Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 218 (2008) 649–652
ADVANCES IN
Mathematics
www.elsevier.com/locate/aim

Families of varieties of general type over compact bases ☆

Stefan Kebekus [a], Sándor J. Kovács [b,*]

[a] *Mathematisches Institut, Universität zu Köln, Weyertal 86–90, 50931 Köln, Germany*
[b] *University of Washington, Department of Mathematics, Box 354350, Seattle, WA 98195, USA*

Received 29 May 2007; accepted 15 January 2008

Available online 24 March 2008

Communicated by Karen Smith

Abstract

Let $f : X \to Y$ be a smooth family of canonically polarized complex varieties over a smooth base. Generalizing the classical Shafarevich hyperbolicity conjecture, Viehweg conjectured that Y is necessarily of log general type if the family has maximal variation. A somewhat stronger and more precise version of Viehweg's conjecture was shown by the authors in [S. Kebekus, S.J. Kovács, Families of canonically polarized varieties over surfaces, preprint math.AG/0511378; Invent. Math. (2008), doi: 10.1007/s00222-008-0128-8; S. Kebekus, S.J. Kovács, The structure of surfaces mapping to the moduli stack of canonically polarized varieties, arXiv: 0707.2054v1 [math.AG], 2007] in the case where Y is a quasi-projective surface. Assuming that the minimal model program holds, this very short paper proves the same result for projective base manifolds Y of arbitrary dimension.

Keywords: Families; Varieties of general type; Shafarevich's conjecture; Viehweg's conjecture

☆ Stefan Kebekus was supported in part by the DFG-Forschergruppe "Classification of Algebraic Surfaces and Compact Complex Manifolds". Sándor Kovács was supported in part by NSF Grant DMS-0554697 and the Craig McKibben and Sarah Merner Endowed Professorship in Mathematics.

* Corresponding author.
E-mail addresses: stefan.kebekus@math.uni-koeln.de (S. Kebekus), kovacs@math.washington.edu (S.J. Kovács).
URLs: http://www.mi.uni-koeln.de/~kebekus (S. Kebekus), http://www.math.washington.edu/~kovacs (S.J. Kovács).

0001-8708/$ – see front matter
doi:10.1016/j.aim.2008.01.005

1. Introduction

Let $f : X \to Y$ be a smooth family of canonically polarized complex varieties over a smooth base. Generalizing the classical Shafarevich hyperbolicity conjecture, Viehweg conjectured that Y is necessarily of log general type if the family has maximal variation. We refer to [2] for a precise formulation, for background and for details about these notions. A somewhat stronger and more precise version of Viehweg's conjecture was shown in [2] and [3] in the case where Y is a quasi-projective surface. Assuming that the minimal model program holds, here we prove the same result for projective base manifolds Y of arbitrary dimension.

We recall the two relevant standard conjectures of higher dimensional algebraic geometry first.

Conjecture 1.1 *(Minimal model program and abundance for $\kappa = 0$). Let Y be a smooth projective variety such that $\kappa(Y) = 0$. Then there exists a birational map $\lambda : Y \dashrightarrow Y_\lambda$ such that the following holds.*

(1.1.1) *Y_λ is $\mathbb{Q}$-factorial and has at worst terminal singularities.*
(1.1.2) *There exists a number n such that nK_{Y_λ} is trivial, i.e., $\mathscr{O}_{Y_\lambda}(nK_{Y_\lambda}) = \mathscr{O}_{Y_\lambda}$.*

Conjecture 1.2 *(Abundance for $\kappa = -\infty$). Let Y be a smooth projective variety. If $\kappa(Y) = -\infty$, then Y is uniruled.*

Remark 1.3. Conjectures 1.1 and 1.2 are known to hold for all varieties of dimension $\dim Y \leqslant 3$.

The main result of this paper is now the following, cf. [2, Conjecture 1.6].

Theorem 1.4. *Let Y be a smooth projective variety and $f : X \to Y$ a smooth family of canonically polarized varieties. Assume that Conjectures* 1.1 *and* 1.2 *hold for all varieties F of dimension* $\dim F \leqslant \dim Y$. *Then the following holds.*

(1.4.1) *If $\kappa(Y) = -\infty$, then* $\operatorname{Var}(f) < \dim Y$.
(1.4.2) *If $\kappa(Y) \geqslant 0$, then* $\operatorname{Var}(f) \leqslant \kappa(Y)$.

Remark 1.5. The argumentation of Section 2 actually shows a slightly stronger result. If $\kappa(Y) = -\infty$, it suffices to assume that Conjecture 1.2 holds for Y. If $\kappa(Y) \geqslant 0$, we need to assume that Conjecture 1.1 holds for all varieties F of dimension $\dim F = \dim Y - \kappa(Y)$.

See Theorem 3.1 below for further generalizations.

Theorem 1.4 and Remark 1.3 immediately imply the following.

Corollary 1.6. *Viehweg's conjecture holds for smooth families of canonically polarized varieties over projective base manifolds of dimension $\leqslant 3$.*

2. Proof of Theorem 1.4

2.1. The case when $\kappa(Y) = -\infty$

The assertion follows immediately from Conjecture 1.2 and from the fact that families of canonically polarized varieties over rational curves are necessarily isotrivial [6, Thm. 1].

