THE STRUCTURE OF SURFACES AND THREEFOLDS MAPPING TO THE MODULI STACK OF CANONICALLY POLARIZED VARIETIES

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Abstract

Generalizing the well-known Shafarevich hyperbolicity conjecture, it has been conjectured by Viehweg that a quasi-projective manifold that admits a generically finite morphism to the moduli stack of canonically polarized varieties is necessarily of log general type. Given a quasi-projective threefold Y° that admits a nonconstant map to the moduli stack, we employ extension properties of logarithmic pluriforms to establish a strong relationship between the moduli map and the minimal model program of Y°: in all relevant cases the minimal model program leads to a fiber space whose fibration factors the moduli map. A much-refined affirmative answer to Viehweg's conjecture for families over threefolds follows as a corollary. For families over surfaces, the moduli map can often be described quite explicitly. Slightly weaker results are obtained for families of varieties with trivial or more generally semiample canonical bundle.

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DUKE MATHEMATICAL JOURNAL
Vol. 155, No. 1, © 2010 DOI 10.1215/00127094-2010-049
Received 12 December 2008. Revision received 1 December 2009.
2000 *Mathematics Subject Classification.* Primary 14J10; Secondary 14D22.
Kebekus's work supported in part by the DFG-Forschergruppe "Classification of Algebraic Surfaces and Compact Complex Manifolds."
Kovács's work supported in part by National Science Foundation grants DMS-0554697 and DMS-0856185, and the Craig McKibben and Sarah Merner Endowed Professorship in Mathematics.

1. Introduction and main results

1.A. Introduction

Let Y° be a quasi-projective manifold that admits a morphism $\mu : Y^\circ \to \mathfrak{M}$ to the moduli stack of canonically polarized varieties. Generalizing the classical Shafarevich hyperbolicity conjecture (see [S1]), Viehweg conjectured in [V, Problem 6.3] that Y° is necessarily of log general type if μ is generically finite. Equivalently, if $f^\circ : X^\circ \to Y^\circ$ is a smooth family of canonically polarized varieties, then Y° is of log general type if the variation of f° is maximal, that is, $\mathrm{Var}(f^\circ) = \dim Y^\circ$. We refer to [KK] for the relevant notions, for detailed references, and for a brief history of the problem, but see also [KS].

Viehweg's conjecture was confirmed for two-dimensional manifolds Y° in [KK] by using explicit surface geometry. Here, we employ recent extension theorems for logarithmic forms to study families over threefolds. If $\dim Y^\circ \leq 3$, we establish a strong relationship between the moduli map μ and the logarithmic minimal model program of Y°: in all relevant cases, any logarithmic minimal model program (MMP) necessarily terminates with a fiber space whose fibration factors the moduli map. This allows us to prove a much-refined version of Viehweg's conjecture for families over surfaces and threefolds and give a positive answer to the conjecture even for families of varieties with only semiample canonical bundle. If Y° is a surface, we recover the results of [KK] in a more sophisticated manner. In fact, going far beyond those results, we give a complete geometric description of the moduli map in those cases when the variation cannot be maximal.

The proof of our main result is rather conceptual and independent of the arguments in [KK], which essentially relied on a combinatorial analysis of curve arrangements on surfaces and on Keel-McKernan's solution to the Miyanishi conjecture in dimension two (see [KeMc]). Many of the techniques introduced here generalize well to higher dimensions, most others at least conjecturally.

Throughout the present article we work over the field of complex numbers.

1.B. Main results

The main results of this article are summarized in the following theorems which describe the geometry of families over threefolds under increasingly strong hypothesis.

THEOREM 1.1 (Viehweg's conjecture for families over threefolds)
Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of varieties with semiample canonical bundle over a quasi-projective manifold Y° of dimension $\dim Y^\circ \leq 3$. If f° has

maximal variation, then Y° is of log general type. In other words,

$$\mathrm{Var}(f^\circ) = \dim Y^\circ \;\Rightarrow\; \kappa(Y^\circ) = \dim Y^\circ.$$

Remark 1.1.1
The definition of Kodaira dimension $\kappa(Y^\circ)$ for quasi-projective manifolds is recalled in Notation 2.3 below.

For families of *canonically* polarized varieties, we can say much more. The following much stronger theorem gives an explicit geometric explanation of Theorem 1.1.

THEOREM 1.2 (Relationship between the moduli map and the MMP)
Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of canonically polarized varieties over a quasi-projective manifold Y° of dimension $\dim Y^\circ \leq 3$. *Let Y be a smooth compactification of Y° such that $D := Y \setminus Y^\circ$ is a divisor with simple normal crossings.*

Then any run of the MMP of the pair (Y, D) will terminate in a Kodaira or Mori fiber space whose fibration factors the moduli map birationally.

Remark 1.2.1
If $\kappa(Y^\circ) = 0$ in the setup of Theorem 1.2, then any run of the MMP will terminate in a Kodaira fiber space that maps to a single point. Since this map to a point factors the moduli map birationally, Theorem 1.2 asserts that the family f° is necessarily isotrivial if $\kappa(Y^\circ) = 0$.

Remark 1.2.2
Neither the compactification Y nor the MMP discussed in Theorem 1.2 is unique. When running the MMP, one often needs to choose the extremal ray that is to be contracted.

In the setup of Theorem 1.2, if $\kappa(Y^\circ) \geq 0$, then the MMP terminates in a Kodaira fiber space whose base has dimension $\kappa(Y^\circ)$. The following refined version of Viehweg's conjecture is therefore an immediate corollary of Theorem 1.2.

COROLLARY 1.3 (Refined Viehweg's conjecture for families over threefolds [KK, Conjecture 1.6])
Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of canonically polarized varieties over a quasi-projective manifold Y° of dimension $\dim Y^\circ \leq 3$. *Then either*

(i) $\kappa(Y^\circ) = -\infty$ *and* $\mathrm{Var}(f^\circ) < \dim Y^\circ$, *or*

(ii) $\kappa(Y^\circ) \geq 0$ *and* $\mathrm{Var}(f^\circ) \leq \kappa(Y^\circ)$. □

As a further application of Theorem 1.2, we describe the family $f^\circ : X^\circ \to Y^\circ$ explicitly if the base manifold Y° is a surface and the variation is not maximal.

THEOREM 1.4 (Description of the family in case of $\operatorname{Var}(f^\circ) = 1$)
Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of canonically polarized varieties over a quasi-projective manifold Y° of dimension $\dim Y^\circ = 2$. *If $\kappa(Y^\circ) < 2$ and* $\operatorname{Var}(f^\circ) = 1$*, then one of the following holds.*

(1.4.1) *$\kappa(Y^\circ) = 1$, and there exists an open set $U \subseteq Y^\circ$ and a Cartesian diagram of one of the following two types*

$$\begin{tikzcd} \widetilde{U} \arrow[rr, "\gamma", "\text{étale}"'] \arrow[d, "\substack{\widetilde{\pi}\\ \text{elliptic}\\ \text{fibration}}"'] & & U \arrow[d, "\substack{\pi\\ \text{elliptic}\\ \text{fibration}}"] \\ \widetilde{V} \arrow[rr, "\text{étale}"'] & & V \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \widetilde{U} \arrow[rr, "\gamma", "\text{étale}"'] \arrow[d, "\substack{\widetilde{\pi}\\ \text{smooth}\\ \text{algebraic}\\ \mathbb{C}^*\text{-bundle}}"'] & & U \arrow[d, "\substack{\pi\\ \text{smooth}\\ \text{algebraic}\\ \mathbb{C}^*\text{-bundle}}"] \\ \widetilde{V} \arrow[rr, equal] & & V \end{tikzcd}$$

such that $f^\circ_{\widetilde{U}} : X^\circ \times_U \widetilde{U} \to \widetilde{U}$ is the pullback of a family over $\widetilde{V}$, or

(1.4.2) *$\kappa(Y^\circ) = -\infty$, and there exists an open set $U \subseteq Y^\circ$ of the form $U = V \times \mathbb{A}^1$ such that $X^\circ\big|_U$ is the pullback of a family over V.*

To complete the description of families with nonmaximal variation over two-dimensional bases, we include the following well-known statement.

THEOREM 1.5 (Description of the family in case of $\operatorname{Var}(f^\circ) = 0$)
Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of canonically polarized varieties over a quasi-projective manifold Y°. If $\operatorname{Var}(f^\circ) = 0$*, then there exists an open set $U \subseteq Y^\circ$ such that $X^\circ\big|_U$ is isotrivial, and further there exists a finite étale cover $\widetilde{U} \to U$ such that $f^\circ_{\widetilde{U}} : X^\circ \times_U \widetilde{U} \to \widetilde{U}$ is trivial.* $\square$

Remark 1.5.1
Theorem 1.5 is not a deep result. Unlike Theorems 1.1–1.4, it follows from simple abstract arguments, as illustrated in the proof of Lemma 7.4 below.

1.C. Outline of proof, outline of article
The main results of this article are shown in Sections 8–10, where we consider the cases $\kappa(Y^\circ) = -\infty$, $\kappa(Y^\circ) = 0$, and $\kappa(Y^\circ) > 0$ separately, the most difficult case being when $\kappa(Y^\circ) = 0$. To keep the proofs readable, we have chosen to present many of the more technical results separately in the preparatory Sections 2–7. These may be of some independent interest. The reader who is primarily interested in a broad

outline of the argument may likely want to take the technicalities on faith and move directly to Sections 8–10 on the first reading.

Section 2 introduces notation used in the remainder of this article. In Section 3, we discuss certain classes of singularities that appear in the MMP and recall the Bogomolov vanishing result for log canonical threefolds. The standard construction of the global index-one cover for good minimal models of Kodaira dimension zero is recalled and summarized in Section 4.

Viehweg and Zuo have shown that the base of a family of positive variation often carries an invertible sheaf of pluridifferentials whose Kodaira-Iitaka dimension is at least the variation of the family. These *Viehweg-Zuo sheaves*, which play a crucial role in our arguments, are introduced and discussed in Section 5. The existence of a Viehweg-Zuo sheaf of positive Kodaira-Iitaka dimension has strong consequences for the geometry of the underlying space: these are discussed in Section 6. We end the preparatory part of the paper with Section 7, where we discuss how families $f^\circ : X^\circ \to Y^\circ$ over a fibered base $\pi^\circ : Y^\circ \to C^\circ$ that are isotrivial over the π°-fibers often come from a family over C°, at least after passing to an étale cover.

Part I. Techniques

2. Notation and conventions

2.A. Reflexive tensor operations

When dealing with sheaves that are not necessarily locally free, we frequently use square brackets to indicate taking the reflexive hull.

Notation 2.1 (Reflexive tensor product)

Let Z be a normal variety, and let $\mathcal{A}$ be a coherent sheaf of $\mathcal{O}_Z$-modules. Given a number $n \in \mathbb{N}$, set $\mathcal{A}^{[n]} := (\mathcal{A}^{\otimes n})^{**}$. In a similar vein, we write $\Omega_Z^{[p]} := \big(\Omega_Z^p\big)^{**}$ and $\Omega_Z^{[p]}(\log \Delta) := \big(\Omega_Z^p(\log \Delta)\big)^{**}$ whenever $\Delta \subset Z$ is a reduced divisor. Likewise, if $\pi : Z' \to Z$ is a morphism of normal varieties, set $\pi^{[*]}(\mathcal{A}) := \big(\pi^*(\mathcal{A})\big)^{**}$.

If $\mathcal{A}$ is reflexive of rank one, we say that $\mathcal{A}$ is $\mathbb{Q}$-*Cartier* if there exists a number n such that $\mathcal{A}^{[n]}$ is invertible.

Remark 2.1.1

Recall from [K1, Section 16] that a *Weil divisorial sheaf* on a normal variety Z is a reflexive subsheaf $\mathcal{L} \subseteq \mathcal{K}(Z)$ of rank one, where $\mathcal{K}$ is the sheaf of total quotient rings introduced in [H, p. 176f]. These sheaves are in one-to-one correspondence with Weil divisors, and addition of Weil divisors corresponds to reflexive tensor products. This justifies Notation 2.1, where we extend the notion of $\mathbb{Q}$-*Cartier* to reflexive sheaves.

