]).

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Lemma 3.12. *Let $\pi : X' \to X$ be a proper birational morphism, $\Sigma \subseteq X$ a closed subset and denote by G the reduced pre-image of Σ via π. Assume that π is chosen such that X' is smooth and G has simple normal crossings. Then $\pi_* \omega_{X'}(G)$ on X is independent of the choice of π up to natural isomorphism.*

Remark 3.13. This result is analogous to the fact that a multiplier ideal is independent of the resolution used to compute it.

Proof. Since any two proper birational morphisms mapping to X with the required properties can be dominated by a third such, it is sufficient to prove the following: Let X'' be a smooth variety and $\phi : X'' \to X'$ a proper birational morphism and let H be the reduced pre-image of Σ via $\pi \circ \phi$. Then $\phi_* \omega_{X''}(H) \simeq \omega_{X'}(G)$.

The fact that X' is smooth and G is a simple normal crossing divisor implies that the pair (X', G) has log canonical singularities. Furthermore, the support of the strict transform of G on X'' is contained in H by definition, and the rest of the components of H are ϕ-exceptional. Therefore,

$$\omega_{X''}(H) \simeq \phi^*\big(\omega_{X'}(G)\big)(F)$$

for an appropriate effective ϕ-exceptional divisor F. Applying the projection formula yields

$$\phi_* \omega_{X''}(H) \simeq \omega_{X'}(G) \otimes \phi_* \mathscr{O}_{X''}(F),$$

and since F is effective and ϕ-exceptional, $\phi_* \mathscr{O}_{X''}(F) \simeq \mathscr{O}_{X'}$ [16, Lemma 1-3-2]. $\square$

We now turn our attention to using this criterion to prove that log canonical singularities are Du Bois. First note that the statement is reasonably straightforward if X is Gorenstein so it would be tempting to try to take a canonical cover, at least in the $\mathbb{Q}$-Gorenstein case. However, it is not clear that the canonical cover of a Cohen–Macaulay log canonical singularity is also Cohen–Macaulay. Examples of rational singularities with non-Cohen–Macaulay canonical covers in [28] suggest that this might be too much to hope for. Therefore, a different technique will be used.

Lemma 3.14. *Let X be a normal irreducible variety and let $\varrho : X' \to X$ be a log resolution of X. Let B be an effective integral divisor on X, $B' = \varrho_*^{-1} B$ (the strict transform of B on X'), and denote the reduced exceptional divisor of ϱ by G. Then there exists a natural injection,*

$$\varrho_* \omega_{X'}\big(B' + G\big) \hookrightarrow \omega_X(B).$$

Proof. Let $\iota : U \hookrightarrow X$ denote the embedding of the open set over which ϱ is an isomorphism. As X is normal, we have that $\operatorname{codim}_X(X \setminus U) \geqslant 2$, and hence the following natural morphisms of sheaves:

$$\varrho_* \omega_{X'}\big(B' + G\big) \hookrightarrow \big(\varrho_* \omega_{X'}\big(B' + G\big)\big)^{**} \simeq \iota_*\big(\varrho_* \omega_{X'}\big(B' + G\big)\big|_U\big) \simeq \omega_X(B),$$

where $(\ )^* = \mathcal{H}om_X(\_, \mathscr{O}_X)$ denotes the dual of a sheaf. The isomorphisms follow because X is S_2 and ϱ is an isomorphism over U. $\square$

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Lemma 3.15. *Let X be a normal irreducible variety with an effective $\mathbb{Q}$-divisor D such that (X, D) has log canonical singularities, and let $\varrho : X' \to X$ be a log resolution of (X, D). Let B be an effective integral divisor on X with $B \leqslant D$. Denote the reduced exceptional divisor of ϱ by G and let $B' = \varrho_*^{-1} B$ and $D' = \varrho_*^{-1} D$. Then the following natural isomorphism holds:*

$$\varrho_* \omega_{X'}\big(B' + G\big) \simeq \omega_X(B).$$

Proof. By Lemma 3.14 there exists a natural inclusion $\iota : \varrho_* \omega_{X'}(B' + G) \hookrightarrow \omega_X(B)$, so the question is local. We may assume that X is affine and need only prove that every section of $\omega_X(B)$ is already contained in $\varrho_* \omega_{X'}(B' + G)$. Note that ι restricted to the naturally embedded subsheaf $\omega_{X'} \subseteq \omega_{X'}(B' + G)$ gives the usual natural inclusion $\varrho_* \omega_{X'} \hookrightarrow \omega_X$.

Next, choose a canonical divisor $K_{X'}$ and let $K_X = \varrho_* K_{X'}$. This is the divisor corresponding to the image of the section of $\omega_{X'}$ corresponding to $K_{X'}$ via ι. As $D' = \varrho_*^{-1} D$, it follows that the divisors $K_{X'} + D'$ and $\varrho_*^{-1}(K_X + D)$ may only differ in exceptional components. We emphasize that these are actual divisors, not just equivalence classes (and so are B and B').

Since X and X' are birationally equivalent, their function fields are isomorphic. Let us identify $K(X)$ and $K(X')$ via ϱ^* and denote them by K. Further let $\mathscr{K}$ and $\mathscr{K}'$ denote the K-constant sheaves on X and X' respectively.

Now we have the following inclusions:

$$\Gamma\big(X, \varrho_* \omega_{X'}\big(B' + G\big)\big) \subseteq \Gamma\big(X, \omega_X(B)\big) \subseteq \Gamma(X, \mathscr{K}) = K,$$

and we need to prove that the first inclusion is actually an equality. Let $g \in \Gamma(X, \omega_X(B))$. By assumption, $B \leqslant D$, so

$$0 \leqslant \operatorname{div}_X(g) + K_X + B \leqslant \operatorname{div}_X(g) + K_X + D. \tag{3.15.1}$$

As (X, D) is log canonical and thus $K_X + D$ is $\mathbb{Q}$-Cartier, there exists an $m \in \mathbb{N}$ such that $mK_X + mD$ is a Cartier divisor and hence can be pulled back to a Cartier divisor on X'. By the choices we made earlier, we have that $\varrho^*(mK_X + mD) = mK_{X'} + mD' + \Theta$ where Θ is an exceptional divisor. Again we emphasize that these are actual divisors and the two sides are actually equal, not just equivalent. We would also like to point out that Θ is not necessarily divisible by m, neither is $\varrho^*(mK_X + mD)$.