2.2. The case when $\kappa(Y) = 0$

In this case, we need to show that the family f is isotrivial. We argue by contradiction and assume that $\operatorname{Var}(f) \geqslant 1$. By [8, Thm. 1.4.i], this implies that there exists a number n and an invertible subsheaf $\mathscr{A} \subset \operatorname{Sym}^n \Omega^1_Y$ of Kodaira–Iitaka dimension $\kappa(\mathscr{A}) \geqslant \operatorname{Var}(f) \geqslant 1$.

By assumption, there exists a birational map $\lambda : Y \dashrightarrow Y_\lambda$ as discussed in Conjecture 1.1. Resolving the indeterminacies of λ and pulling back the family f, we may assume without loss of generality that λ is a morphism, i.e., defined everywhere.

Let $C_\lambda \subset Y_\lambda$ be a general complete intersection curve. Then C_λ will avoid the singularities of Y_λ. In particular, the restriction $\Omega^1_{Y_\lambda}|_{C_\lambda}$ is a vector bundle of degree

$$\deg \Omega^1_{Y_\lambda}\big|_{C_\lambda} = K_{Y_\lambda} \cdot C_\lambda = 0. \tag{2.2.1}$$

Claim 2.1. *The vector bundle $\Omega^1_{Y_\lambda}|_{C_\lambda}$ is not semistable.*

Proof of Claim 2.1. Observe that the curve C_λ avoids the fundamental points of λ, and hence that λ is an isomorphism in a neighborhood of C_λ. Setting $C := \lambda^{-1}(C_\lambda)$, the morphism λ induces an isomorphism $\Omega^1_{Y_\lambda}|_{C_\lambda} \cong \Omega^1_Y|_C$. This shows that $\Omega^1_{Y_\lambda}|_{C_\lambda}$ cannot be semistable, for if it was, its symmetric product $\operatorname{Sym}^n \Omega^1_{Y_\lambda}|_{C_\lambda}$ would also be semistable of degree 0. However, this contradicts the existence of the subsheaf $\mathscr{A}$ whose restriction to C has positive degree. □

To end the proof, observe that (2.2.1) and Claim 2.1 together imply that $\Omega^1_{Y_\lambda}|_{C_\lambda}$ has an invertible quotient of negative degree. In this setup, Miyaoka's uniruledness criterion, cf. [7, Cor. 8.6], [5] or [4, Chapt. 2.1], applies to show that Y is uniruled, contradicting the assumption that $\kappa(Y) = 0$.

2.3. The case when $\kappa(Y) > 0$

In this case, consider the Iitaka fibration of Y, $\mathrm{i} : Y' \to Z$, cf. [1, Thm. 10.3]. Since the Iitaka model is only determined birationally, we may assume that there exists a birational morphism $Y' \to Y$. Pulling the family $f : X \to Y$ back to Y', we may assume that $Y' = Y$, and hence we may assume that there exists a fibration $\mathrm{i} : Y \to Z$ such that $\dim Z = \kappa(Y)$ and $\kappa(F) = 0$ for the general fiber F of i [1, Thm. 11.8]. We have seen in Section 2.2 that $f|_F$ is isotrivial and hence $\operatorname{Var}(f) \leqslant \dim Y - \dim F = \dim Z = \kappa(Y)$. This finishes the proof of Theorem 1.4.

3. Families of varieties of general type

Using [8, Thm. 1.4.iii], the argumentation of Section 2 immediately gives the following, somewhat weaker, result for families of varieties of general type.

Theorem 3.1. *Let Y be a smooth projective variety and $f : X \to Y$ a smooth family of varieties of general type of maximal variation, i.e.,* $\mathrm{Var}(f) = \dim Y$*. If Conjectures* 1.1 *and* 1.2 *hold for all varieties F of dimension* $\dim F \leqslant \dim Y$*, then Y is of general type.*

References

[1] S. Iitaka, Algebraic Geometry: An Introduction to Birational Geometry of Algebraic Varieties, Grad. Texts in Math., vol. 76, Springer-Verlag, New York, 1982, North-Holland Math. Lib., 24. MR637060 (84j:14001).

[2] S. Kebekus, S.J. Kovács, Families of canonically polarized varieties over surfaces, preprint math.AG/0511378; Invent. Math. (2008), doi: 10.1007/s00222-008-0128-8.

[3] S. Kebekus, S.J. Kovács, The structure of surfaces mapping to the moduli stack of canonically polarized varieties, arXiv: 0707.2054v1 [math.AG], 2007.

[4] S. Kebekus, L. Solá Conde, Existence of rational curves on algebraic varieties, minimal rational tangents, and applications, in: Global Aspects of Complex Geometry, Springer, Berlin, 2006, pp. 359–416. MR2264116.

[5] S. Kebekus, L. Solá Conde, M. Toma, Rationally connected foliations after Bogomolov and McQuillan, J. Algebraic Geom. 16 (1) (2007) 65–81.

[6] S.J. Kovács, Smooth families over rational and elliptic curves, J. Algebraic Geom. 5 (2) (1996) 369–385. MR1374712 (97c:14035).

[7] Y. Miyaoka, Deformations of a morphism along a foliation and applications, in: Algebraic Geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), in: Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 245–268. MR927960 (89e:14001).

[8] E. Viehweg, K. Zuo, Base spaces of non-isotrivial families of smooth minimal models, in: Complex Geometry, Göttingen, 2000, Springer, Berlin, 2002, pp. 279–328. MR1922109 (2003h:14019).