We later discuss the Kodaira dimension of singular pairs and the Kodaira-Iitaka dimension of reflexive sheaves on normal spaces. Since this is perhaps not quite standard, we recall the definition here.

Notation 2.2 (Kodaira-Iitaka dimension of a sheaf)
Let Z be a normal projective variety, and let $\mathscr{A}$ be a reflexive sheaf of rank one on Z. If $h^0\big(Z,\ \mathscr{A}^{[n]}\big) = 0$ for all $n \in \mathbb{N}$, then we say that $\mathscr{A}$ has *Kodaira-Iitaka dimension* $\kappa(\mathscr{A}) := -\infty$. Otherwise, set

$$M := \big\{n \in \mathbb{N} \,\big|\, h^0(Z,\ \mathscr{A}^{[n]}) > 0\big\},$$

recall that the restriction of $\mathscr{A}$ to the smooth locus of Z is locally free and consider the natural rational mapping

$$\phi_n : Z \dashrightarrow \mathbb{P}\big(H^0(Z,\ \mathscr{A}^{[n]})^*\big), \quad \forall n \in M.$$

The Kodaira-Iitaka dimension of $\mathscr{A}$ is then defined as

$$\kappa(\mathscr{A}) := \max_{n \in M}\big(\dim \overline{\phi_n(Z)}\,\big).$$

Notation 2.3 (Kodaira dimension of a quasi-projective variety)
If Z° is a quasi-projective manifold and if Z is a smooth compactification such that $\Delta := Z \setminus Z^\circ$ is a divisor with at most simple normal crossings, define the Kodaira dimension of Z° as $\kappa(Z^\circ) := \kappa\big(\mathscr{O}_Z(K_Z + \Delta)\big)$. Recall the standard fact that this number is independent of the choice of the compactification.

2.B. Reduced pairs
The following fundamental definitions of logarithmic geometry are used in the remainder of the article.

Definition 2.4 (Reduced pair)
A *reduced pair* or *logarithmic pair* (Z, Δ) consists of a normal variety Z and a reduced, but not necessarily irreducible Weil divisor $\Delta \subset Z$. A *morphism of reduced pairs*, written as $\gamma : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$, is a morphism $\gamma : \widetilde{Z} \to Z$ such that $\gamma^{-1}(\Delta) = \widetilde{\Delta}$ set-theoretically.

Definition 2.5 (Snc pairs)
Let (Z, Δ) be a reduced pair, and let $z \in Z$ be a point. We say that (Z, Δ) is *snc* at z, if there exists a Zariski open neighborhood U of z such that U is smooth and such that $\Delta \cap U$ has only simple normal crossings. The pair (Z, Δ) is snc if it is snc at all points.

Given a reduced pair (Z, Δ), let $(Z, \Delta)_{\text{reg}}$ be the maximal open set of Z where (Z, Δ) is snc, and let $(Z, \Delta)_{\text{sing}}$ be its complement, with the induced reduced subscheme structure.

Definition 2.6 (Log resolution)
A *log resolution* of (Z, Δ) is a birational morphism of pairs $\pi : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$ such that the π-exceptional set $\operatorname{Exc}(\pi)$ is of pure codimension one, such that $\big(\widetilde{Z}, \operatorname{supp}(\widetilde{\Delta} + \operatorname{Exc}(\pi))\big)$ is snc and such that π is isomorphic along $(Z, \Delta)_{\text{reg}}$.

If (Z, Δ) is a reduced pair, a log resolution is known to exist (see [K2]).

2.C. Minimal model program
We use the definitions and apply the techniques of the MMP frequently, sometimes without explicit references. On these occasions, the reader is referred to [KM] for background and details.

In particular, we use the fact that the MMP asserts the existence of *extremal contractions* (see [KM, Theorem 3.7 (3.31)]) on nonminimal varieties. These extremal contractions come in three different kinds: *divisorial*, *small*, and *of fiber type*. The first gives a birational morphism that contracts a divisor, the second leads to a *flip* (see [KM, Definition 2.8]), and the third gives a fiber space. Recall that a fiber space $\pi : Y \to Z$ is called *proper* if the general fiber F is of dimension $0 < \dim F < \dim Y$. We call an extremal contraction of fiber type *nontrivial* if the resulting fiber space is proper. Finally, recall that extremal contractions of divisorial or fiber type have relative Picard number one (see [KM, Proposition 3.36]).

Further note that since we are working in dimension at most three, we do not need to appeal to the recent phenomenal advances in the MMP by Hacon-McKernan and Birkar-Cascini-Hacon-McKernan (see [C], [BCHM]). However, these results give us reasonable hope that the methods here may extend to all dimensions.

3. Singularities of the MMP

3.A. Dlt singularities of index one
If (Z, Δ) is an snc pair of dimension $\dim Z \leq 3$, the MMP yields a birational map to a pair $(Z_\lambda, \Delta_\lambda)$, where Z_λ is $\mathbb{Q}$-factorial and where $(Z_\lambda, \Delta_\lambda)$ is dlt (see [KM, Definition 2.37] for the definition of dlt). We remark for later use that dlt pairs of index one are snc in codimension two.

LEMMA 3.1
Let (Z, Δ) be a dlt pair of index one, that is, a pair where $K_Z + \Delta$ is Cartier. Then

$$\operatorname{codim}_Z\big((Z, \Delta)_{\text{sing}} \cap \Delta\big) \geq 3. \tag{3.1.1}$$

Remark 3.1.2

It is important to note that (Z, Δ) has *simple* normal crossings away from $(Z, \Delta)_{\mathrm{sing}}$, whereas having only normal crossings would give a much weaker result. This, for example, implies that the components of Δ are smooth in codimension one, which is not true for a boundary with only normal crossings (see [KM, Remark 2.38]).

Proof

We prove the statement by induction on the dimension.

Start of induction. First assume that $\dim Z = 2$. Then by definition of dlt singularities (see [KM, Definition 2.37]), there exists a finite subset $T \subset Z$ such that $(Z, \Delta)_{\mathrm{sing}} \subseteq T$ and such that Z is log terminal at the points of T; that is, the discrepancy of any divisor E that lies over T is $a(E, Z, \Delta) > -1$. But since $K_Z + \Delta$ is Cartier, this number must be an integer, so $a(E, Z, \Delta) \geq 0$. This shows that (Z, Δ) is canonical at the points of T. Therefore, it follows by [KM, Theorem 4.5] that $T \cap \Delta = \emptyset$. In particular, (3.1.1) holds.

Inductive step. Now let Z be of arbitrary dimension, and let $H \subseteq Z$ be a general hyperplane section. Set $\Delta_H := \Delta \cap H$. Since a Cartier divisor being smooth at a point implies that the ambient space is also smooth at that point, it follows that for any $z \in H$, the pair (H, Δ_H) is snc at z if and only if (Z, Δ) is snc at z. In other words, $(H, \Delta_H)_{\mathrm{sing}} = (Z, \Delta)_{\mathrm{sing}} \cap H$ and

$$\operatorname{codim}_H\big((H, \Delta_H)_{\mathrm{sing}} \cap \Delta_H\big) = \operatorname{codim}_Z\big((Z, \Delta)_{\mathrm{sing}} \cap \Delta\big).$$

Notice further that (H, Δ_H) is dlt of index one. The claim thus follows by induction. $\square$

3.B. Dlc singularities

Given an snc pair of Kodaira dimension zero, the MMP terminates with a dlt pair (Z, Δ), where Δ is $\mathbb{Q}$-Cartier and where $K_Z + \Delta$ is torsion. Many of the arguments in Section 9 are based on the following observation.

> If $\Delta \neq \emptyset$ and if $\varepsilon \in \mathbb{Q}^+$ sufficiently small, then $\big(Z, (1-\varepsilon)\Delta\big)$ is a dlt pair of Kodaira dimension $-\infty$. Therefore, it admits at least one further extremal contraction.

Using the thinned down boundary to push the MMP further, we end with a reduced pair (Z', Δ') that might no longer be dlt, but still has manageable singularities.

Definition 3.2 (*Dlc pairs*)
A reduced pair (Z', Δ') is called *dlc* if (Z', Δ') is log canonical, Δ' is $\mathbb{Q}$-Cartier, and for any sufficiently small positive number $\varepsilon \in \mathbb{Q}^+$, the pair $\big(Z', (1-\varepsilon)\Delta'\big)$ is dlt.

Dlc singularities are of interest to us because sheaves of reflexive differentials on dlc surface pairs enjoy good pullback properties (see Theorem 5.3 below). For future reference, we recall the relationship between dlc and another notion of singularity.

Definition 3.3 (Boundary-lc pair [GKK, Definition 3.6])
A reduced pair (Z, Δ) is called *boundary-lc* if (Z, Δ) is log canonical and if $(Z \setminus \Delta, \emptyset)$ is log terminal.

Remark 3.4 (Relationship between dlc and boundary-lc pairs)
By definition, a dlc pair (Z, Δ) is boundary-lc. If $\dim Z = 2$, then this implies that (Z, Δ) is finitely dominated by analytic snc pairs (see [GKK, Lemma 3.9]). In other words, every point $z \in Z$ admits an analytic neighborhood and a finite, surjective morphism of reduced pairs $(\widetilde{U}, D) \to (U, \Delta \cap U)$, where $\widetilde{U}$ is smooth and where the divisor D has only simple normal crossings.

3.C. Bogomolov-Sommese vanishing on singular spaces
If (Z, Δ) is an snc pair, the well-known Bogomolov-Sommese vanishing theorem asserts that for any number $1 \leq p \leq \dim Z$, any invertible subsheaf $\mathscr{C} \subseteq \Omega^p_Z(\log \Delta)$ has Kodaira-Iitaka dimension at most p. (See [EV, Section 6] for a thorough discussion.) Many of the arguments in this article are deeply based on the fact that similar results also hold for reflexive sheaves of differentials on pairs with dlc, or more generally log canonical singularities.

The formulation of the general result we expect to be true is the following.

CONJECTURE 3.5 (Bogomolov-Sommese vanishing for log canonical varieties)
Let (Z, Δ) be a reduced pair, and assume that (Z, Δ) is log canonical. Let $\mathscr{A} \subseteq \Omega^{[p]}_Z(\log \Delta)$ be any reflexive subsheaf of rank one. If $\mathscr{A}$ is $\mathbb{Q}$-Cartier, then $\kappa(\mathscr{A}) \leq p$.

At this time, Conjecture 3.5 has been verified with the additional assumption $\dim Z \leq 3$ in [GKK] (see also [GKKP]).

THEOREM 3.6 (Bogomolov-Sommese vanishing for log canonical threefolds [GKK, Theorem 1.4])
Let (Z, Δ) be a reduced pair of dimension $\dim Z \leq 3$, *and assume that (Z, Δ) is log canonical. Let $\mathscr{A} \subseteq \Omega^{[p]}_Z(\log \Delta)$ be any reflexive subsheaf of rank one. If $\mathscr{A}$ is $\mathbb{Q}$-Cartier, then $\kappa(\mathscr{A}) \leq p$.* □

4. Global index-one covers for varieties of Kodaira dimension zero

In this section, we consider good minimal models of pairs with Kodaira dimension zero. We briefly recall the main properties of the global index-one cover, as described in [KM, Definition 2.52] or [R, Section 3.6f].

