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Doubly transitive permutation groups with abelian stabilizers

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Summary. We prove that any doubly transitive permutation group with abelian stabilizers is the group of linear functions over a suitable field. The result is not new: for finite groups it is well known, for infinite groups it follows from a more general theorem of W. Kerby and H. Wefelscheid on sharply doubly transitive groups in which the stabilizers have finite commutator subgroups. We give a direct and elementary proof.

The sharply doubly transitive groups of finite degree were determined by H. Zassenhaus [5] in 1936. They are the groups of linear functions over near-fields. Under additional assumptions the result has been extended to infinite groups, see M. Hall [1, p. 382]. In this note we will consider the case when the stabilizers are abelian.

In a somewhat more general situation, namely if the stabilizers have finite commutator subgroups, W. Kerby and H. Wefelscheid [2, Corollary 3] proved that the group must be the group of linear functions over a near-field. Although our result is a special case of theirs, we think our short proof might be of some interest.

THEOREM. *Let G be a doubly transitive permutation group. If the stabilizer of a point is abelian then G is the group of linear functions over a field.*

The same result has been obtained independently by V. D. Mazurov [4].

Observe that a transitive abelian group is regular, hence G is sharply doubly transitive, i.e., if two permutations from G agree at two points, they are equal.

As a corollary to our theorem we obtain that the stabilizers in G are isomorphic to the multiplicative group of a field. That group cannot be an infinite cyclic group, so we get a negative answer to a problem of Ja. P. Sysak from the Kourovka Notebook [3, 10.63].

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We introduce some notation. Let F be the set of permuted elements, and fix two elements $0, 1 \in F$. Let $A = G_0$, the stabilizer of 0, which is abelian by assumption. A acts regularly on the set of nonzero elements of F, hence we can identify any $a \in A$ with $a(1) \in F$.

We are going to define a field structure on F. First we define the multiplication in the obvious way, namely $xy = z$ for nonzero $x, y, z \in F$ iff this holds for the corresponding elements in A, and $0x = x0 = 0$ for all $x \in F$. Now, under the above mentioned identification, we have $a(x) = ax$ for all $a \in A$, $x \in F$.

We will distinguish two cases.

Case I. A contains a permutation of order 2. Since any two permutations of order 2 are conjugate in G (see [1, Lemma 20.7.2]), A contains a unique such element. We will denote this permutation, and also the corresponding element of F, by -1. We will write $-x$ for $(-1)x$.

Case II. A contains no permutations of order 2. (This will correspond to the case when F has characteristic 2.) For notational convenience we will write $-1 = 1$, $-x = x$ in this case.

By double transitivity, there exists a $t \in G$ interchanging 0 and 1. Our definition of the addition is motivated by the fact that in the group of linear functions $t(x) = 1 - x$. So let

$$x + y = \begin{cases} xt\left(-\dfrac{y}{x}\right), & \text{if } x \neq 0; \\ y, & \text{if } x = 0 \end{cases}$$

for all $x, y \in F$.

LEMMA 1. *For every $x, y \in F$, $x \neq 0, 1$, we have*

$$t(t(y)) = y, \tag{1}$$

$$t(xy) = t(x) \cdot t\left(\frac{t(y)}{t(1/x)}\right), \tag{2}$$

$$t\left(\frac{1}{t(x)}\right) = \frac{1}{t(1/x)}, \tag{3}$$

$$t(xy) = t(x) \cdot t\left(t(y) \cdot t\left(\frac{1}{t(x)}\right)\right). \tag{4}$$

Proof. (1) Since G is sharply doubly transitive and t^2 fixes both 0 and 1, it follows that t has order two.

(2) Using the identification of $F\backslash\{0\}$ and A we can define $t(x) \in A$ for all $x \in A$, $x \neq 1$. We show that

$$t \cdot x = t(x) \cdot t \cdot t(x^{-1})^{-1} \cdot t$$

holds in G. Indeed, it is easy to check that these permutations both map 1 to $t(x)$ and x^{-1} to 0, and as G is sharply doubly transitive, they should be equal. Evaluating the two permutations for an arbitrary $y \in F$ we obtain (2).

