GENERIC VANISHING FAILS FOR SINGULAR VARIETIES AND IN CHARACTERISTIC $p > 0$

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Dedicated to Rob Lazarsfeld on the occasion of his sixtieth birthday.

1. Introduction

In recent years there has been considerable interest in understanding the geometry of irregular varieties, i.e., varieties admitting a nontrivial morphism to an abelian veriety. One of the central results in the area is the following result conjectured by M. Green and R. Lazarsfeld (cf. [GL91, 6.2]) and proven in [Hac04] and [PP09].

Theorem 1.1. *Let $\lambda : X \to A$ be a generically finite (onto its image) morphism from a compact Kähler manifold to a complex torus. If $\mathscr{L} \to X \times \operatorname{Pic}^0(A)$ is the universal family of topologically trivial line bundles, then*

$$R^i \pi_{\operatorname{Pic}^0(A)*} \mathscr{L} = 0 \qquad \text{for } i < n.$$

At first sight, the above result appears to be quite technical however it has many concrete applications (see for example [CH11], [JLT11] and [PP09]). In this paper we will show that (1.1) does not generalize to characteristic $p > 0$ or to singular varieties in characteristic 0.

Notation 1.2. Let A be an abelian variety over an algebraically closed field k, $\widehat{A}$ its dual abelian variety, $\mathscr{P}$ the normalized Poincaré bundle on $A \times \widehat{A}$ and $p_{\widehat{A}} : A \times \widehat{A} \to \widehat{A}$ the projection. Let $\lambda : X \to A$ be a projective morphism, $\pi_{\widehat{A}} : X \times \widehat{A} \to \widehat{A}$ the projection and $\mathscr{L} := (\lambda \times \mathrm{id}_{\widehat{A}})^* \mathscr{P}$ where $(\lambda \times \mathrm{id}_{\widehat{A}}) : X \times \widehat{A} \to A \times \widehat{A}$ is the product morphism.

Theorem 1.3. *Let k be an algebraically closed field. Then, using the notation in* (1.2)*, there exist a projective variety X over k such that*

- *if* $\operatorname{char} k = p > 0$*, then X is smooth, and*
- *if* $\operatorname{char} k = 0$*, then X has isolated Gorenstein log canonical singularities,*

and a separated projective morphism to an abelian variety $\lambda : X \to A$ which is generically finite onto its image and such that

$$R^i \pi_{\widehat{A}*} \mathscr{L} \neq 0 \qquad \text{for some } 0 \le i < n.$$

Remark 1.4. Due to the birational nature of the statement, (1.1) trivially generalizes to the case of X having only rational singularities. Arguably Gorenstein log canonical singularities are the simplest examples of singularities that are not rational. Therefore the characteristic 0 part of (1.3) may be interpreted as saying that generic vanishing does not extended to singular varieties in a non-trivial way.

Remark 1.5. Note that (1.3) seems to contradict the main result of [Par03].

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2. Preliminaries

Let A be a g-dimensional abelian variety over an algebraically closed field k, $\widehat{A}$ its dual abelian variety p_A and $p_{\widehat{A}}$ the projections of $A \times \widehat{A}$ onto A and $\widehat{A}$, and $\mathscr{P}$ the normalized Poincaré bundle on $A \times \widehat{A}$. We denote by $\mathbf{R}\widehat{S} : \mathbf{D}(A) \to \mathbf{D}(\widehat{A})$ the usual Fourier-Mukai functor given by $\mathbf{R}\widehat{S}(\mathscr{F}) = \mathbf{R}p_{\widehat{A}*}(p_A^*\mathscr{F} \otimes \mathscr{P})$ cf. [Muk81]. There is a corresponding functor $\mathbf{R}S : \mathbf{D}(\widehat{A}) \to \mathbf{D}(A)$ such that

$$\mathbf{R}S \circ \mathbf{R}\widehat{S} = (-1_A)^*[-g] \qquad \text{and} \qquad \mathbf{R}\widehat{S} \circ \mathbf{R}S = (-1_{\widehat{A}})^*[-g].$$

Definition 2.1. An object $F \in \mathbf{D}(A)$ is called *WIT-i* if $R^j\widehat{S}(F) = 0$ for all $j \neq i$. In this case we use the notation $\widehat{F} = R^i\widehat{S}(F)$.

Notice that if F is a WIT-i coherent sheaf (in degree 0), then $\widehat{F}$ is a WIT-$(g-i)$ coherent sheaf (in degree i) and $F \simeq (-1_A)^*R^{g-i}S(\widehat{F})$.