PROPOSITION 4.1
Let (Z, Δ) be a reduced pair. Assume that the log canonical divisor $K_Z + \Delta$ is torsion (in particular, that it is $\mathbb{Q}$-Cartier); that is, assume that there exists a number $m \in \mathbb{N}^+$ such that $\mathscr{O}_Z\big(m \cdot (K_Z + \Delta)\big) \cong \mathscr{O}_Z$. Then there exists a morphism of pairs $\eta : (Z', \Delta') \to (Z, \Delta)$, called the index-one cover*, with the following properties.*

(4.1.1) *The morphism η is finite. It is étale wherever Z is smooth. In particular, η is étale in codimension one.*

(4.1.2) *$K_{Z'} + \Delta'$ is Cartier and $\mathscr{O}_{Z'}(K_{Z'} + \Delta') \simeq \mathscr{O}_{Z'}$.*

(4.1.3) *If (Z, Δ) is dlt, then (Z', Δ') is dlt as well. If, furthermore, $z' \in Z'$ is a point where (Z', Δ') is not snc, then (Z', Δ') is canonical at z'.*

Proof
Properties (4.1.1) and (4.1.2) follow directly from the construction (see [KM, 2.50–53]). To prove (4.1.3), assume for the remainder of the proof that (Z, Δ) is dlt. We need to show that (Z', Δ') is dlt as well. Observe that if $z' \in Z'$ is a point such that (Z, Δ) is snc at $\eta(z')$, then (Z', Δ') is snc at z'. The definition of dlt, together with the fact that discrepancies only increase under finite morphisms (see [KM, Proposition 5.20]), then immediately yields the claim.

Finally, if $z' \in Z'$ is any point where (Z', Δ') is not snc, then the discrepancy of any divisor E that lies over z' is $a(E, Z', \Delta') > -1$. But, since $K_{Z'} + \Delta'$ is Cartier, this number must be an integer, so $a(E, Z', \Delta') \geq 0$. It follows that the pair (Z', Δ') is canonical at z'; hence, (4.1.3) is shown. □

COROLLARY 4.2
Under the conditions of Proposition 4.1, if $\gamma : (\widetilde{Z}, \widetilde{\Delta}) \to (Z', \Delta')$ is any log resolution, then $\kappa(K_{\widetilde{Z}} + \widetilde{\Delta}) = 0$.

Proof
Since (Z', Δ') is canonical wherever it is not snc, the definition of canonical singularities (see [KM, Notation 2.26, Definition 2.34]), implies that $K_{\widetilde{Z}} + \widetilde{\Delta}$ is represented by an effective, γ-exceptional divisor. □

5. Viehweg-Zuo sheaves

5.A. Definition of Viehweg-Zuo sheaves

In the setup of Theorem 1.2 and in a few other cases, Viehweg and Zuo have shown in [VZ1, Theorem 1.4] that there exists a number $n \gg 0$ and an invertible sheaf $\mathcal{A} \subseteq \operatorname{Sym}^n \Omega^1_Y(\log D)$ whose Kodaira-Iitaka dimension is at least the variation of f°; that is, $\kappa(\mathcal{A}) \geq \operatorname{Var}(f^\circ)$. The existence of a sheaf like this is a cornerstone of many of our arguments.

For technical reasons, it turns out to be more convenient to view $\mathcal{A}$ as a subsheaf of the tensor product via the injection $\operatorname{Sym}^n \Omega^1_Y(\log D) \hookrightarrow \big(\Omega^1_Y(\log D)\big)^{\otimes n}$. It is also advantageous to extend the study of these sheaves on singular varieties, and then it is natural to allow rank one reflexive sheaves instead of restricting to line bundles. These considerations give rise to the following definition.

Definition 5.1 (Viehweg-Zuo sheaf)

Let (Z, Δ) be a reduced pair. A reflexive sheaf $\mathcal{A}$ of rank one is called a *Viehweg-Zuo sheaf* if there exists a number $n \in \mathbb{N}$ and an embedding $\mathcal{A} \subseteq \big(\Omega^1_Z(\log \Delta)\big)^{[n]}$.

5.B. Pushing forward and pulling back

We often need to compare Viehweg-Zuo sheaves on different birational models of a pair. The following elementary statement shows that the pushforward of a Viehweg-Zuo sheaf under a birational map of pairs is often again a Viehweg-Zuo sheaf.

LEMMA 5.2 (Pushforward of Viehweg-Zuo sheaves)

Let (Z, Δ) be a reduced pair, and assume that there exists a Viehweg-Zuo sheaf $\mathcal{A} \subseteq \big(\Omega^1_Z(\log \Delta)\big)^{[n]}$. If $\lambda : Z \dashrightarrow Z'$ is a birational map whose inverse does not contract any divisor, if Z' is normal, and if Δ' is the (necessarily reduced) cycle-theoretic image of Δ, then there exists a Viehweg-Zuo sheaf $\mathcal{A}' \subseteq \big(\Omega^1_{Z'}(\log \Delta')\big)^{[n]}$ of Kodaira-Iitaka dimension $\kappa(\mathcal{A}') \geq \kappa(\mathcal{A})$.

Proof

The assumption that λ^{-1} does not contract any divisors and the normality of Z' guarantee that $\lambda^{-1} : Z' \dashrightarrow Z$ is a well-defined embedding over an open subset $U \subseteq Z'$ whose complement has codimension $\operatorname{codim}_{Z'}(Z' \setminus U) \geq 2$ (see Zariski's main theorem in [H, Theorem V 5.2]). In particular, $\Delta'\big|_U = \big(\lambda^{-1}\big|_U\big)^{-1}(\Delta)$. Let $\iota : U \hookrightarrow Z'$ denote the inclusion, and set $\mathcal{A}' := \iota_*\big((\lambda^{-1}\big|_U)^*\mathcal{A}\big)$. We obtain an inclusion of sheaves, $\mathcal{A}' \subseteq \big(\Omega^1_{Z'}(\log \Delta')\big)^{[n]}$. By construction, we have that $h^0(Z', \mathcal{A}'^{[m]}) \geq h^0(Z, \mathcal{A}^{[m]})$ for all $m > 0$; hence, $\kappa(\mathcal{A}') \geq \kappa(\mathcal{A})$. □

If Z is a singular space with desingularization $\pi : \widetilde{Z} \to Z$, it follows almost by definition that any differential $\sigma \in H^0\big(Z,\ \Omega^p_Z\big)$ pulls back to a differential $\pi^*(\sigma) \in H^0\big(\widetilde{Z},\ \Omega^p_{\widetilde{Z}}\big)$ [H, II Proposition 8.11]. However, if σ is a reflexive differential, that is, if $\sigma \in H^0\big(Z,\ \Omega^{[p]}_Z\big)$, it is not all clear (and generally false) that $\pi^*(\sigma)$ can be interpreted as a differential on $\widetilde{Z}$. Likewise, if (Z, Δ) is a reduced pair with log resolution $\pi : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$ and if $\mathscr{A} \subseteq \big(\Omega^1_Z(\log \Delta)\big)^{[n]}$ is a Viehweg-Zuo sheaf, it is generally not possible to interpret the reflexive pullback $\pi^{[*]}(\mathscr{A})$ as a Viehweg-Zuo sheaf on $(\widetilde{Z}, \widetilde{\Delta})$. However, if the pair (Z, Δ) is log canonical, the extension theorems for differential forms studied in [GKK] show that an interpretation of $\pi^{[*]}(\mathscr{A})$ as a Viehweg-Zuo sheaf often exists. The following theorem is an immediate consequence of Remark 3.4 and [GKK, Theorem 8.1]. It summarizes the results of [GKK] that are relevant for our arguments.

THEOREM 5.3 (Extension of Viehweg-Zuo sheaves [GKK, Theorem 8.1])
Let (Z, Δ) *be a dlc pair of dimension* $\dim Z \leq 2$*, and assume that there exists a Viehweg-Zuo sheaf* $\mathscr{A}$ *with inclusion* $\iota : \mathscr{A} \hookrightarrow \big(\Omega^1_Z(\log \Delta)\big)^{[n]}$*. If* $\pi : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$ *is a log resolution and if*

$$E := \textit{largest reduced divisor contained in } \pi^{-1}(\Delta) \cup \mathrm{Exc}(\pi),$$

then there exists an invertible Viehweg-Zuo sheaf $\mathscr{C} \subseteq \big(\Omega^1_{\widetilde{Z}}(\log E)\big)^{[n]}$ *with the following property. For an arbitrary* $m \in \mathbb{N}$*, the inclusion pulls back to give a sheaf morphism that factors through* $\mathscr{C}^{\otimes m}$*,*

$$\tilde{\iota}^{[m]} : \pi^{[*]}(\mathscr{A}^{[m]}) \hookrightarrow \mathscr{C}^{\otimes m} \subseteq \big(\Omega^1_{\widetilde{Z}}(\log E)\big)^{[m\cdot n]},$$

where $\pi^{[*]}$ *is the reflexive pullback introduced in Notation 2.1 above. In particular,* $\kappa(\mathscr{C}) \geq \kappa(\mathscr{A})$*.* □

5.C. The reduction lemma

Like regular differentials, logarithmic differentials come with a normal bundle, and the corresponding restriction sequences (see [EV, Properties 2.3], [KK, Lemma 2.13], and their respective references). Since Viehweg-Zuo sheaves live in tensor products of the sheaf of differentials, this does not immediately translate into a sequence for a given Viehweg-Zuo sheaf. This makes the following lemma useful in the remainder of this article.

LEMMA 5.4 (Reduction lemma)
Let Z *be an irreducible variety, let* $\mathscr{E}, \mathscr{F}, \mathscr{G}, \mathscr{H}$ *be locally free sheaves, and let* $\mathscr{A}$ *be a rank one torsion-free sheaf on* Z*. Assume that there exists a short exact sequence*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{E} \longrightarrow \mathscr{G} \longrightarrow 0. \tag{5.4.1}$$

Then

(5.4.2) *If there exists an inclusion $\mathscr{A} \hookrightarrow \mathscr{E}$, then either $\mathscr{A} \hookrightarrow \mathscr{F}$ or $\mathscr{A} \hookrightarrow \mathscr{G}$.*

(5.4.3) *If for some $m \in \mathbb{N}$, there exists an inclusion $\mathscr{A} \hookrightarrow \mathscr{H} \otimes \mathscr{E}^{\otimes m}$, then there exists a $p \in \mathbb{N}$, $0 \leq p \leq m$ such that $\mathscr{A} \hookrightarrow \mathscr{H} \otimes \mathscr{F}^{\otimes p} \otimes \mathscr{G}^{\otimes m-p}$.*

(5.4.4) *If for some $m \in \mathbb{N}$, there exists an inclusion $\mathscr{A} \hookrightarrow \mathscr{E}^{\otimes m}$ and $\mathscr{F} \simeq \mathscr{O}_Z$ (respectively, $\mathscr{G} \simeq \mathscr{O}_Z$), then there exists a $p \in \mathbb{N}$, $0 \leq p \leq m$ such that $\mathscr{A} \hookrightarrow \mathscr{G}^{\otimes p}$ (respectively, $\mathscr{A} \hookrightarrow \mathscr{F}^{\otimes p}$).*

Proof

Suppose that $\mathscr{A} \hookrightarrow \mathscr{E}$, and let $\mathscr{K} = \ker[\mathscr{A} \to \mathscr{G}] \subseteq \mathscr{A}$. If $\mathscr{A} \to \mathscr{G}$ is injective at the general point of Z, then $\mathscr{K}$ is a torsion sheaf and hence zero, so $\mathscr{A} \hookrightarrow \mathscr{G}$. Since $\operatorname{rk} \mathscr{A} = 1$, if $\mathscr{A} \to \mathscr{G}$ is not injective at the general point, then it is zero. However, then $\mathscr{A}/\mathscr{K} \subseteq \mathscr{G}$ is a torsion sheaf and hence zero, so $\mathscr{A} \hookrightarrow \mathscr{F}$. This proves (5.4.2). Taking $\mathscr{H} = \mathscr{O}_Z$, it is easy to see that (5.4.4) is a special case of (5.4.3). To prove (5.4.3), we use induction.

Start of induction. If $m = 1$, assertion (5.4.3) follows from applying (5.4.2) to the short exact sequence obtained by tensoring (5.4.1) with $\mathscr{H}$,

$$0 \to \mathscr{H} \otimes \mathscr{F} \to \mathscr{H} \otimes \mathscr{E} \to \mathscr{H} \otimes \mathscr{G} \to 0.$$

Note that if $m = 1$, then either $p = 0$ or $m - p = 0$.