However, using the fact that (X, D) is log canonical, one obtains that $\Theta \leqslant mG$. Combining this with (3.15.1) gives that

$$0 \leqslant \operatorname{div}_{X'}\big(g^m\big) + \varrho^*(mK_X + mD) \leqslant m\big(\operatorname{div}_{X'}(g) + K_{X'} + D' + G\big),$$

and in particular we obtain that

$$\operatorname{div}_{X'}(g) + K_{X'} + D' + G \geqslant 0.$$

Claim. $\operatorname{div}_{X'}(g) + K_{X'} + B' + G \geqslant 0.$

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Proof. By construction

$$\operatorname{div}_{X'}(g) + K_{X'} + B' + G = \varrho_*^{-1}\big(\underbrace{\operatorname{div}_X(g) + K_X + B}_{\geqslant 0}\big) + \underbrace{F + G}_{\text{exceptional}} . \tag{3.15.2}$$

Where F is an appropriate exceptional divisor, though it is not necessarily effective. We also have that

$$\operatorname{div}_{X'}(g) + K_{X'} + B' + G = \underbrace{\operatorname{div}_{X'}(g) + K_{X'} + D' + G}_{\geqslant 0} - \underbrace{\big(D' - B'\big)}_{\text{non-exceptional}} . \tag{3.15.3}$$

Now let A be an arbitrary irreducible component of $\operatorname{div}_{X'}(g) + K_{X'} + B' + G$. If A were not effective, it would have to be exceptional by (3.15.2) and non-exceptional by (3.15.3). Hence A must be effective and the claim is proven. □

It follows that $g \in \Gamma(X', \omega_{X'}(B' + G)) = \Gamma(X, \varrho_*\omega_{X'}(B' + G))$, completing the proof.

Now we are in a position to prove that Cohen–Macaulay log canonical singularities are Du Bois.

Theorem 3.16. *Suppose X is normal and Cohen–Macaulay and $\Delta \subset X$ an effective $\mathbb{Q}$-divisor such that the pair $K_X + \Delta$ is $\mathbb{Q}$-Cartier. If (X, Δ) is log canonical, then X has Du Bois singularities.*

Proof. Use Lemma 3.15 setting $B = 0$ and note that the map of Corollary 3.10 is an isomorphism outside the singular locus. Furthermore, both sheaves are reflexive (since they are abstractly isomorphic by Lemma 3.15), completing the proof. □

4. The non-normal case

The aim of this section is to show that Cohen–Macaulay semi-log canonical singularities are Du Bois. Let us begin by recalling some of the relevant definitions.

Definition 4.1. A reduced scheme X of finite type over $\mathbb{C}$ is said to be *seminormal* if every finite morphism $X' \to X$ of reduced finite type schemes over $\mathbb{C}$ that is a bijection on points is an isomorphism.

Remark 4.2. If one is not working over an algebraically closed field of characteristic zero, one needs to alter the above definition somewhat. See [8] for details.

Definition 4.3. If X is a reduced scheme of finite type over $\mathbb{C}$ with normalization $\eta : X^N \to X$, then the *conductor ideal sheaf of X in its normalization* is defined to be the ideal sheaf $\operatorname{Ann}_{\mathscr{O}_X}(\eta_*\mathscr{O}_{X^N}/\mathscr{O}_X)$.

Remark 4.4. Consider the affine case where R is a subring of its normalization R^N (in its total field of fractions). Then the conductor ideal is the largest ideal of R^N that is contained in R. This

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implies that, with the previous notation, if I_C is the conductor ideal sheaf of X in its normalization and if I_B is the extension of I_C to the normalization (that is $I_B = I_C\mathscr{O}_{X^N}$), then $\eta_* I_B = I_C$.

Remark 4.5. If X is seminormal, then the conductor ideal sheaf of X in its normalization is a radical ideal sheaf, even when extended to the normalization, see [33, Lemma 1.3]. If X is S_2, then all the associated primes of the conductor I_C are height one, see [8, Lemma 7.4], thus all the associated primes of I_B are also height one (cf. [5, 9.2]). Therefore, if X is seminormal and S_2, then $\operatorname{supp} C$ is exactly the codimension 1 locus of the singular set of X.

Definition 4.6. Let X be a reduced equidimensional scheme of finite type over $\mathbb{C}$. Assume that X satisfies the following conditions:

(i) X is S_2, and
(ii) X is seminormal.

These conditions imply that the conductor of X, in its normalization X^N, is a reduced ideal sheaf corresponding to an effective divisor on X^N (cf. (4.5)). We let B denote this divisor on X^N and let C denote the corresponding divisor on X (by construction, these divisors have the same ideal sheaf in the normalization of $\mathscr{O}_X$). Further, let Δ be an effective $\mathbb{Q}$-divisor on X and assume that Δ and C have no common components. Then (X, Δ) is said to be *weakly semi-log canonical* if the pair $(X^N, B + \eta_*^{-1}\Delta)$ is log canonical.

Remark 4.7. The log canonical assumption implies that B has to be reduced, but actually this does not impose any new conditions because simply from the fact that X is seminormal, it follows by [8, 1.4] that both B and C are reduced.

Remark 4.8. The usual notion of *semi-log canonical* also adds the additional condition

(iii-a) X is Gorenstein in codimension 1.

This condition, under the previous seminormality assumption is equivalent to

(iii-b) X has simple double normal crossings in codimension 1.

We won't need this condition, so we will leave it out. Note that many easy to construct schemes satisfy conditions (i) and (ii) but not (iii). For example, the reduced scheme consisting of the three axes in $\mathbb{A}^3$ does not have double crossings in codimension 1, but is both S_2 and seminormal.