One easily sees that if F and G are arbitrary objects, then

$$\operatorname{Hom}_{\mathbf{D}(A)}(F, G) = \operatorname{Hom}_{\mathbf{D}(\widehat{A})}(\mathbf{R}\widehat{S}F, \mathbf{R}\widehat{S}G).$$

An easy consequence (cf. [Muk81, 2.5]) is that if F is a WIT-i sheaf and G is a WIT-j sheaf (or if F is a WIT-i locally free sheaf and G is a WIT-j object – not necessarily a sheaf), then

$$\begin{aligned}(2.1.1)\quad \mathrm{Ext}^k_{\mathscr{O}_A}(F,G) &\simeq \mathrm{Hom}_{\mathbf{D}(A)}(F,G[k]) \simeq\\ &\simeq \mathrm{Hom}_{\mathbf{D}(\widehat{A})}(\mathbf{R}\widehat{S}F,\mathbf{R}\widehat{S}G[k]) =\\ &= \mathrm{Hom}_{\mathbf{D}(\widehat{A})}(\widehat{F}[-i],\widehat{G}[k-j]) \simeq \mathrm{Ext}^{k+i-j}_{\mathscr{O}_{\widehat{A}}}(\widehat{F},\widehat{G}).\end{aligned}$$

Let L be any ample line bundle on $\widehat{A}$, then $\mathbf{R}S(L) = R^0S(L) = \widehat{L}$ is a vector bundle on A of rank $h^0(L)$. For any $x \in A$, let $t_x : A \to A$ be the translation by x and let $\phi_L : \widehat{A} \to A$ be the isogeny determined by $\phi_L(\widehat{x}) = t^*_{\widehat{x}}L \otimes L^\vee$, then $\phi^*_L(\widehat{L}) = \bigoplus_{h^0(L)} L^\vee$.

Let $\lambda : X \to A$ be a projective morphism of normal varieties, and $\mathscr{L} = (\lambda \times \mathrm{id}_{\widehat{A}})^*\mathscr{P}$. We let $\mathbf{R}\Phi : \mathbf{D}(X) \to \mathbf{D}(\widehat{A})$ be the functor defined by $\mathbf{R}\Phi(F) = \mathbf{R}\pi_{\widehat{A}*}(\pi^*_X F \otimes \mathscr{L})$ where π_X and $\pi_{\widehat{A}}$ denote the projections of $X \times \widehat{A}$ on to the first and second factor. Note that

$$\begin{aligned}(2.1.2)\quad \mathbf{R}\Phi(F) &= \mathbf{R}\pi_{\widehat{A}*}(\pi^*_X F \otimes \mathscr{L}) \simeq^1\\ &\simeq \mathbf{R}p_{\widehat{A}*}\mathbf{R}(\lambda \times \mathrm{id}_{\widehat{A}})_*(\pi^*_X F \otimes (\lambda \times \mathrm{id}_{\widehat{A}})^*\mathscr{P}) \simeq^2\\ &\simeq \mathbf{R}p_{\widehat{A}*}\big(\mathbf{R}(\lambda \times \mathrm{id}_{\widehat{A}})_*(\pi^*_X F) \otimes \mathscr{P}\big) \simeq^3\\ &\simeq \mathbf{R}p_{\widehat{A}*}(p^*_A\mathbf{R}\lambda_*F \otimes \mathscr{P}) \simeq \mathbf{R}\widehat{S}(\mathbf{R}\lambda_*F),\end{aligned}$$

where $\simeq^1$ follows by composition of derived functors [Har66, II.5.1], $\simeq^2$ follows by the projection formula [Har66, II.5.6], and $\simeq^3$ follows by flat base change [Har66, II.5.12].

We also define $\mathbf{R}\Psi : \mathbf{D}(\widehat{A}) \to \mathbf{D}(X)$ by $\mathbf{R}\Psi(F) = \mathbf{R}\pi_{X*}(\pi^*_{\widehat{A}}F \otimes \mathscr{L})$. Notice that if F is a locally free sheaf, then $\pi^*_{\widehat{A}}F \otimes \mathscr{L}$ is also a locally free sheaf. In particular, for any $i \in \mathbb{Z}$, we have that

$$(2.1.3)\qquad R^i\Psi(F) \simeq R^i\pi_{X*}(\pi^*_{\widehat{A}}F \otimes \mathscr{L}).$$

We will need the following fact (which is also proven during the proof of Theorem B of [PP11]).

Lemma 2.2. *Let L be an ample line bundle on $\widehat{A}$, then*

$$\mathbf{R}\Psi(L^\vee) = R^g\Psi(L^\vee) = \lambda^*\widehat{L^\vee}.$$

Proof. Since L is ample, $H^i(\widehat{A}, L^\vee \otimes \mathscr{L}_x) = H^i(\widehat{A}, L^\vee \otimes \mathscr{P}_{\lambda(x)}) = 0$ for $i \neq g$ where $\mathscr{P}_{\lambda(x)} = \mathscr{P}|_{\lambda(x)\times\widehat{A}}$ and $\mathscr{L}_x = \mathscr{L}|_{x\times\widehat{A}}$ are isomorphic. By cohomology and base change $\mathbf{R}\Psi(L^\vee) = R^g\Psi(L^\vee)$ (resp. $\widehat{L^\vee}$) is a vector bundle of rank $h^g(\widehat{A}, L^\vee)$ on X (resp. on A).