Induction step. Now assume that the statement is true for all numbers $m' < m$. Consider the short exact sequence obtained by tensoring (5.4.1) with $\mathscr{H} \otimes \mathscr{E}^{\otimes(m-1)}$,

$$0 \to \mathscr{H} \otimes \mathscr{F} \otimes \mathscr{E}^{\otimes(m-1)} \to \mathscr{H} \otimes \mathscr{E}^{\otimes m} \to \mathscr{H} \otimes \mathscr{G} \otimes \mathscr{E}^{\otimes(m-1)} \to 0.$$

Applying (5.4.2) for this short exact sequence yields that either $\mathscr{A} \hookrightarrow (\mathscr{H} \otimes \mathscr{F}) \otimes \mathscr{E}^{\otimes(m-1)}$ or $\mathscr{A} \hookrightarrow (\mathscr{H} \otimes \mathscr{G}) \otimes \mathscr{E}^{\otimes(m-1)}$. Setting $\mathscr{H}' := \mathscr{H} \otimes \mathscr{F}$ or $\mathscr{H}' := \mathscr{H} \otimes \mathscr{G}$, respectively, and applying the induction hypothesis to the sequence

$$0 \to \mathscr{H}' \otimes \mathscr{F} \otimes \mathscr{E}^{\otimes(m-2)} \to \mathscr{H}' \otimes \mathscr{E}^{\otimes(m-1)} \to \mathscr{H}' \otimes \mathscr{G} \otimes \mathscr{E}^{\otimes(m-2)} \to 0,$$

we obtain a number $p \in \mathbb{N}$, $0 \leq p \leq m-1$ such that either $\mathscr{A} \hookrightarrow (\mathscr{H} \otimes \mathscr{F}) \otimes \mathscr{F}^{\otimes p} \otimes \mathscr{G}^{\otimes m-1-p}$ or $\mathscr{A} \hookrightarrow (\mathscr{H} \otimes \mathscr{G}) \otimes \mathscr{F}^{\otimes p} \otimes \mathscr{G}^{\otimes m-1-p}$. This proves (5.4.3). □

6. Viehweg-Zuo sheaves on minimal models

The existence of a Viehweg-Zuo sheaf of positive Kodaira-Iitaka dimension clearly has consequences for the geometry of the underlying space. In case the underlying space is the end product of the MMP, we summarize the two most important consequences below: when $\kappa = -\infty$ and when $\kappa = 0$.

6.A. The Picard number of minimal models with nonpositive Kodaira dimension

The following theorem is used later to show that a given pair is a Mori-Fano fiber space. This turns out to be a crucial step in the proof of our main results.

THEOREM 6.1

Let (Z, Δ) be a log canonical reduced pair, where Z is a projective $\mathbb{Q}$-factorial variety of dimension at most three. Assume that the following holds:

(6.1.1) *there exists a Viehweg-Zuo sheaf $\mathscr{A} \subseteq \left(\Omega^1_Z(\log \Delta)\right)^{[n]}$ of positive Kodaira-Iitaka dimension, and*

(6.1.2) *the antilog canonical divisor $-(K_Z + \Delta)$ is nef.*

Then the Picard number of Z is greater than one, $\rho(Z) > 1$.

Proof

We argue by contradiction and assume that $\rho(Z) = 1$. Let $C \subseteq Z$ be a general complete intersection curve. Since C is general, it avoids the singular locus $(Z, \Delta)_{\mathrm{sing}}$. By (6.1.2), the restriction $\Omega^1_Z(\log \Delta)\big|_C$ is a vector bundle of nonpositive degree

$$\deg \Omega^1_Z(\log \Delta)\big|_C = (K_Z + \Delta) \cdot C \le 0. \tag{6.1.3}$$

We claim that the restriction $\Omega^1_Z(\log \Delta)\big|_C$ is not anti-nef, that is, that the dual vector bundle $\mathscr{T}_Z(-\log \Delta)\big|_C$ is not nef. Equivalently, we claim that $\Omega^1_Z(\log \Delta)\big|_C$ admits an invertible subsheaf of positive degree. Indeed, if $\Omega^1_Z(\log \Delta)\big|_C$ was anti-nef, then none of its products $\left(\Omega^1_Z(\log \Delta)\big|_C\right)^{[n]}$ could contain a subsheaf of positive degree. However, since C is general, the restriction of the Viehweg-Zuo sheaf to C is a locally free subsheaf $\mathscr{A}\big|_C \subseteq \left(\Omega^1_Z(\log \Delta)\big|_C\right)^{[n]}$ of positive Kodaira-Iitaka dimension, and hence of positive degree. This proves the claim.

As a consequence of the claim and of (6.1.3), we obtain that $\Omega^{[1]}_Z(\log \Delta)$ is not semistable, and that if $\mathscr{B} \subseteq \Omega^{[1]}_Z(\log \Delta)$ denotes the maximal destabilizing subsheaf, then its slope $\mu(\mathscr{B})$ is positive. The assumption that $\rho(Z) = 1$ and the $\mathbb{Q}$-factoriality of Z then guarantee that $\det \mathscr{B}$ is a $\mathbb{Q}$-Cartier and $\mathbb{Q}$-ample sheaf of p-forms. Notice that by its choice the rank of $\mathscr{B}$ has to be strictly less than the rank of $\Omega^{[1]}_Z(\log \Delta)$; hence, $p < \dim Z$. However, this leads to a contradiction. Because $\mathscr{B}$ is $\mathbb{Q}$-ample, it follows that $\kappa(\det \mathscr{B}) = \dim Z$, violating the Bogomolov-Sommese vanishing theorem (Theorem 3.6). □

In the case when Z is a surface, this theorem immediately gives a criterion to guarantee that Viehweg-Zuo sheaves of positive Kodaira-Iitaka dimension cannot exist.

COROLLARY 6.2

Let (Z, Δ) be a projective reduced dlt pair of dimension two, and assume that $-(K_Z + \Delta)$ is $\mathbb{Q}$-ample. If $\mathscr{A}$ is any Viehweg-Zuo sheaf on Z, then its Kodaira-Iitaka dimension is nonpositive; that is, $\kappa(\mathscr{A}) \leq 0$.

Proof

First recall from [KM, Proposition 4.11] that Z is $\mathbb{Q}$-factorial. The MMP then yields a morphism $\lambda : (Z, \Delta) \to (Z_\lambda, \Delta_\lambda)$ to a $\mathbb{Q}$-Cartier model that does not admit any divisorial contractions. Note that the MMP for surfaces does not involve flips. Let $\mathscr{A}_\lambda$ be the associated Viehweg-Zuo sheaf on Z_λ, as given by Lemma 5.2. It suffices to show that $\kappa(\mathscr{A}_\lambda) \leq 0$.

To this end, observe that $-(K_{Z_\lambda} + \Delta_\lambda)$ is still $\mathbb{Q}$-ample. Theorem 6.1 and the cone theorem [KM, Theorem 3.7] then imply that there are at least two distinct contractions of fiber type, say $\pi_1 : Z_\lambda \to C_1$ and $\pi_2 : Z_\lambda \to C_2$. If F is a general fiber of π_1, then $F \cong \mathbb{P}^1$, the fiber F is entirely contained inside the snc locus of $(Z_\lambda, \Delta_\lambda)$, and F intersects the boundary divisor Δ_λ transversely in no more than one point. It follows from standard short exact sequences (see [KK, Lemma 2.13]) that

$$\Omega^{[1]}_{Z_\lambda}(\log \Delta_\lambda)|_F \cong \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(a), \quad \text{with } a \leq 0.$$

In particular, $\Omega^{[1]}_{Z_\lambda}(\log \Delta_\lambda)|_F$ is anti-nef, and $\mathscr{A}_\lambda|_F$ is necessarily trivial. But the same holds for the restriction of $\mathscr{A}_\lambda$ to general fibers of π_2. It follows that $\kappa(\mathscr{A}_\lambda) \leq 0$, as claimed. □

6.B. Viehweg-Zuo sheaves on good minimal models for varieties of logarithmic Kodaira dimension zero

Recall that a log canonical pair (Z, Δ) is called a *good minimal model* if there exists a number m such that $m(K_Z + \Delta)$ is Cartier and has a base-point-free linear system. If (Z, Δ) is a good minimal model of Kodaira dimension zero, the existence of a Viehweg-Zuo sheaf of positive Kodaira-Iitaka dimension implies that Z is uniruled. This is shown next.

THEOREM 6.3

Let (Z, Δ) be a reduced pair where Z is projective. Assume that the following holds:

(6.3.1) *there exists a Viehweg-Zuo sheaf $\mathscr{A} \subseteq \big(\Omega^1_Z(\log \Delta)\big)^{[n]}$ of positive Kodaira-Iitaka dimension, and*

(6.3.2) *the log canonical divisor $K_Z + \Delta$ is $\mathbb{Q}$-Cartier and numerically trivial.*

Then Z is uniruled.

Proof
We argue by contradiction and assume that Z is *not* uniruled. If $\pi : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$ is any log resolution, this is equivalent to assuming that $K_{\widetilde{Z}}$ is pseudoeffective (see [BDPP, Corollary 0.3]). (See also [L, Section 11.4.C].) Again by [BDPP, Theorem 0.2], this is in turn equivalent to the assumption that $K_{\widetilde{Z}} \cdot \widetilde{C} \geq 0$ for all moving curves $\widetilde{C} \subset \widetilde{Z}$.

As a first step, we show that the assumption implies that the (Weil) divisor Δ is zero. To this end, choose a polarization of Z and consider a general complete intersection curve $C \subset Z$. Because C is a complete intersection curve, it intersects the support of the effective divisor Δ nontrivially if the support is not empty. By the generality of the complete intersection, the curve C is contained in the snc locus of (Z, Δ) and avoids the indeterminacy locus of π^{-1}. Its preimage $\widetilde{C} := \pi^{-1}(C)$ is then a moving curve in $\widetilde{Z}$ which intersects $\widetilde{\Delta}$ positively if and only if the Weil divisor Δ is not zero. But

$$0 = \underbrace{(K_Z + \Delta)}_{\text{num. triv.}} \cdot C = (K_{\widetilde{Z}} + \widetilde{\Delta}) \cdot \widetilde{C} = \underbrace{K_{\widetilde{Z}} \cdot \widetilde{C}}_{\geq 0,\text{ as } \widetilde{C} \text{ is moving}} + \underbrace{\widetilde{\Delta} \cdot \widetilde{C}}_{\geq 0,\text{ as } \widetilde{C} \not\subseteq \widetilde{\Delta}},$$

so $\widetilde{\Delta} \cdot \widetilde{C} = 0$, and then $\Delta = \emptyset$ as claimed. Combined with Assumption (6.3.2), this implies that the canonical divisor K_Z is itself numerically trivial. The restrictions $\Omega^1_Z\big|_C$ and $\mathcal{T}_Z\big|_C$ are locally free sheaves of degree zero, and so is the product $\big(\Omega^1_Z\big|_C\big)^{\otimes n}$. On the other hand, the restriction $\mathcal{A}\big|_C \subseteq \big(\Omega^1_Z\big|_C\big)^{\otimes n}$ has positive degree. In particular, $\big(\Omega^1_Z\big|_C\big)^{[n]}$ is not semistable. Since products of semistable vector bundles are again semistable (see [HL, Corollary 3.2.10]), this implies that $\Omega^1_Z\big|_C$ and $\mathcal{T}_Z\big|_C$ are likewise not semistable. In particular, the maximal destabilizing subsheaf of $\mathcal{T}_Z\big|_C$ is semistable and of positive degree, hence ample. In this setup, a variant (see [KST, Corollary 5]) of Miyaoka's uniruledness criterion (see [Mi, Corollary 8.6]) applies to give the uniruledness of Z, contrary to our assumption. For more details on this criterion, see the survey [KS]. □

As a corollary, we obtain a criterion to guarantee that the boundary is not empty. This allows us to apply the ideas in Section 3.B above.

COROLLARY 6.4
In the setup of Theorem 6.3, if (Z, Δ) is dlc, then the boundary divisor Δ is not empty.