The key ingredient in the proof of the normal case is Proposition 3.4, the injectivity of a certain map. We will now prove a strengthening of a special case of that injectivity that we will need in this section. First we need a lemma.

Lemma 4.9. *Let $d = \dim X$ and $\underline{\Omega}^{\times}_X$ a complex that completes α to an exact triangle:*

$$\mathscr{O}_X \xrightarrow{\alpha} \underline{\Omega}^0_X \longrightarrow \underline{\Omega}^{\times}_X \xrightarrow{+1} .$$

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Then $\dim \operatorname{Supp}(h^i(\underline{\Omega}^\times_X)) \leqslant d-i-1$ *for* $i \geqslant 0$ *and* $h^i(\underline{\Omega}^\times_X)=0$ *for* $i<0$.

Proof. The statement follows from the definition for $i<0$, so we may assume that $i \geqslant 0$. Furthermore, for $i=0$ the result also follows since X is reduced and thus X is generically smooth and hence generically Du Bois, so $\mathcal{O}_X \to h^0(\underline{\Omega}^0_X)$ is an isomorphism outside a set of codimension 1. We proceed by induction on the dimension of X. Clearly it is true for zero (or even one) dimensional varieties (see [4, 4.9] for the one-dimensional case). Let Σ denote the singular set of X and let $\pi : \widetilde{X} \to X$ be a resolution of X coming from an embedded resolution as in (3.2) (so that π is an isomorphism over the smooth locus of X). Let $E=\pi^{-1}(\Sigma)_{\mathrm{red}}$, that is, E is the reduced pre-image of the singular set. We then have the exact triangle (cf. (3.2)),

$$\mathcal{R}\pi_*\mathcal{O}_{\widetilde{X}}(-E) \longrightarrow \underline{\Omega}^0_X \longrightarrow \underline{\Omega}^0_\Sigma \xrightarrow{+1} \cdot \tag{4.9.4}$$

The case $i=0$ follows from the fact that $h^0(\underline{\Omega}^0_X)$ is the structure sheaf of the seminormalization of X by [23, 5.2] or [26, 5.6]. For $i>0$, we note that it is sufficient to prove the statement for $h^i(\underline{\Omega}^0_X) \cong h^i(\underline{\Omega}^\times_X)$. Observe that

$$\dim \operatorname{Supp}\big(\mathcal{R}^i\pi_*\mathcal{O}_{\widetilde{X}}(-E)\big) \leqslant d-i-1$$

because $\pi|_{\widetilde{X}}$ is birational and the dimension of the exceptional set cannot be "too" big (cf. [11, III.11.2]). By the inductive hypothesis the statement holds for $h^i(\underline{\Omega}^0_\Sigma)$ and so the long exact sequence coming from the triangle (4.9.4) completes the proof. □

Now we are in position to prove the desired injectivity statement.

Proposition 4.10. *In addition to* (3.2), *let* $\dim X = d$. *Then the map* $\mathcal{R}^{-d}\pi_*\omega^\cdot_{\widetilde{X}} \to h^{-d}(\omega^\cdot_X)$ *is injective.*

Proof. By local duality, it is enough to show that $H^d_x(X,\mathcal{O}_X) \to \mathbb{H}^d_x(X,\underline{\Omega}^0_X)$ is surjective for every closed point $x \in X$. Considering the exact triangle,

$$\mathcal{O}_X \longrightarrow \underline{\Omega}^0_X \longrightarrow \underline{\Omega}^\times_X \xrightarrow{+1}$$

shows that it is enough to prove that $\mathbb{H}^d_x(X,\underline{\Omega}^\times_X)=0$. First observe that $H^p_x(X,h^q(\underline{\Omega}^\times_X))=0$ for $p>\dim \operatorname{Supp}(h^q(\underline{\Omega}^\times_X))$. Furthermore, by Lemma 4.9, we obtain that $H^p_x(X,h^q(\underline{\Omega}^\times_X))=0$ for $p>d-q-1$, and therefore $\mathbb{H}^d_x(X,\underline{\Omega}^\times_X)=0$ since for $p+q=d>d-1$ we see that every term in the standard spectral sequence that might contribute is already zero. □

The next proposition is the key step in our proof that Cohen–Macaulay semi-log canonical singularities are Du Bois. This can be thought of as a generalization of certain aspects of the Kempf-like criterion, Theorem 3.1.

We work in the following setting: X is a reduced d-dimensional Cohen–Macaulay scheme of finite type over $\mathbb{C}$. Let $\pi : \widetilde{X} \to X$ be a log resolution of X, and $\Sigma \subseteq X$ a reduced closed subscheme such that π is an isomorphism outside Σ (in particular $\Sigma \supseteq \operatorname{Sing} X$). Let F denote

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the reduced pre-image of Σ in $\widetilde{X}$. Consider the natural map $\mathscr{I}_\Sigma \to \mathcal{R}\pi_*\mathscr{O}_{\widetilde{X}}(-F)$ where $\mathscr{I}_\Sigma$ is the ideal sheaf of Σ. Apply $\mathcal{R}\mathcal{H}om^{\boldsymbol{\cdot}}_X(\_, \omega^{\boldsymbol{\cdot}}_X) = \mathcal{R}\mathcal{H}om^{\boldsymbol{\cdot}}_X(\_, \omega_X[d])$, which gives us a map

$$\mathcal{R}\mathcal{H}om^{\boldsymbol{\cdot}}_X\big(\mathcal{R}\pi_*\mathscr{O}_{\widetilde{X}}(-F), \omega_X[d]\big) \to \mathcal{R}\mathcal{H}om^{\boldsymbol{\cdot}}_X\big(\mathscr{I}_\Sigma, \omega_X[d]\big).$$

Then, by Grothendieck duality,

$$\mathcal{R}\mathcal{H}om^{\boldsymbol{\cdot}}_X\big(\mathcal{R}\pi_*\mathscr{O}_{\widetilde{X}}(-F), \omega_X[d]\big) \cong \mathcal{R}\pi_*\mathcal{R}\mathcal{H}om^{\boldsymbol{\cdot}}_{\widetilde{X}}\big(\mathscr{O}_{\widetilde{X}}(-F), \omega_{\widetilde{X}}[d]\big) \cong \mathcal{R}\pi_*\omega_{\widetilde{X}}(F)[d].$$

Taking the $-d$th cohomology gives us a natural map

$$h^{-d}\big(\mathcal{R}\pi_*\omega_{\widetilde{X}}(F)[d]\big) \cong \pi_*\omega_{\widetilde{X}}(F) \to \mathcal{H}om_X(\mathscr{I}_\Sigma, \omega_X).$$

We will use properties of this map to deduce that X has DB singularities.