(3) For $y = 0$, Equation (2) turns into

$$1 = t(x) \cdot t\left(\frac{1}{t(1/x)}\right),$$

which, by substituting $1/x$ for x, immediately implies (3).

(4) Combining (2) and (3) we get the last equation.

LEMMA 2. *For every $x \in F$, $x \neq 0$, we have*

$$t(x) = -x \cdot t\left(\frac{1}{x}\right). \tag{5}$$

Proof. The equation trivially holds for $x = -1$. First we prove (5) in Case I. Let $u, v \in F$, $u, v \neq 0, 1$. Applying (4) twice, we obtain by (1), (2), and (3)

$$\begin{aligned} t(uv) &= t(u) \cdot t\left(t(v) \cdot t\left(\frac{1}{t(u)}\right)\right) \\ &= t(u) \cdot v \cdot t\left(\frac{1}{t(u)} \cdot t\left(\frac{1}{v}\right)\right) \\ &= t(u) \cdot v \cdot \frac{t(1/uv)}{t(1/u)}. \end{aligned}$$

Now let $u = -1$, $v = -x$, then the previous equation turns into (5).

Now assume that the Case II occurs. Let $x \in F$, $x \neq 0, 1$. Then

$$x^2 = t(t(x \cdot x)) = t\left(t(x) \cdot t\left(t(x) \cdot t\left(\frac{1}{t(x)}\right)\right)\right)$$

$$= t\left(t\left(t(x) \cdot t\left(\frac{1}{t(x)}\right)\right) \cdot t(x)\right)$$

$$= t(x) \cdot t\left(\frac{1}{t(x)}\right) \cdot t\left(x \cdot t\left(\frac{1}{t(x) \cdot t(1/t(x))}\right)\right)$$

$$= \left[t(x) \cdot t\left(\frac{1}{t(x)}\right)\right]^2.$$

Since A does not contain any element of order two, this implies that

$$x = t(x) \cdot t\left(\frac{1}{t(x)}\right),$$

and then (5) follows using (3), as we have $-1 = 1$ in this case.

PROPOSITION. *F is a field under the operations $+$ and $\cdot$.*

Proof. The axioms for multiplication are obviously satisfied, since A is an abelian group. It is easy to check that $x + 0 = x$ and $x + (-x) = 0$ for all $x \in F$. Hence, in proving the commutativity of addition, we may assume that $x, y \neq 0$. Then (5) yields

$$x + y = x \cdot t\left(-\frac{y}{x}\right) = x \cdot \frac{y}{x} \cdot t\left(-\frac{x}{y}\right) = y \cdot t\left(-\frac{x}{y}\right) = y + x.$$

Now let us show that $+$ is associative. The equation

$$x + (y + z) = (x + y) + z$$

trivially holds if $x = 0$ or $y = 0$, and it is also straightforward to check it if $y = -x$. Hence we may assume that $x \neq 0$, $y \neq 0$, $x + y \neq 0$. Now applying (2), (1), and (5) we obtain

$$x + (y + z) = x + y \cdot t\left(-\frac{z}{y}\right) = x \cdot t\left(-\frac{y}{x} \cdot t\left(-\frac{z}{y}\right)\right)$$

$$= x \cdot t\left(-\frac{y}{x}\right) \cdot t\left(\frac{-z/y}{t(-x/y)}\right)$$

$$= x \cdot t\left(-\frac{y}{x}\right) \cdot t\left(\frac{-z/y}{\frac{x}{y} \cdot t(-y/x)}\right)$$

$$= x \cdot t\left(-\frac{y}{x}\right) \cdot t\left(-\frac{z}{x \cdot t(-y/x)}\right)$$

$$= x \cdot t\left(-\frac{y}{x}\right) + z = (x + y) + z.$$

Showing distributivity is again an easy task.

Proof of the Theorem. Now

$$1 - x = 1 + (-x) = 1 \cdot t\left(-\frac{-x}{1}\right) = t(x).$$

Also, for $a \in A$ $a(x) = ax$ holds. These linear functions generate the group of all linear functions over F, as for $a, b \in F \backslash \{0\}$ we have

$$ax + b = b \cdot t\left(-\frac{a}{b}x\right).$$

This group is sharply doubly transitive, hence it is equal to G.

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