Proof
We argue by contradiction and assume $\Delta = \emptyset$. By the definition of dlc, the pair $(Z, \emptyset)$ is then dlt. Let $\eta : (Z', \emptyset) \to (Z, \emptyset)$ be the index-one cover discussed in Proposition 4.1. Since η is finite and étale in codimension one, there obviously exists

an injection

$$\eta^{[*]}(\mathscr{A}) \subseteq (\Omega^1_{Z'})^{[n]}.$$

An application of Theorem 6.3, using the sheaf $\eta^{[*]}(\mathscr{A})$ as a Viehweg-Zuo sheaf on $(Z', \emptyset)$ then shows that Z' is uniruled. If $\widetilde{Z} \to Z'$ is a resolution, then Z' is likewise uniruled. But Corollary 4.2 would then assert that $\kappa(K_{\widetilde{Z}}) = 0$, in contradiction to uniruledness. □

7. Unwinding families

We consider projective families $g : Y \to T$, where the base T itself admits a fibration $\varrho : T \to B$ such that g is isotrivial on all ϱ-fibers. It is of course generally false that g is then the pullback of a family defined over B. However, we show that in some situations the family g does become a pullback after a suitable base change. Most results here are known to experts. We include full statements and proofs for the reader's convenience. We do not claim originality for any of the statements in this section. The ideas here already appear in [Mu, Section 3], [CHM, Section 5.2], and in the authoritative reference [DG, Exposé IV].

We use the following notation for fibered products that appear in our setup.

Notation 7.1

Let T be a scheme, let Y and Z be schemes over T, and let $h : Y \to Z$ be a T-morphism. If $t \in T$ is any point, let Y_t and Z_t denote the fibers of Y and Z over t. Furthermore, let h_t denote the restriction of h to Y_t. More generally, for any T-scheme $\widetilde{T}$, let

$$h_{\widetilde{T}} : \underbrace{Y \times_T \widetilde{T}}_{=: Y_{\widetilde{T}}} \to \underbrace{Z \times_T \widetilde{T}}_{=: Z_{\widetilde{T}}}$$

denote the pullback of h to $\widetilde{T}$. The situation is summarized in the following commutative diagram.

$$\begin{tikzcd}
Y_{\widetilde{T}} \arrow[rr, "h_{\widetilde{T}}"'] \arrow[rrr, bend left] \arrow[dr] & & Z_{\widetilde{T}} \arrow[rrr, bend left] \arrow[dl] & Y \arrow[rr, "h"'] \arrow[dr] & & Z \arrow[dl] \\
& \widetilde{T} \arrow[rrr] & & & T. &
\end{tikzcd}$$

The setup of the current section is then formulated as follows.

ASSUMPTION 7.2
For the remainder of this section, consider a sequence of morphisms between algebraic varieties

$$Y \xrightarrow[\text{smooth, projective}]{g} T \xrightarrow[\substack{\text{smooth, rel. dim.}=1\\ \text{quasi-projective}}]{\varrho} B,$$

where T and B are irreducible, g is a smooth projective family, and ϱ is smooth quasiprojective of relative dimension one. Assume further that for all $b \in B$, there exists a smooth variety F_b such that for all $t \in T_b$, there exists an isomorphism $Y_t \simeq F_b$.

7.A. Relative isomorphisms of families over the same base
To start, recall the well-known fact that an isotrivial family of varieties of general type over a curve becomes trivial after passing to an étale cover of the base. As we are not aware of an adequate reference, we include a proof here.

LEMMA 7.3
Let $b \in B$ and assume that $\mathrm{Aut}(F_b)$ is finite. Then the natural morphism $\iota : I = \mathrm{Isom}_{T_b}(Y_b, T_b \times F_b) \to T_b$ is finite and étale. Furthermore, pullback to I yields an isomorphism of I-schemes $Y_I \simeq I \times F_b$.

Proof
Consider the T_b-scheme

$$H := \mathrm{Hilb}_{T_b}\big(Y_b \times_{T_b} (T_b \times F_b)\big) \simeq \mathrm{Hilb}_{T_b}\big(Y_b \times F_b\big).$$

By Assumption 7.2, $H_t \simeq \mathrm{Hilb}(F_b \times F_b)$ for all $t \in T_b$. Similarly, $I_t \simeq \mathrm{Aut}(F_b)$; hence, I is one-dimensional, $\mathrm{length}(I_t)$ is constant on T_b, and $I \to T_b$ is dominant. Since I is open in H, the closure of I in H, denoted by H^I, consists of a union of components of H. Therefore, H^I is also one-dimensional, and since $H^I \to T$ is dominant, it is quasifinite.

Recall that $H \to T_b$ is projective, so $H^I \to T_b$ is also projective, hence finite. Since $H \to T_b$ is flat, $\mathrm{length}(H^I_t)$ is constant. Furthermore, $I \subseteq H^I$ is open, so $H^I_t = I_t$ and, hence, $\mathrm{length}(H^I_t) = \mathrm{length}(I_t)$ for a general $t \in T_b$. However, we observed above that $\mathrm{length}(I_t)$ is also constant, so we must have that $\mathrm{length}(H^I_t) = \mathrm{length}(I_t)$ for all $t \in T_b$, and since $I \subseteq H^I$, this means that $I = H^I$ and $\iota : I \to T_b$ is finite and unramified, hence étale.

To prove the global triviality of Y_I, consider $\mathrm{Isom}_I(Y_I, I \times F_b)$. Recall that taking Hilb and Isom commutes with base change, and so we obtain an isomorphism

$$\mathrm{Isom}_I(Y_I, I \times F_b) \simeq I \times_{T_b} \mathrm{Isom}_{T_b}(Y_b, T_b \times F_b) \simeq I \times_{T_b} I.$$

This scheme admits a natural section over I, namely its diagonal, which induces an I-isomorphism between Y_I and $I \times F_b$. $\square$

Lemma 7.3 above can be used to compare two families whose associated moduli maps agree. In our setup, any two such families become globally isomorphic after base change.

LEMMA 7.4

In addition to Assumption 7.2, suppose that there exists another projective morphism $Z \to T$ with the following property: for any $b \in B$ and any $t \in T_b$, we have $Y_t \simeq Z_t \simeq F_b$. Then

(7.4.1) *there exists a surjective morphism $\tau : \widetilde{T} \to T$ such that the pullback families of Y and Z to $\widetilde{T}$ are isomorphic as $\widetilde{T}$-schemes; that is, we have a commutative diagram as follows:*

$$\begin{tikzcd}
Y_{\widetilde{T}} \arrow[rr, leftrightarrow, "\widetilde{T}-\mathit{isom.}"'] \arrow[dr] \arrow[rrr, bend left] & & Z_{\widetilde{T}} \arrow[dl] \arrow[rrr, bend left] & Y \arrow[dr] & & Z \arrow[dl] \\
& \widetilde{T} \arrow[rrr, "\tau"] & & & T \arrow[d, "\varrho"] & \\
& & & & B. &
\end{tikzcd}$$

Furthermore, if for all $b \in B$, the group $\mathrm{Aut}(F_b)$ is finite, then $\widetilde{T}$ can be chosen such that the following holds. Let $\widetilde{T}' \subseteq \widetilde{T}$ be any irreducible component. Then

(7.4.2) *τ is quasifinite,*

(7.4.3) *the image set $\tau(\widetilde{T}')$ is a union of ϱ-fibers, and*

(7.4.4) *if $\widetilde{T}'$ dominates B, then there exists an open subset $B^\circ \subseteq (\varrho \circ \tau)(\widetilde{T}')$ such that $\tau\big|_{\widetilde{T}'}$ is finite and étale over B°. More precisely, if we set $T^\circ := \varrho^{-1}(B^\circ)$ and $\widetilde{T}^\circ := \tau^{-1}(T^\circ) \cap \widetilde{T}'$, then the restriction $\tau\big|_{\widetilde{T}^\circ} : \widetilde{T}^\circ \to T^\circ$ is finite and étale.*

Remark 7.4.5

In Lemma 7.4, we do not claim that $\widetilde{T}$ is irreducible or connected.

Proof of Lemma 7.4

Set $\widetilde{T} := \mathrm{Isom}_T(Y, Z)$, and let $\tau : \widetilde{T} \to T$ be the natural morphism. Again, taking Isom commutes with base change, and we have an isomorphism $\widetilde{T} \times_T \widetilde{T} \simeq \mathrm{Isom}_{\widetilde{T}}(Y_{\widetilde{T}}, Z_{\widetilde{T}})$. Similarly, for all $b \in B$ and for all $t \in T_b$, there is a natural one-to-one correspondence between $\widetilde{T}_t$ and $\mathrm{Aut}(F_b)$. In particular, we obtain that τ is surjective. As before, observe that $\widetilde{T} \times_T \widetilde{T}$ admits a natural section, the diagonal. This shows (7.4.1).

If for all $b \in B$, $\mathrm{Aut}(F_b)$ is finite, then the restriction of τ to any ϱ-fiber $\tau_b : \widetilde{T}_b \to T_b$ is finite étale by Lemma 7.3. This shows (7.4.2) and (7.4.3). Furthermore, it implies that if $\widetilde{T}' \subseteq \widetilde{T}$ is a component that dominates B, neither the ramification locus of $\tau|_{\widetilde{T}'}$ nor the locus where $\tau|_{\widetilde{T}'}$ is not finite dominates B.

Let $\widehat{B} \subseteq T$ be a rational multisection of $\varrho : T \to B$, that is, a closed subvariety that dominates B and is of equal dimension. In particular, the morphism $\varrho|_{\widehat{B}} : \widehat{B} \to B$ is quasifinite. The scheme $\mathrm{Isom}_{\widehat{B}}(Y, Y)$ is quasifinite and quasiprojective over $\widehat{B}$, hence, over B as well. Then there exists an open subset $B^\circ \subseteq B$ where $\mathrm{length}\big(\mathrm{Isom}_{\widehat{B}}(Y, Y)\big)_b$ is constant for $b \in B$. It is easy to see that (7.4.4) holds for B°. □

7.B. Families where ϱ has a section

In addition to Assumption 7.2, assume that the morphism ϱ admits a section $\sigma : B \to T$. Using $\sigma : B \to T$, define $Y_B := Y \times_T B$ and let $Z := Y_B \times_B T$ be the pullback of Y_B to T. With these definitions, Lemma 7.4 applies to the families $Y \to T$ and $Z \to T$. As a corollary, we show below that in this situation $\widetilde{T}$ contains a component $\widetilde{T}'$ such that the pullback family $Y_{\widetilde{T}'}$ comes from B. Better still, the restriction $\tau|_{\widetilde{T}'}$ is *relatively étale* in the sense that $\tau|_{\widetilde{T}'}$ is étale and that $\varrho \circ \tau|_{\widetilde{T}'}$ has connected fibers.

COROLLARY 7.5

Under the conditions of Lemma 7.4, assume that ϱ admits a section $\sigma : B \to T$ and that $Z = Y_B \times_B T$. Then there exists an irreducible component $\widetilde{T}' \subseteq \widetilde{T}$ such that

(7.5.1) *$\widetilde{T}'$ surjects onto B, and*

(7.5.2) *the restricted morphism $\widetilde{\varrho} := \varrho \circ \tau|_{\widetilde{T}'} : \widetilde{T}' \to B$ has connected fibers.*

Proof

It is clear from the construction that $Y_B \simeq Z_B$. This isomorphism corresponds to a morphism $\widetilde{\sigma} : B \to \mathrm{Isom}_T(Y, Z) = \widetilde{T}$. Let $\widetilde{T}' \subseteq \widetilde{T}$ be an irreducible component that contains the image of $\widetilde{\sigma}$. Observe that $\widetilde{\sigma}$ is a section of $\widetilde{\varrho} : \widetilde{T}' \to B$ and that the existence of a section guarantees that $\widetilde{\varrho}$ is surjective and its fibers are connected. □

One particular setup where a section is known to exist is when T is a birationally ruled surface over B. The following becomes important later.