Theorem 4.11. *Suppose we are in the setting described above. If the natural map $\varrho : \pi_*\omega_{\widetilde{X}}(F) \to \mathcal{H}om_X(\mathscr{I}_\Sigma, \omega_X)$ is an isomorphism, then X has DB singularities.*

Remark 4.12. Notice that if X is not normal then Σ contains the conductor.

Proof. By Lemma 3.12 the isomorphism class of $\pi_*\omega_{\widetilde{X}}(F)$ is independent of the choice of the resolution, thus we may assume it came from an embedded resolution of X in some Y as in (3.2). In particular we have $\pi : \overline{X} \to X$, where E is the reduced pre-image of Σ in $\widetilde{Y}$ (i.e., $E|_{\widetilde{X}} = F$). First, consider the following map of exact triangles,

$$\begin{tikzcd}
\mathscr{I}_\Sigma \arrow[r] \arrow[d] & \mathscr{O}_X \arrow[r] \arrow[d] & \mathscr{O}_\Sigma \arrow[r, "+1"] \arrow[d] & {} \\
\mathcal{R}\pi_*\mathscr{O}_{\widetilde{X}}(-F) \arrow[r] & \mathcal{R}\pi_*\mathscr{O}_{\overline{X}} \arrow[r] & \mathcal{R}\pi_*\mathscr{O}_E \arrow[r, "+1"] & {}\,.
\end{tikzcd}$$

Applying $\mathcal{R}\mathcal{H}om^{\boldsymbol{\cdot}}_X(\_, \omega^{\boldsymbol{\cdot}}_X)$ produces

$$\begin{tikzcd}
\mathcal{R}\pi_*\omega^{\boldsymbol{\cdot}}_E \arrow[r] \arrow[d] & \mathcal{R}\pi_*\omega^{\boldsymbol{\cdot}}_{\overline{X}} \arrow[r] \arrow[d] & \mathcal{R}\pi_*\omega^{\boldsymbol{\cdot}}_{\widetilde{X}}(F) \arrow[r, "+1"] \arrow[d] & {} \\
\omega^{\boldsymbol{\cdot}}_\Sigma \arrow[r] & \omega^{\boldsymbol{\cdot}}_X \arrow[r] & \mathcal{R}\mathcal{H}om^{\boldsymbol{\cdot}}_X(\mathscr{I}_\Sigma, \omega^{\boldsymbol{\cdot}}_X) \arrow[r, "+1"] & {}\,.
\end{tikzcd}$$

Considering the long exact cohomology sequence and using Corollary 3.7 leads to the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & h^{-d}(\mathcal{R}\pi_*\omega^{\boldsymbol{\cdot}}_{\overline{X}}) \arrow[r] \arrow[d, "\alpha"'] & h^{-d}(\mathcal{R}\pi_*\omega^{\boldsymbol{\cdot}}_{\widetilde{X}}(F)) \arrow[r] \arrow[d, "\beta"'] & h^{-d+1}(\mathcal{R}\pi_*\omega^{\boldsymbol{\cdot}}_E) \arrow[r] \arrow[d, "\gamma"'] & \cdots \\
0 \arrow[r] & h^{-d}(\omega^{\boldsymbol{\cdot}}_X) \arrow[r] & h^{-d}(\mathcal{R}\mathcal{H}om^{\boldsymbol{\cdot}}_X(\mathscr{I}_\Sigma, \omega^{\boldsymbol{\cdot}}_X)) \arrow[r] & h^{-d+1}(\omega^{\boldsymbol{\cdot}}_\Sigma) \arrow[r] & {}\,.
\end{tikzcd}$$

Note that β is simply the map ϱ and thus it is surjective by hypothesis. Note further that γ is injective by Proposition 4.10, thus α is surjective by the five lemma. Combining this with Proposition 3.4 completes the proof. $\square$

We also need the following two lemmata.

Notation 4.13. Let S be a reduced quasi-projective scheme of finite type of dimension e over $\mathbb{C}$. Then denote by S_e the union of the e-dimensional irreducible components of S, and by $S_{<e}$ the union of the irreducible components of S whose dimension is strictly less than e.

Lemma 4.14. *Let Σ be a reduced quasi-projective scheme of finite type of dimension e over $\mathbb{C}$. Then $h^{-e}(\omega^{\cdot}_{\Sigma}) \simeq \omega_{\Sigma_e} = h^{-e}(\omega^{\cdot}_{\Sigma_e})$.*

Proof. Obviously, $\dim \Sigma_e = e$, $\dim \Sigma_{<e} < e$ and $\dim(\Sigma_e \cap \Sigma_{<e}) < e-1$. Consider the short exact sequence

$$0 \to \mathscr{O}_{\Sigma} \to \mathscr{O}_{\Sigma_e} \oplus \mathscr{O}_{\Sigma_{<e}} \to \mathscr{O}_{\Sigma_e \cap \Sigma_{<e}} \to 0$$

where $\Sigma_e \cap \Sigma_{<e}$ is not necessarily a reduced scheme. Next, apply $\mathcal{R}\mathcal{H}om^{\cdot}_{\Sigma}(\_, \omega^{\cdot}_{\Sigma})$ to get a long exact sequence:

$$\cdots \to h^{-e}\big(\omega^{\cdot}_{\Sigma_e \cap \Sigma_{<e}}\big) \to h^{-e}\big(\omega^{\cdot}_{\Sigma_e} \oplus \omega^{\cdot}_{\Sigma_{<e}}\big) \to h^{-e}\big(\omega^{\cdot}_{\Sigma}\big) \to h^{-e+1}\big(\omega^{\cdot}_{\Sigma_e \cap \Sigma_{<e}}\big) \to \cdots.$$