COROLLARY 7.6

In addition to Assumption 7.2, suppose that B is a smooth curve and that the general ϱ-fiber is isomorphic to $\mathbb{P}^1$, $\mathbb{A}^1$, or $(\mathbb{A}^1)^* = \mathbb{A}^1 \setminus \{0\}$. Then there exist nonempty Zariski open sets $B^\circ \subseteq B$, $T^\circ := \varrho^{-1}(B^\circ)$ and a commutative diagram

$$\begin{tikzcd} \widetilde{T}^\circ \arrow[r, "\tau", "\text{étale}"'] \arrow[dr, "\text{conn. fibers}"'] & T^\circ \arrow[d, "\varrho"] \\ & B^\circ \end{tikzcd}$$

such that

(7.6.1) *the fibers of $\varrho \circ \tau$ are again isomorphic to $\mathbb{P}^1$, $\mathbb{A}^1$, or $(\mathbb{A}^1)^*$, respectively, and*

(7.6.2) *the pullback family $Y_{\widetilde{T}^\circ}$ comes from B°, that is, there exists a projective family $Z \to B^\circ$ and a $\widetilde{T}^\circ$-isomorphism*

$$Y_{\widetilde{T}^\circ} \simeq Z_{\widetilde{T}^\circ}.$$

Remark 7.6.3

If the general ϱ-fiber is isomorphic to $\mathbb{P}^1$ or $\mathbb{A}^1$, the morphism τ is necessarily an isomorphism. Shrinking B° further, if necessary, $\varrho : T^\circ \to B^\circ$ then even becomes a trivial $\mathbb{P}^1$- or $\mathbb{A}^1$-bundle, respectively.

Proof

Shrinking B, if necessary, we may assume that all ϱ-fibers are isomorphic to $\mathbb{P}^1$, $\mathbb{A}^1$, or $(\mathbb{A}^1)^*$, and hence that T is smooth. Then it is always possible to find a relative smooth compactification of T, that is, a smooth B-variety $\overline{T} \to B$ and a smooth divisor $D \subset T$ such that $\overline{T} \setminus D$ and T are isomorphic B-schemes.

By Tsen's theorem (see [S2, p. 73]), there exists a section $\sigma : B \to \overline{T}$. In fact, there exists a positive dimensional family of sections, so that we may assume without loss of generality that $\sigma(B)$ is not contained in D.

Let $B^\circ \subseteq B$ be the open subset such that for all $b \in B^\circ$, $\overline{T}_b \simeq \mathbb{P}^1$, T_b is isomorphic to $\mathbb{P}^1$, $\mathbb{A}^1$, or $(\mathbb{A}^1)^*$, respectively, and $\sigma(b) \notin D$. Using that any connected finite étale cover of T_b is again isomorphic to T_b, and shrinking B° further, Corollary 7.5 yields the claim. $\square$

Part II. The proofs of Theorems 1.1, 1.2, and 1.4

8. The case $\kappa(Y^\circ) = -\infty$, the Kodaira dimension is minus infinity

8.A. Setup

Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of varieties with semiample canonical bundle, over a quasi-projective manifold Y° of dimension $\dim Y^\circ \leq 3$ and logarithmic Kodaira dimension $\kappa(Y^\circ) = -\infty$.

Consider a smooth compactification Y of Y°, where $D := Y \setminus Y^\circ$ is a divisor with simple normal crossings. Let $\lambda : Y \dashrightarrow Y_\lambda$ be a sequence of extremal divisorial contractions and flips given by the (Y, D)-MMP, and let $D_\lambda \subset Y_\lambda$ be the cycle-theoretic image of D. We may assume that (Y_λ, D_λ) satisfies the following properties.

8.1 Properties of (Y_λ, D_λ)

(8.1.1) The variety Y_λ is $\mathbb{Q}$-factorial, and (Y_λ, D_λ) is a reduced dlt pair.

(8.1.2) The pair (Y_λ, D_λ) does not admit a divisorial or small extremal contraction.

(8.1.3) As $\kappa(Y^\circ) = -\infty$, either
- (a) $\rho(Y_\lambda) = 1$ and (Y_λ, D_λ) is $\mathbb{Q}$-Fano, or
- (b) $\rho(Y_\lambda) > 1$ and (Y_λ, D_λ) admits a nontrivial extremal contraction of fiber type.

8.B. Proof of Theorem 1.2

To prove Theorem 1.2, assume that f° is a family of canonically polarized varieties and that f° has positive variation $\mathrm{Var}(f^\circ) > 0$. By [VZ1, Theorem 1.4] and Lemma 5.2, this implies that there exists a Viehweg-Zuo sheaf $\mathcal{A}_\lambda$ of positive Kodaira-Iitaka dimension $\kappa(\mathcal{A}_\lambda) \geq \mathrm{Var}(f^\circ) > 0$ on (Y_λ, D_λ) . Since (Y_λ, D_λ) is $\mathbb{Q}$-factorial and dlt, in particular log canonical, Theorem 6.1 implies that $\rho(Y_\lambda) > 1$. Therefore by (8.1.3), there exists an extremal contraction of fiber type $\pi : Y_\lambda \to C$. Let $F \subset Y_\lambda$ be a general π-fiber, and let $D_{\lambda,F} := D_\lambda\big|_F$ be the restriction of the boundary divisor.

We now push the family f° down to F, to the maximum extent possible. Since the inverse map λ^{-1} does not contract any divisor, we may use λ^{-1} to pull the family $f^\circ : X^\circ \to Y^\circ$ back to obtain a smooth family of canonically polarized varieties

$$f_\lambda : X_\lambda \to Y_\lambda \setminus (D_\lambda \cup T), \quad \text{where } \operatorname{codim}_{Y_\lambda} T \geq 2.$$

Let $f_{\lambda,F} := f_\lambda|_F$ be the restriction of this family to F. To prove Theorem 1.2 in our context, it suffices to show that the family $f_{\lambda,F}$ is isotrivial. We carry this out next.

8.B.1. Proof of Theorem 1.2 when F is a curve

If F is a curve, it is entirely contained inside the snc locus of (Y_λ, D_λ) and does not intersect T. Furthermore, it follows from the adjunction formula that $F \cong \mathbb{P}^1$ and that $D_{\lambda,F}$ contains no more than one point. In this situation, the isotriviality of $f_{\lambda,F}$ is well known (see [Ko, Theorem 0.2] and [VZ2, Theorem 0.1]). This shows that the variation $\mathrm{Var}(f^\circ)$ cannot be maximal and finishes the proof of Theorem 1.2.

8.B.2. Proof of Theorem 1.2 when F is a surface

Again, we need to show that $f_{\lambda,F}$ is isotrivial. We argue by contradiction and assume that this is *not* the case. By general choice of F, the pair $(F, D_{\lambda,F})$ is again dlt and

$$\operatorname{codim}_F T_F = \operatorname{codim}_{Y_\lambda} T \geq 2, \quad \text{where } T_F := T \cap F.$$

We claim that there exists a Viehweg-Zuo sheaf $\mathscr{B}_\lambda$ on $(F, D_{\lambda,F})$ which is of positive Kodaira-Iitaka dimension. In fact, an embedded resolution of $D_{\lambda,F} \cup T_F \subseteq F$ provides an snc pair $(\widetilde{F}, \widetilde{D})$ and a proper morphism $\eta : \widetilde{F} \to F$ such that $\eta(\widetilde{D}) = D_{\lambda,F} \cup T_F$. The family $f_{\lambda,F}$ pulls back to a family on $\widetilde{F} \setminus \widetilde{D}$, and [VZ1, Theorem 1.4] asserts the existence of a Viehweg-Zuo sheaf $\mathscr{B}$ on $(\widetilde{F}, \widetilde{D})$ with $\kappa(\mathscr{B}) > 0$. The existence of a Viehweg-Zuo sheaf $\mathscr{B}_\lambda$ on $(F, D_{\lambda,F})$ with $\kappa(\mathscr{B}_\lambda) \geq \kappa(\mathscr{B}) > 0$ then follows from Lemma 5.2.

On the other hand, $-(K_F + D_{\lambda,F})$ is $\mathbb{Q}$-ample because π is an extremal contraction of fiber type. Corollary 6.2 thus asserts that $\kappa(\mathscr{B}_\lambda) \leq 0$, which is a contradiction. This finishes the proof of Theorem 1.2 in case $\kappa(Y^\circ) = -\infty$. □

8.C. Proof of Theorem 1.4

We maintain the notation and assumptions made in Section 8.B above and assume in addition that Y is a surface. The minimal model map λ is then a morphism. As we have seen in Section 8.B.1, the general fiber F' of $\pi \circ \lambda$ is again a rational curve which intersects the boundary in at most one point and that then the restriction of the family f° to the fiber $F' \cap Y^\circ$ is necessarily isotrivial. The detailed descriptions of Y° and of the moduli map in case $\kappa(Y^\circ) = -\infty$ which are asserted in Theorem 1.4 then follow from Corollary 7.6 and Remark 7.6.3. This finishes the proof of Theorem 1.4 in case $\kappa(Y^\circ) = -\infty$. □

8.D. Proof of Theorem 1.1

To prove Theorem 1.1, we argue by contradiction and assume that $\kappa(Y^\circ) = -\infty$ and that $\mathrm{Var}(f^\circ) = \dim Y^\circ$. Lemma 5.2 and [VZ1, Theorem 1.4] then give the existence of a big Viehweg-Zuo sheaf $\mathscr{A}_\lambda$ on (Y_λ, D_λ). The arguments of Section 8.B apply verbatim and show the existence of a proper fibration of $\pi : Y_\lambda \to C$ such that the induced family is isotrivial when restricted to the general π-fiber. That, however, contradicts the assumption that the variation is maximal. Theorem 1.1 is thus shown in case $\kappa(Y^\circ) = -\infty$. □

9. The case $\kappa(Y^\circ) = 0$, the Kodaira dimension is zero

9.A. Setup

Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of varieties with semiample canonical bundle over a quasi-projective variety Y° of dimension $\dim Y^\circ \leq 3$ and logarithmic Kodaira dimension $\kappa(Y^\circ) = 0$. To prove Theorems 1.1 and 1.2 in this case, it suffices to show that f° is not of maximal variation, and even isotrivial if its fibers are canonically polarized. Since those families give rise to Viehweg-Zuo sheaves of positive Kodaira-Iitaka dimension by [VZ1, Theorem 1.4], Theorems 1.1 and 1.2 immediately follow from the following proposition.

PROPOSITION 9.1

Let (Z, Δ) be a reduced dlt pair. Assume that Z is a $\mathbb{Q}$-factorial variety of dimension $\dim Z \leq 3$ *and that* $\kappa(K_Z + \Delta) = 0$*. If $\mathscr{A}$ is any Viehweg-Zuo sheaf on* (Z, Δ)*, then* $\kappa(\mathscr{A}) \leq 0$.

Observe that once Theorem 1.2 holds, the assertion of Theorem 1.4 is vacuous in our case. Accordingly, we do not consider Theorem 1.4 here.

We show Proposition 9.1 in the remainder of the present section. The proof proceeds by induction on $\dim Z$. If $\dim Z = 1$, the statement of Proposition 9.1 is obvious. We therefore assume throughout the proof that $\dim Z > 1$ and that the following holds.

Induction Hypothesis 9.2

Proposition 9.1 is already shown for all pairs (Z', Δ') of dimension $\dim Z' < \dim Z$.

We argue by contradiction and assume the following.

Assumption 9.3

There exists a Viehweg-Zuo sheaf $\mathscr{A}$ of positive Kodaira-Iitaka dimension $\kappa(\mathscr{A}) > 0$.