As $\dim(\Sigma_e \cap \Sigma_{<e}) < e-1$, it follows that $h^{-e+1}(\omega^{\cdot}_{\Sigma_e \cap \Sigma_{<e}}) = h^{-e}(\omega^{\cdot}_{\Sigma_e \cap \Sigma_{<e}}) = 0$, and hence the statement holds. $\square$

Lemma 4.15. *Under the conditions of Theorem* 4.11 *and using the notation of* (4.5) *let $\eta : X^N \to X$ be the normalization of X and assume that $\Sigma_e = C$ where $e = \dim \Sigma$. Then*

$$\eta_* \omega_{X^N}(B) \simeq \eta_* \mathcal{H}om_{X^N}(I_B, \omega_{X^N}) \simeq \mathcal{H}om_X(I_C, \omega_X) \simeq \mathcal{H}om_X(\mathscr{I}_{\Sigma}, \omega_X).$$

Proof. The first isomorphism holds because ω_{X^N} is a reflexive sheaf and X^N is normal. The second follows from Grothendieck duality applied to the finite morphism η (and the definition of the conductor). To prove the last isomorphism, consider the map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{I}_{\Sigma} \arrow[r] \arrow[d] & \mathscr{O}_X \arrow[r] \arrow[d, equal] & \mathscr{O}_{\Sigma} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_C \arrow[r] & \mathscr{O}_X \arrow[r] & \mathscr{O}_C \arrow[r] & 0
\end{tikzcd}$$

and apply $\mathcal{R}\mathcal{H}om^{\cdot}_X(\_, \omega^{\cdot}_X)$. By Lemma 4.14, we obtain

$$h^{-d+1}\big(\mathcal{R}\mathcal{H}om^{\cdot}_X\big(\mathscr{O}_{\Sigma}, \omega^{\cdot}_X\big)\big) = h^{-d+1}\big(\omega^{\cdot}_{\Sigma}\big) \simeq h^{-d+1}\big(\omega^{\cdot}_C\big) = h^{-d+1}\big(\mathcal{R}\mathcal{H}om^{\cdot}_X\big(\mathscr{O}_C, \omega^{\cdot}_X\big)\big),$$

thus we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & h^{-d}(\omega^{\cdot}_X) \arrow[r] \arrow[d, equal] & h^{-d}(\mathcal{R}\mathcal{H}om^{\cdot}_X(I_C, \omega^{\cdot}_X)) \arrow[r] \arrow[d] & h^{-d+1}(\omega^{\cdot}_C) \arrow[r] \arrow[d, "\simeq"] & h^{-d+1}(\omega^{\cdot}_X) \arrow[d, equal] \\
0 \arrow[r] & h^{-d}(\omega^{\cdot}_X) \arrow[r] & h^{-d}(\mathcal{R}\mathcal{H}om^{\cdot}_X(\mathscr{I}_\Sigma, \omega^{\cdot}_X)) \arrow[r] & h^{-d+1}(\omega^{\cdot}_\Sigma) \arrow[r] & h^{-d+1}(\omega^{\cdot}_X).
\end{tikzcd}$$

The statement then follows from the five lemma. □

Theorem 4.16. *If (X, Δ) is a Cohen–Macaulay weakly semi-log canonical pair, then X has DB singularities.*

Proof. In our setting we may assume that X is non-normal but that it is S_2 and seminormal. Thus C, the non-normal locus of X, is a codimension 1 subset of X (see Remark 4.5), in particular $C \neq 0$. Using the same notation as above, let $\Sigma = \operatorname{Sing} X$. Also observe that any log resolution $\pi : \widetilde{X} \to X$ factors through η:

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\zeta"] \arrow[rr, bend right, "\pi"'] & X^N \arrow[r, "\eta"] & X.
\end{tikzcd}$$

Therefore $\pi_*\omega_{\widetilde{X}}(F) = \eta_*\zeta_*\omega_{\widetilde{X}}(F)$. Recall from Remark 4.7 that C is reduced, so $\Sigma_e = C$, and then by Lemma 4.15, $\eta_*\omega_{X^N}(B) \simeq \mathcal{H}om_X(\mathscr{I}_\Sigma, \omega_X)$. Then by Theorem 4.11 it is sufficient to show that $\zeta_*\omega_{\widetilde{X}}(F) = \omega_{X^N}(B)$. Notice that by definition $F = \zeta_*^{-1}B + G$ where G is the reduced exceptional divisor of ζ (cf. Remark 4.12). By assumption $(X^N, B + \eta_*^{-1}\Delta)$ is log canonical, so Lemma 3.15 implies that $\zeta_*\omega_{\widetilde{X}}(F) = \omega_{X^N}(B)$. □

5. Cohomologically insignificant degenerations

DB singularities were originally defined by Steenbrink in a Hodge-theoretic context and they admit many interesting properties. In particular, they are strongly connected with cohomologically insignificant degenerations.

Inspired by Mumford's definition of insignificant surface singularities Dolgachev [3] defined a *cohomologically insignificant degeneration* as follows:

Let $f : \mathscr{X} \to S$ be a proper holomorphic map from a complex space $\mathscr{X}$ to the unit disk S that is smooth over the punctured disk $S \setminus \{0\}$. For $t \in S$ denote $\mathscr{X}_t = f^{-1}(t)$. The fiber $\mathscr{X}_0$, the *special fiber*, may be considered as a degeneration of any fiber $\mathscr{X}_t$, $t \neq 0$. Let $\beta_t : H^j(\mathscr{X}) \to H^j(\mathscr{X}_t)$ be the restriction map of the jth-cohomology spaces with real coefficients. Because $\mathscr{X}_0$ is a strong deformation retract of $\mathscr{X}$ the map β_0 is bijective. The composite map

$$\mathrm{sp}_t^j = \beta_t \circ \beta_0^{-1} : H^j(\mathscr{X}_0) \to H^j(\mathscr{X}_t)$$

is called the *specialization map* and plays an important role in the theory of degenerations of algebraic varieties. According to Deligne for every complex algebraic variety Y the cohomology space $H^n(Y)$ admits a canonical and functorial mixed Hodge structure. However in general sp_t^j is not a morphism of these mixed Hodge structures. On the other hand, Schmid [24] and Steenbrink [29] introduced a mixed Hodge structure on $H^i(\mathscr{X}_0)$, the *limit Hodge structure*, with respect to which sp_t^j becomes a morphism of mixed Hodge structures.

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In the above setup, $\mathscr{X}_0$ is called a *cohomologically j-insignificant degeneration* if sp_t^j induces an isomorphism of the (p,q)-components of the mixed Hodge structures with $pq=0$. Note that this definition is independent of the choice of $t \neq 0$. Finally, $\mathscr{X}_0$ is called a *cohomologically insignificant degeneration* if it is cohomologically j-insignificant for every j.

For us the relevance of this notion is that Steenbrink [30] proved that every proper, flat degeneration f over the unit disk S is cohomologically insignificant provided $f^{-1}(0)$ has DB singularities. As a combination of Steenbrink's result and our main theorem, we obtain the following.

Theorem 5.1. *Let X be a proper algebraic variety over $\mathbb{C}$ with Cohen–Macaulay semi-log canonical singularities. Then every proper flat degeneration f over the unit disk with $f^{-1}(0)=X$ is cohomologically insignificant.*

6. Kodaira vanishing

Following ideas of Kollár [17, §9], we give a proof of Kodaira vanishing for log canonical singularities. This was recently also proven by Fujino using a different technique [7, Corollary 5.11]. Our proof applies to (weakly) semi-log canonical singularities as well, while Fujino's does not use the Cohen–Macaulay assumption. Partial results were also obtained by Kollár [17, 12.10] and Kovács [21, 2.2].

Convention 6.1. For the rest of the section, all cohomologies are in the Euclidean topology. Nonetheless, the results remain true for coherent cohomology by Serre's GAGA principle.

First we need a variation on an important theorem.

Theorem 6.2. *(See [17, Theorem 9.12].) Let X be a proper variety and $\mathscr{L}$ a line bundle on X. Let $\mathscr{L}^n \simeq \mathscr{O}_X(D)$, where $D=\sum d_i D_i$ is an effective divisor (the D_i are the irreducible components of D). We also assume that the generic point of each D_i is a smooth point of X. Let s be a global section of $\mathscr{L}^n$ whose zero divisor is D. Assume that $0<d_i<n$ for every i. Let Z be the scheme obtained by taking the nth root of s (that is, $Z=X[\sqrt{s}]$ using the notation from* [17, 9.4]*). Assume further that*

$$H^j(Z,\mathbb{C}_Z) \to H^j(Z,\mathscr{O}_Z)$$

is surjective. Then for any collection of $b_i \geqslant 0$ the natural map

$$H^j\Big(X,\mathscr{L}^{-1}\Big(-\sum b_i D_i\Big)\Big) \to H^j\big(X,\mathscr{L}^{-1}\big)$$

is surjective.

The aforementioned variation of this theorem differs from the version stated above in that Z was defined to be the normalization of $X[\sqrt{s}\,]$. However, as we are dealing with possibly non-normal schemes (e.g., *slc*) we need this version. In some sense, the proof of this formulation is actually easier and we sketch the argument below. The strategy is the same as in [17, Theorem 9.12], but as Z is not normalized, some steps and ingredients are different.

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Proof. Let $Z = X[\sqrt{s}\,] = \mathrm{Spec}_X \sum_{t=0}^{n-1} \mathscr{L}^{-t}$ and $p : Z \to X$ denote the natural map. By construction there is a decomposition $p_* \mathscr{O}_Z \simeq \sum_{t=0}^{n-1} \mathscr{L}^{-t}$. Fixing an nth root of unity ζ and considering the associated $\mathbb{Z}/n$-action on Z (coming from the cyclic cover), we see that $\mathbb{Z}/n$ acts on the summand $\mathscr{L}^{-t}$ by multiplication by ζ^{-t}, so $p_*\mathscr{O}_Z \simeq \sum \mathscr{L}^{-t}$ is actually the eigensheaf decomposition of $p_*\mathscr{O}_Z$, cf. [17, Proposition 9.8]. One also has an eigensheaf decomposition, $p_*\mathbb{C}_Z \simeq \sum_{t=0}^{n-1} \mathscr{G}_t$. For ease of reference, set $\mathscr{G}_t$ to be the eigensheaf corresponding to the eigenvalue ζ^{-t}. In particular, $\mathscr{G}_t \subseteq \mathscr{L}^{-t}$.

With these decompositions, note that we have a surjective map

$$\sum_t H^j(X, \mathscr{G}_t) \simeq H^j(Z, \mathbb{C}_Z) \twoheadrightarrow H^j(Z, \mathscr{O}_Z) \simeq \sum_t H^j\big(X, \mathscr{L}^{-t}\big)$$

and so in particular we have a surjection

$$H^j(X, \mathscr{G}_1) \twoheadrightarrow H^j\big(X, \mathscr{L}^{-1}\big) \tag{6.2.1}$$

for every j. We now claim that $\mathscr{G}_1$ is a subsheaf of $\mathscr{L}^{-1}(-\sum b_i D_i)$. As they are both subsheaves of $\mathscr{L}^{-1}$, this is a local question and it is enough to show that for every connected open set $U \subseteq X$, the inclusion $\gamma : \Gamma(U, \mathscr{G}_1) \hookrightarrow \Gamma(U, \mathscr{L}^{-1})$ factors through the inclusion $\delta : \Gamma(U, \mathscr{L}^{-1}(-\sum b_i D_i)) \hookrightarrow \Gamma(U, \mathscr{L}^{-1})$.