We run the MMP and obtain a birational map $\lambda : Z \dashrightarrow Z_\lambda$, where Z_λ is $\mathbb{Q}$-factorial. If Δ_λ is the cycle-theoretic image, the pair $(Z_\lambda, \Delta_\lambda)$ is dlt and $K_{Z_\lambda} + \Delta_\lambda$ is semiample. Since $\kappa(K_{Z_\lambda} + \Delta_\lambda) = 0$, the divisor $K_{Z_\lambda} + \Delta_\lambda$ is $\mathbb{Q}$-torsion; that is,

$$\exists m \in \mathbb{N} \text{ such that } \mathscr{O}_{Z_\lambda}\big(m(K_{Z_\lambda} + \Delta_\lambda)\big) \cong \mathscr{O}_{Z_\lambda}. \tag{9.3.1}$$

Lemma 5.2 guarantees the existence of a Viehweg-Zuo sheaf $\mathscr{A}_\lambda$ on $(Z_\lambda, \Delta_\lambda)$ with $\kappa(\mathscr{A}_\lambda) > 0$. Raising $\mathscr{A}$ and $\mathscr{A}_\lambda$ to a suitable reflexive power if necessary, we assume without loss of generality that $\mathscr{A}_\lambda$ is invertible and that $h^0(Z_\lambda, \mathscr{A}_\lambda) > 0$.

9.B. Outline of the proof

Since the proof of Proposition 9.1 is slightly more complicated than most other proofs here, we outline the main strategy for the convenience of the reader.

The main idea is to apply induction, using a component of the boundary divisor Δ_λ. For that, we show in Section 9.E that $\mathscr{A}_\lambda$ is not trivial on the boundary and that there exists a component $\Delta'_\lambda \subseteq \Delta_\lambda$ such that $\kappa(\mathscr{A}_\lambda\big|_{\Delta'_\lambda}) > 0$. Passing to the index-one cover, in Section 9.F we then construct a Viehweg-Zuo sheaf of positive Kodaira-Iitaka dimension on the associated boundary component and verify that this component with its natural boundary satisfies all the requirements of Proposition 9.1. This clearly contradicts Induction Hypothesis 9.2 and finishes the proof.

To find Δ'_λ, we need to analyze the geometry of Z_λ in more detail. For that, in Section 9.C we show that the minimal model Z_λ admits further contractions if one is willing to modify the coefficients of the boundary. A second application of the MMP then brings us to a reduced dlc pair (Z_μ, Δ_μ) that shares many of the good properties of $(Z_\lambda, \Delta_\lambda)$. In addition, it turns out in Section 9.D that Z_λ has the structure of a Mori fiber space. An analysis of the Viehweg-Zuo sheaf along the fibers is essential.

9.C. Minimal models of $(Z_\lambda, (1-\varepsilon)\Delta_\lambda)$

As a first step in the program outlined in Section 9.B, we need the following claim.

CLAIM 9.4

The boundary Δ_λ is not empty, $\Delta_\lambda \neq \emptyset$.

Proof

Using (9.3.1) and the existence of the Viehweg-Zuo sheaf $\mathscr{A}_\lambda$ of positive Kodaira-Iitaka dimension, this follows immediately from Corollary 6.4. □

In particular, (9.3.1) implies that $K_{Z_\lambda} \equiv -\Delta_\lambda$, and it follows that for any rational number $0 < \varepsilon < 1$,

$$\kappa\big(K_{Z_\lambda} + (1-\varepsilon)\Delta_\lambda\big) = \kappa(\varepsilon K_{Z_\lambda}) = \kappa(Z_\lambda) = -\infty. \tag{9.4.1}$$

Now choose one ε and run the log MMP for the dlt pair $\big(Z_\lambda, (1-\varepsilon)\Delta_\lambda\big)$. This way one obtains a birational map $\mu : Z_\lambda \dashrightarrow Z_\mu$. Let Δ_μ be the cycle-theoretic image of Δ_λ. The variety Z_μ is $\mathbb{Q}$-factorial and the pair $\big(Z_\mu, (1-\varepsilon)\Delta_\mu\big)$ is then dlt.

CLAIM 9.5

The reduced pair (Z_μ, Δ_μ) is dlc.

Proof
By (9.3.1), some positive multiples of K_{Z_λ} and $-\Delta_\lambda$ are numerically equivalent. For any two rational numbers $0 < \varepsilon', \varepsilon'' < 1$, the divisors $K_{Z_\lambda} + (1-\varepsilon')\Delta_\lambda$ and $K_{Z_\lambda} + (1-\varepsilon'')\Delta_\lambda$ are thus again numerically equivalent up to a positive rational multiple.

The birational map μ is therefore an MMP for the pair $\big(Z_\lambda, (1-\varepsilon)\Delta_\lambda\big)$, independently of the number ε chosen in its construction. It follows that $\big(Z_\mu, (1-\varepsilon')\Delta_\mu\big)$ has dlt singularities for all $0 < \varepsilon' < 1$, so (Z_μ, Δ_μ) is indeed dlc. □

9.D. The fiber space structure of Z_μ
Since the Kodaira-dimension of $\big(Z_\lambda, (1-\varepsilon)\Delta_\lambda\big)$ is negative by (9.4.1), either $\rho(Z_\mu) = 1$ or $\rho(Z_\mu) > 1$, and the pair $\big(Z_\mu, (1-\varepsilon)\Delta_\mu\big)$ admits an extremal contraction of fiber type. We apply Theorem 6.1 to show that the Picard number cannot be one.

PROPOSITION 9.6
The Picard number of Z_μ is not one. The pair $\big(Z_\mu, (1-\varepsilon)\Delta_\mu\big)$ therefore admits a nontrivial extremal contraction of fiber type, $\pi : Z_\mu \to W$.

Proof
As the birational map μ is a sequence of extremal divisorial contractions and flips, the inverse of μ does not contract any divisors. This has two consequences. First, the divisor $K_{Z_\mu} + \Delta_\mu$ is torsion, and $-(K_{Z_\mu} + \Delta_\mu)$ is nef. On the other hand, Lemma 5.2 applies and shows the existence of a Viehweg-Zuo sheaf $\mathcal{A}_\mu$ of positive Kodaira-Iitaka dimension. Since we have seen in Claim 9.5 that (Z_μ, Δ_μ) is dlc, in particular log canonical, and since we know that Z_μ is $\mathbb{Q}$-factorial, Theorem 6.1 then gives that $\rho(Z_\mu) > 1$, as desired. □

Now let $F \subset Z_\mu$ be a general fiber of π, and set $\Delta_F := \Delta_\mu \cap F$. Since normality is preserved when passing to general elements of base-point-free systems (see [BS, Theorem 1.7.1]), and since discrepancies only increase, the reduced pair (F, Δ_F) is again dlc.

Remark 9.7
The adjunction formula gives that $K_F + \Delta_F$ is torsion. On the other hand, π is an extremal contraction so $-\big(K_F + (1-\varepsilon)\Delta_F\big)$ is π-ample. It follows that the boundary divisor of F cannot be empty, $\Delta_F \neq \emptyset$. It is not clear to us whether in general F is necessarily $\mathbb{Q}$-factorial.

9.E. Nontriviality of $\mathscr{A}_\lambda\big|_{\Delta_\lambda}$

As in Section 9.A, Lemma 5.2 guarantees the existence of a Viehweg-Zuo sheaf $\mathscr{A}_\mu$ on (Z_μ, Δ_μ) with $\kappa(\mathscr{A}_\mu) \geq \kappa(\mathscr{A}_\lambda) > 0$. Again, passing to a suitable reflexive power, we can assume that $\mathscr{A}_\mu$ is invertible and that $h^0(Z_\mu,\ \mathscr{A}_\mu) > 0$.

PROPOSITION 9.8

The restriction $\mathscr{A}_\mu\big|_F$ has Kodaira-Iitaka dimension zero, $\kappa(\mathscr{A}_\mu\big|_F) = 0$.

Proof

Consider the open set $F^\circ := (F, \Delta_F)_{\mathrm{reg}} \cap (Z_\mu, \Delta_\mu)_{\mathrm{reg}}$. The fiber F being general, it is clear that $\operatorname{codim}_F(F \setminus F^\circ) \geq 2$. On F°, the standard conormal sequence (see [KK, Lemma 2.13]) for logarithmic differentials then gives a short exact sequence of locally free sheaves

$$0 \longrightarrow \underbrace{\pi^*\big(\Omega^1_W\big)\big|_{F^\circ}}_{\text{trivial}} \longrightarrow \Omega^1_{Z_\mu}(\log \Delta_\mu)\big|_{F^\circ} \longrightarrow \Omega^1_F(\log \Delta_F)\big|_{F^\circ} \longrightarrow 0. \tag{9.8.1}$$

By the definition of a "Viehweg-Zuo sheaf," there exists a number $n \in \mathbb{N}$ and an embedding $\mathscr{A}_\mu\big|_{F^\circ} \to \big(\Omega^1_{Z_\mu}(\log \Delta_\mu)\big|_{F^\circ}\big)^{\otimes n}$. The first term in (9.8.1) being trivial, Lemma 5.4 gives a number $m \leq n$ and an injection

$$\mathscr{A}_\mu\big|_{F^\circ} \hookrightarrow \big(\Omega^1_F(\log \Delta_F)\big|_{F^\circ}\big)^{\otimes m}. \tag{9.8.2}$$

Recall that $\mathscr{A}_\mu$ is invertible. Then by (9.8.2), we obtain an injection between the reflexive hulls $\mathscr{A}_\mu\big|_F \hookrightarrow \big(\Omega^1_F(\log \Delta_F)\big)^{[m]}$; that is, we realize $\mathscr{A}_\mu\big|_F$ as a Viehweg-Zuo sheaf on (F, Δ_F).

The log canonical divisor $K_F + \Delta_F$ being torsion, Proposition 9.8 follows immediately if F is a curve. We thus assume for the remainder of the proof that $\dim F = 2$.

It remains to show that the Viehweg-Zuo sheaf $\mathscr{A}_\mu\big|_F$ on (F, Δ_F) has Kodaira-Iitaka dimension $\kappa(\mathscr{A}_\mu\big|_F) \leq 0$. The fact that $\kappa(\mathscr{A}_\mu) > 0$ then implies that $\kappa(\mathscr{A}_\mu\big|_F) = 0$, as claimed. To do this, consider a log resolution $\psi : (\widetilde{F}, \widetilde{\Delta}_F) \to (F, \Delta_F)$. Setting

$$E := \text{maximal reduced divisor in } \psi^{-1}(\Delta_F) \cup \operatorname{Exc}(\psi),$$

it follows immediately from the definition of dlc that $K_{\widetilde{F}} + E$ is represented by the sum of a torsion divisor and an effective, ψ-exceptional divisor. In particular, $\kappa(K_{\widetilde{F}} + E) = 0$, and Theorem 5.3 gives the existence of a Viehweg-Zuo sheaf $\mathscr{C}$ on the snc pair $(\widetilde{F}, E)$ with $\kappa(\mathscr{C}) \geq \kappa(\mathscr{A}_\mu\big|_F)$. However, this contradicts Induction Hypothesis 9.2, which asserts that $\kappa(\mathscr{C}) \leq 0$. □

COROLLARY 9.9

The restriction $\mathscr{A}_\mu\big|_F$ is trivial; that is, $\mathscr{A}_\mu\big|_F \cong \mathscr{O}_F$.

Proof
Since $\mathcal{A}_\mu$ is invertible and since $h^0(Z_\mu,\ \mathcal{A}_\mu) > 0$, there exists an effective Cartier divisor D on Z_μ with $\mathcal{A}_\mu \cong \mathcal{O}_{Z_\mu}(D)$. Decompose $D = D^h + D^v$, where D^h consists of those components that dominate W, and where D^v consists of those components that do not. We need to show that $D^h = 0$. Again, we argue by contradiction and assume that D^h is nontrivial.

Recall that $\pi : Z_\mu \to W$ is a contraction of an extremal ray and that the relative Picard number $\rho(Z_\mu/W)$ is therefore one. The divisor D^h is thus relatively ample, contradicting Proposition 9.8. □

COROLLARY 9.10
There exists a component $\Delta_{\lambda,1} \subseteq \Delta_\lambda$ such that $\kappa(\mathcal{A}_\lambda|_{\Delta_{\lambda,1}}) > 0$.

Proof
We have seen in Remark 9.7 that $\Delta_F = \Delta_\mu \cap F$ is not empty. So, there exists a component $\Delta_{\mu,1} \subseteq \Delta_\mu$ that intersects all π-fibers. Let $\Delta_{\lambda,1} \subseteq \Delta_\lambda$ be its strict transform. Since the birational map μ does not contract $\Delta_{\lambda,1}$, and since μ^{-1} does not contract any divisors, μ induces an isomorphism of open sets $U_\lambda \subseteq Z_\lambda$ and $U_\mu \subseteq Z_\mu$ such that $\Delta^\circ_{\lambda,1} := \Delta_{\lambda,1} \cap U_\lambda$ and $\Delta^\circ_{\mu,1} := \Delta_{\mu,1} \cap U_\mu$ are both nonempty.

For an arbitrary $m \in \mathbb{N}$, we obtain a commutative diagram of linear maps

$$\begin{array}{ccccc}
H^0\big(Z_\lambda,\ \mathcal{A}_\lambda^{\otimes m}\big) & \xrightarrow[\text{restr.}]{\alpha_1} & H^0\big(\Delta_{\lambda,1},\ \mathcal{A}_\lambda^{\otimes m}\big|_{\Delta_{\lambda,1}}\big) & \xrightarrow[\text{restr.}]{\alpha_2} & H^0\big(\Delta^\circ_{\lambda,1},\ \mathcal{A}_\lambda^{\otimes m}\big|_{\Delta^\circ_{\lambda,1}}\big) \\
\downarrow{\scriptstyle \mu_1} & & & & \downarrow{\scriptstyle \mu_2} \\
H^0\big(Z_\mu,\ \mathcal{A}_\mu^{\otimes m}\big) & \xrightarrow[\text{restr.}]{\beta_1} & H^0\big(\Delta_{\mu,1},\ \mathcal{A}_\mu^{\otimes m}\big|_{\Delta_{\mu,1}}\big) & \xrightarrow[\text{restr.}]{\beta_2} & H^0\big(\Delta^\circ_{\mu,1},\ \mathcal{A}_\mu^{\otimes m}\big|_{\Delta^\circ_{\mu,1}}\big),
\end{array}$$

where the μ_i, $i = 1, 2$ are the obvious pushforward morphisms coming from the construction of $\mathcal{A}_\mu$ in Lemma 5.2. Since μ_1 and β_2 are clearly injective, Corollary 9.10 follows once we show that β_1 is injective as well. Now let $\sigma \in H^0\big(Z_\mu,\ \mathcal{A}_\mu^{\otimes m}\big)$, and assume that σ is in the kernel of β_1. By choice of $\Delta_{\mu,1}$, any general fiber F intersects $\Delta_{\mu,1}$ in at least one point. The triviality of $\mathcal{A}_\mu|_F$ asserted in Corollary 9.9 then implies that σ vanishes along F. The fiber F being general, we obtain that $\sigma = 0$ on all of Z_μ. Corollary 9.10 follows. □

9.F. Existence of pluriforms on the boundary
Now consider the index-one cover $\gamma : (Z'_\lambda, \Delta'_\lambda) \to (Z_\lambda, \Delta_\lambda)$, as described in Proposition 4.1. The pair $(Z'_\lambda, \Delta'_\lambda)$ is then dlt, the log canonical divisor is trivial, $\mathcal{O}_{Z'_\lambda}\big(K_{Z'_\lambda} + \Delta'_\lambda\big) \cong \mathcal{O}_{Z'_\lambda}$, and the pullback $\mathcal{A}'_\lambda := \gamma^*(\mathcal{A}_\lambda)$ is an invertible

Viehweg-Zuo sheaf on $(Z'_\lambda, \Delta'_\lambda)$ with $\kappa(\mathscr{A}'_\lambda) > 0$. Better still, if $\Delta'_{\lambda,1} \subseteq \gamma^{-1}\big(\Delta_{\lambda,1}\big)$ is any component, Corollary 9.10 immediately implies that $\kappa(\mathscr{A}'_\lambda|_{\Delta'_{\lambda,1}}) > 0$.

Now recall from Lemma 3.1 that $(Z'_\lambda, \Delta'_\lambda)$ is snc along the boundary away from a closed subset W with $\mathrm{codim}_Z(W \cap \Delta'_\lambda) \geq 3$. The divisor Δ'_λ is therefore Cartier in codimension two and inversion of adjunction applies (see [KM, Section 5.4]). Setting

$$\Delta''_{\lambda,1} := (\Delta'_\lambda - \Delta'_{\lambda,1})\big|_{\Delta'_{\lambda,1}},$$

this yields the following:

(9.10.1) the subvariety $\Delta'_{\lambda,1}$ is normal [KM, Corollary 5.52], and

(9.10.2) the pair $(\Delta'_{\lambda,1}, \Delta''_{\lambda,1})$ is again reduced and dlt [KM, Proposition 5.59].

Observation 9.11

It follows from the adjunction formula that the log canonical divisor $K_{\Delta'_{\lambda,1}} + \Delta''_{\lambda,1}$ is trivial.

PROPOSITION 9.12

The pair $(\Delta'_{\lambda,1}, \Delta''_{\lambda,1})$ admits a Viehweg-Zuo sheaf of positive Kodaira-Iitaka dimension.

Proof

Consider the standard conormal sequence for logarithmic differentials [KK, Lemma 2.13] on the open subset $\Delta'^{\circ}_{\lambda,1} := (\Delta'_{\lambda,1}, \Delta''_{\lambda,1})_{\mathrm{reg}}$,

$$0 \longrightarrow \Omega^1_{\Delta'^{\circ}_{\lambda,1}}(\log \Delta''_{\lambda,1}) \longrightarrow \Omega^1_{Z'_\lambda}(\log \Delta'_\lambda)\big|_{\Delta'^{\circ}_{\lambda,1}} \longrightarrow \mathscr{O}_{\Delta'^{\circ}_{\lambda,1}} \longrightarrow 0. \qquad (9.12.1)$$

The last term in (9.12.1) being trivial, Lemma 5.4 gives a number $m \leq n$ and an injection

$$\mathscr{A}'_\lambda\big|_{\Delta'^{\circ}_{\lambda,1}} \hookrightarrow \big(\Omega^1_{\Delta'^{\circ}_{\lambda,1}}(\log \Delta''_{\lambda,1})\big)^{\otimes m}.$$

Using that $\mathscr{A}'_\lambda$ is invertible and that $\mathrm{codim}_{\Delta'_{\lambda,1}}(W \cap \Delta'_{\lambda,1}) \geq 2$, we pass to reflexive hulls and realize $\mathscr{A}'_\lambda$ as a Viehweg-Zuo sheaf on $\Delta'_{\lambda,1}$,

$$\mathscr{A}'_\lambda\big|_{\Delta'_{\lambda,1}} \subseteq \big(\Omega^1_{\Delta'_{\lambda,1}}(\log \Delta''_{\lambda,1})\big)^{[m]}. \qquad \square$$

9.G. Completion of the proof

Recall that we have seen that the pair $(\Delta'_{\lambda,1}, \Delta''_{\lambda,1})$ is dlt, has trivial log canonical class, and admits a Viehweg-Zuo sheaf of positive Kodaira-Iitaka dimension. Since $\dim \Delta'_{\lambda,1} \leq 2$, being dlt implies that the variety $\Delta'_{\lambda,1}$ is $\mathbb{Q}$-factorial [KM, Proposition 4.11]. This clearly contradicts Induction Hypothesis 9.2. Assumption 9.3 is therefore absurd. This finishes the proof of Proposition 9.1. Consequently, Theorems 1.1 and 1.2 are shown in case $\kappa(Y^\circ) = 0$. $\square$

10. The case $\kappa(Y^\circ) > 0$, in case of positive Kodaira dimension

10.A. Setup

Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of varieties with semiample canonical bundle over a quasi-projective variety Y° of dimension $\dim Y^\circ \leq 3$ and logarithmic Kodaira dimension $\kappa(Y^\circ) > 0$.

Again, let Y be a compactification of Y°, where $D := Y \setminus Y^\circ$ is a divisor with simple normal crossings, and let $\lambda : (Y, D) \dashrightarrow (Y_\lambda, D_\lambda)$ be the map to a minimal model. The divisor $K_{Y_\lambda} + D_\lambda$ is then semiample by the log abundance theorem [KMM] and defines a map $\pi : Y_\lambda \to C$ with $\dim C = \kappa(Y^\circ)$.

10.B. Proof of Theorem 1.2

To prove Theorem 1.2, assume f° is a family of canonically polarized manifolds. We may also assume without loss of generality that the family f° is not isotrivial and that $\kappa(Y^\circ) < \dim Y$. Blowing up Y and pulling back the family, we obtain a diagram as follows:

$$\begin{tikzcd}
\widetilde{X}^\circ \arrow[r, "\widetilde{f}^\circ", "\text{family of canon. pol. var.}"'] \arrow[d, "\text{pullback}"'] & (\widetilde{Y}, \widetilde{D}) \arrow[d, "\text{blow up}"] \arrow[rrd, bend left, "\widetilde{\pi}"] & & \\
X^\circ \arrow[r, "f^\circ", "\text{family of canon. pol. var.}"'] & (Y, D) \arrow[r, dashed, "\lambda", "\text{MMP}"'] & (Y_\lambda, D_\lambda) \arrow[r, "\pi"'] & C.
\end{tikzcd}$$

If $\widetilde{F} \subset \widetilde{Y}$ is the general $\widetilde{\pi}$-fiber, recall the standard fact that $\kappa(K_{\widetilde{F}} + \widetilde{D}|_{\widetilde{F}}) = 0$ (see [I, Section 11.6]). We saw in Section 9 that then the family $\widetilde{f}^\circ$ must be isotrivial over $\widetilde{F}$. This shows that the fibration π factors the moduli map birationally and proves Theorem 1.2 in case $\kappa(Y^\circ) > 0$. $\square$

10.C. Proof of Theorem 1.4

It remains to prove Theorem 1.4 and give a detailed description of the moduli map if Y is a surface.

To this end, we maintain the notation and assumptions made in Section 10.A above and assume in addition that Y is a surface, that $\mathrm{Var}(f^\circ) > 0$, and that $\kappa(Y^\circ) = 1$. As there are no flipping contractions in dimension two, λ is a birational morphism, and $K_{Y_\lambda} + D_\lambda$ is trivial on the general π-fiber $F_\lambda \subset Y_\lambda$. In particular, one of the following holds:

(10.0.1) F_λ is an elliptic curve and no component of D_λ dominates C, or
(10.0.2) F_λ is isomorphic to $\mathbb{P}^1$ and intersects D_λ in exactly two points.

If the general fibers of π are isomorphic to $(\mathbb{A}^1)^*$, Corollary 7.6 gives the statement of Theorem 1.4.

Otherwise, let $V \subseteq C$ be an open subset such that π is a smooth elliptic fibration over V. Let $\widetilde{V} \subset Y_\lambda$ be a general hyperplane section. Restricting V further if necessary, we may assume that $\widetilde{V}$ is étale over V. Taking a base change to $\widetilde{V}$, we obtain a section $\sigma : \widetilde{V} \to \widetilde{U} := U \times_V \widetilde{V}$. Finally, set $\widetilde{X} := X \times_U \widetilde{U}$ and $Z := \widetilde{V} \times_\sigma \widetilde{X}$. Shrinking V further if necessary, an application of Lemma 7.4 completes the proof. □

10.D. Proof of Theorem 1.1

To prove Theorem 1.1, we argue by contradiction and assume that $0 < \kappa(Y^\circ) < \dim Y^\circ$ and that $\operatorname{Var}(f^\circ) = \dim Y^\circ$. The arguments of Section 10.B apply verbatim and show the existence of a proper fibration of $\widetilde{\pi} : \widetilde{Y} \to C$ such that the family $\widetilde{f}^\circ$ is isotrivial when restricted to the general $\widetilde{\pi}$-fiber. That, however, contradicts the assumption that the variation is maximal. Theorem 1.1 is thus shown in case $\kappa(Y^\circ) > 0$. □

Acknowledgments. We thank Eckart Viehweg and Chengyang Xu for numerous discussions that motivated the problem and helped to make this a better article. We also thank the anonymous referees for a detailed and knowledgeable report and a number of comments that certainly helped to improve the presentation.

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