If U is such that $U \cap D = \emptyset$, then δ is an isomorphism and so the statement holds trivially.

If U is such that $U \cap D \neq \emptyset$, then $\Gamma(U, \mathscr{G}_1) = 0$ by (6.2.2), and so the statement again holds trivially.

Claim 6.2.2. *Let $U \subseteq X$ be a connected open set such that $U \cap D \neq \emptyset$. Then $\Gamma(U, \mathscr{G}_1) = 0$.*

Proof. We give two short proofs of this claim.

First, let $\widetilde{Z}$ denote the normalization of Z, $\widetilde{p} : \widetilde{Z} \to X$ the induced map, and $\widetilde{\mathscr{G}_t}$ the eigensheaf of the $\mathbb{Z}/n$ action on $\widetilde{p}_*\mathbb{C}_{\widetilde{Z}}$ corresponding to the eigenvalue ζ^{-t}. Then $p_*\mathbb{C}_Z \hookrightarrow \widetilde{p}_*\mathbb{C}_{\widetilde{Z}}$ naturally, so in particular, $\mathscr{G}_t \subseteq \widetilde{\mathscr{G}_t}$ for all t and hence the statement follows by [17, 9.11.3].

Alternatively, one can give a direct proof as follows. The assumptions imply that there exists a dense open subset $U' \subseteq U$ such that each $x \in U' \cap D$ has a neighborhood where X is smooth and D is defined by a power of a coordinate function. Then the computation in [17, 9.9] shows that $p^{-1}U'$ and therefore $p^{-1}U$ are connected and the claim follows easily. □

Therefore $\mathscr{G}_1$ is indeed a subsheaf of $\mathscr{L}^{-1}(-\sum b_i D_i)$ and one obtains a factorization

$$\begin{tikzcd} H^j(X, \mathscr{G}_1) \arrow[r] \arrow[rr, bend left, two heads] & H^j(X, \mathscr{L}^{-1}(-\sum b_i D_i)) \arrow[r] & H^j(X, \mathscr{L}^{-1}), \end{tikzcd}$$

where the composition is surjective by (6.2.1) and so the second arrow is surjective as well. □

Theorem 6.3 *(Serre's vanishing). (See [19, 5.72].) Let X be a projective scheme over a field k of pure dimension n with ample Cartier divisor D. Then the following are equivalent:*

(1) *X is Cohen–Macaulay,*
(2) *$H^j(X, \mathscr{O}_X(-rD)) = 0$ for every $j < n$ and $r \gg 0$.*

In order to use the "usual" covering trick, we need to establish that our assumptions are inherited by the covers. Examples of rational singularities with non-Cohen–Macaulay canonical covers in [28] suggest that this is not entirely obvious.

First we need the following construction.

Notation 6.4. Let $\tau : S \to T$ be a finite morphism between reduced schemes of finite type over $\mathbb{C}$ and assume that T and S are normal. Let $i : U = T \setminus \operatorname{Sing} T \hookrightarrow T$ and $j : V = \tau^{-1}U \hookrightarrow S$. Since T is normal and τ is finite, $\operatorname{codim}_T(T \setminus U) \geqslant 2$ and $\operatorname{codim}_S(S \setminus V) \geqslant 2$. Let $D \subseteq T$ be an effective Weil divisor. Then $D|_U$ is a Cartier divisor corresponding to the invertible sheaf $\mathscr{L}$ on U and D induces a section of $\mathscr{L}$: $\delta : \mathscr{O}_U \hookrightarrow \mathscr{L}$. Furthermore, $\mathscr{O}_T(D) \simeq i_*\mathscr{L}$. Then the pullback of δ induces a section $\tau^*\delta : \mathscr{O}_V \hookrightarrow \tau^*\mathscr{L}$, which in turn induces a section of the rank 1 reflexive sheaf $j_*\tau^*\mathscr{L}$ on S. Denote the corresponding Weil divisor by $\tau^{[*]}D$, i.e., $\mathscr{O}_S(\tau^{[*]}D) \simeq j_*\tau^*\mathscr{L}$. An alternative way to obtain $\tau^{[*]}D$ is to take the closure (in S) of the divisor $\tau^*(D|_U) \subseteq V$.

Lemma 6.5. *Let (X, Δ) be a projective log variety with weakly semi-log canonical singularities and $\sigma : Z \to X$ a cyclic cover of X induced by a general section of a sufficiently large power of an ample line bundle as in* [19, 2.50]. *Then there exists an effective $\mathbb{Q}$-divisor Γ on Z such that (Z, Γ) is weakly semi-log canonical. Furthermore, if X is Cohen–Macaulay, then so is Z.*

Proof. Let $\mathscr{L}$ be an ample line bundle on X, $m \gg 0$ and $s \in H^0(X, \mathscr{L}^m)$ a general section. Then $D = (s = 0)$ is reduced. As before, let $\eta : X^N \to X$ denote the normalization of X and $B \subset X^N$ the extension of the conductor to X^N (cf. (4.5)). Then, by assumption, $(X^N, B + \eta_*^{-1}\Delta)$ is log canonical. Observe that as η is finite, $\eta^*\mathscr{L}$ is also ample. Therefore, by [19, 5.17.(2)], $(X^N, B + \eta_*^{-1}\Delta + \eta^*D)$ is also log canonical.

Let $\mathscr{A} = \bigoplus_{i=0}^{m-1} \mathscr{L}^{-i}$ with the $\mathscr{O}_X$-algebra structure induced by s. Let $\sigma : Z = \operatorname{Spec}_X \mathscr{A} \to X$ be as in [19, 2.50] and similarly $\widetilde{\sigma} : W = \operatorname{Spec}_X \eta^*\mathscr{A} \to X^N$.

$$\begin{tikzcd} W \arrow[r, "\widetilde{\sigma}"] \arrow[d, "\tau"'] & X^N \arrow[d, "\eta"] \\ Z \arrow[r, "\sigma"'] & X. \end{tikzcd}$$

By assumption X^N is normal, i.e., R_1 and S_2 by Serre's criterion. Then W is also R_1 by [19, 2.51] and furthermore $\widetilde{\sigma}_*\mathscr{O}_W \simeq \eta^*\mathscr{A}$ is also S_2 (as it is locally free), so we see that W is also S_2 by [19, 5.4]. Therefore W is normal by Serre's criterion and hence τ factors through the normalization of Z:

$$\begin{tikzcd} W \arrow[r, "\widetilde{\tau}"'] \arrow[rr, bend left, "\tau"] & \widetilde{Z} \arrow[r] & Z. \end{tikzcd}$$

From the construction it is clear that $\tau : W \to Z$ is birational, and hence so is $\widetilde{\tau}$. However, then it must be an isomorphism by Zariski's Main Theorem, and hence W is the normalization of Z.

As X is S_2, so is $\sigma_*\mathscr{O}_Z \simeq \mathscr{A}$, and then Z is S_2 as well.

Let $T = (\operatorname{Sing} X \cap D)_{\mathrm{red}} \subseteq X$ and $\widetilde{T} = \sigma^{-1}T \subseteq Z$ closed subsets in X and Z respectively.

Then by construction $\operatorname{codim}_X T \geqslant 2$ and hence $\operatorname{codim}_Z \widetilde{T} \geqslant 2$. Observe that for any $z \in Z \setminus \widetilde{T}$ either Z is smooth at z or σ is étale in a neighborhood of z in Z. Therefore, Z is seminormal in codimension 1 (since X is seminormal) and as we have just shown that Z is also S_2, it follows by [8, 2.7] that Z is seminormal everywhere.

Let $M = (\widetilde{\sigma}^*(\eta^* D))_{\mathrm{red}}$. Since X^N and W are smooth in codimension 1, and $\widetilde{\sigma}$ is ramified exactly along $\eta^* D$ with degree m and ramification index m everywhere, it follows, that

$$\widetilde{\sigma}^*\big(K_{X^N} + B + \eta_*^{-1}\Delta + \eta^* D\big) = K_W + \widetilde{\sigma}^{[*]}\big(B + \eta_*^{-1}\Delta\big) + M.$$

Then $(W, \widetilde{\sigma}^{[*]}(B + \eta_*^{-1}\Delta) + M)$ is log canonical by [19, 5.20(4)].

Let B_Z denote the subscheme defined by the conductor of Z in W. We claim that B_Z is contained in the proper transform of B, i.e., $B_Z \subseteq \widetilde{\sigma}^{[*]} B$ (cf. (4.4)). To see this, note that since Z is S_2 and seminormal, the conductor is simply the codimension 1 part of the non-smooth locus of Z. But the non-smooth locus of Z is the pre-image of the non-smooth locus of X by [19, 2.51] and so the claim follows.

Then there exists an effective $\mathbb{Q}$-divisor Θ on W satisfying $\widetilde{\sigma}^{[*]}(B + \eta_*^{-1}\Delta) = B_Z + \Theta$. Now, if we choose $\Gamma = \tau_*(\Theta + M)$, then (Z, Γ) has weakly semi-log canonical singularities.

If X is Cohen–Macaulay, then so is $\mathscr{A} \simeq \pi_* \mathscr{O}_Z$ and Z is Cohen–Macaulay by [19, 5.4]. □

Corollary 6.6. *Kodaira vanishing holds for Cohen–Macaulay weakly semi-log canonical varieties: Let (X, Δ) be a projective Cohen–Macaulay weakly semi-log canonical pair and $\mathscr{L}$ an ample line bundle on X. Then $H^i(X, \mathscr{L}^{-1}) = 0$ for $i < \dim X$.*

Proof. We will use the notation from (6.5). Z is Cohen–Macaulay and (Z, Γ) is weakly semi-log canonical. Therefore by (4.16) Z is Du Bois, and it follows from (6.2) that $H^i(X, \mathscr{L}^{-m}) \to H^i(X, \mathscr{L}^{-1})$ is surjective for all $m \geqslant 0$. Serre's vanishing (6.3) implies that $H^i(X, \mathscr{L}^{-m}) = 0$ for $m \gg 0$ and $i < \dim X$, so the desired statement follows. □

This implies invariance of plurigenera in stable Gorenstein families.

Corollary 6.7. *Let $f : \mathscr{X} \to S$ be a stable Gorenstein family, i.e., a flat projective family of canonically polarized varieties with at most Gorenstein (weakly) semi-log canonical singularities. Then $h^i(\mathscr{X}_t, \omega_{\mathscr{X}_t}^m)$ is independent of $t \in S$ for any $m > 0$ and $i \geqslant 0$.*

Proof. By Serre duality $h^i(\mathscr{X}_t, \omega_{\mathscr{X}_t}^m) = h^{\dim \mathscr{X}_t - i}(\mathscr{X}_t, \omega_{\mathscr{X}_t}^{-m+1})$ and the latter vanishes for $i > 0$ and $m > 1$ by (6.6). Then

$$h^0\big(\mathscr{X}_t, \omega_{\mathscr{X}_t}^m\big) = \chi\big(\mathscr{X}_t, \omega_{\mathscr{X}_t}^m\big)$$

for $m > 1$ and this is independent of t because f is flat.

So we may assume that $m = 1$. Then $h^i(\mathscr{X}_t, \omega_{\mathscr{X}_t}) = h^{\dim \mathscr{X}_t - i}(\mathscr{X}_t, \mathscr{O}_{\mathscr{X}_t})$ and this is independent of t for any i by (4.16) and [4, Théorème 4.6]. □

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