The a natural transformation $\mathrm{id}_{A\times\widehat{A}} \to (\lambda\times\mathrm{id}_{\widehat{A}})_*(\lambda\times\mathrm{id}_{\widehat{A}})^*$ induces a natural morphism,

$$\widehat{L^\vee} = R^g p_{A*}(p_{\widehat{A}}^* L^\vee \otimes \mathscr{P}) \to R^g p_{A*}(\lambda\times\mathrm{id}_{\widehat{A}})_*(\pi_{\widehat{A}}^* L^\vee \otimes \mathscr{L}).$$

Let $\sigma = p_A \circ (\lambda\times\mathrm{id}_{\widehat{A}}) = \lambda\circ\pi_X$. By the Grothendieck spectral sequence associated to $p_{A*}\circ(\lambda\times\mathrm{id}_{\widehat{A}})_*$ there exists a natural morphism

$$R^g p_{A*}(\lambda\times\mathrm{id}_{\widehat{A}})_*(\pi_{\widehat{A}}^* L^\vee \otimes \mathscr{L}) \to R^g\sigma_*(\pi_{\widehat{A}}^* L^\vee \otimes \mathscr{L}),$$

and similarly by the Grothendieck spectral sequence associated to $\lambda_*\circ\pi_{X*}$ there exists a natural morphism

$$R^g\sigma_*(\pi_{\widehat{A}}^* L^\vee \otimes \mathscr{L}) \to \lambda_* R^g\pi_{X*}(\pi_{\widehat{A}}^* L^\vee \otimes \mathscr{L}).$$

Combining the above three morphisms gives a natural morphism

$$\widehat{L^\vee} \to \lambda_* R^g\pi_{X*}(\pi_{\widehat{A}}^* L^\vee \otimes \mathscr{L}) = \lambda_* R^g\Psi(L^\vee),$$

and hence by adjointness a natural morphism,

$$\eta : \lambda^*\widehat{L^\vee} \to R^g\Psi(L^\vee).$$

For any point $x\in X$, by cohomology and base change, the induced morphism on the fiber over x is an isomorphism:

$$\eta_x : \lambda^*\widehat{L^\vee}\otimes\kappa(x) \simeq H^g(\lambda(x)\times\widehat{A}, L^\vee\otimes\mathscr{P}_{\lambda(x)}) \xrightarrow{\simeq} \\ \xrightarrow{\simeq} H^g(x\times\widehat{A}, L^\vee\otimes\mathscr{L}_x) \simeq R^g\Psi(L^\vee)\otimes\kappa(x).$$

Therefore η_x is an isomorphism for all $x\in X$ and hence η is an isomorphism. $\square$

3. Examples

Notation 3.1. Let $T\subseteq\mathbb{P}^n$ be a projective variety. The cone over T in $\mathbb{A}^{n+1}$ will be denoted by $C(T)$. In other words, if $T\simeq\operatorname{Proj} S$, then $C(T)\simeq\operatorname{Spec} S$.

Linear equivalence between (Weil) divisors is denoted by $\sim$ and strict transform of a subvariety T by the inverse of a birational morphism σ is denoted by $\sigma_*^{-1}T$.

Example 3.2. Let k be an algebraically closed field, $V\subseteq\mathbb{P}^n$ and $W\subseteq\mathbb{P}^m$ two smooth projective varieties over k, and $p\in V$ a closed point. Let $x_0,\dots,x_n$ and $y_0,\dots,y_m$ be homogenous coordinates on $\mathbb{P}^n$ and $\mathbb{P}^m$ respectively.

Consider the embedding $V\times W\subset\mathbb{P}^N$ induced by the Segre embedding of $\mathbb{P}^n\times\mathbb{P}^m$. We may choose homogenous coordinates z_{ij} for $i=0,\dots,n$ and $j=0,\dots,m$ on $\mathbb{P}^N$ and in these coordinates $\mathbb{P}^n\times\mathbb{P}^m$

is defined by the equations $z_{\alpha\gamma}z_{\beta\delta} - z_{\alpha\delta}z_{\beta\gamma}$ for all $0 \leq \alpha, \beta \leq n$ and $0 \leq \gamma, \delta \leq m$.

Next let $H \subset W$ such that $\{p\} \times H \subset \{p\} \times W$ is a hyperplane section of $\{p\} \times W$ in $\mathbb{P}^N$. Let $Y = C(V \times W) \subset \mathbb{A}^{N+1}$ and $Z = C(V \times H) \subset Y$ and let $v \in Z \subset Y$ denote the common vertex of Y and Z. If $\dim W = 0$, then $H = \emptyset$. In this case let $Z = \{v\}$ the vertex of Y. Finally let $\mathfrak{m}_v$ denote the ideal of v in the affine coordinate ring of Y. It is generated by all the variables z_{ij}.

Proposition 3.3. *Let $f : X \to Y$ be the blowing up of Y along Z. Then f is an isomorphism over $Y \setminus \{v\}$ and the scheme theoretic preimage of v (whose support is the exceptional locus) is isomorphic to V:*

$$f^{-1}(v) \simeq V.$$

Proof. As Z is of codimension 1 in Y and $Y \setminus \{v\}$ is smooth, it follows that $Z \setminus \{v\}$ is a Cartier divisor in $Y \setminus \{v\}$ and hence f is indeed an isomorphism over $Y \setminus \{v\}$.

To prove the statement about the exceptional locus of f, first assume that $V = \mathbb{P}^n$, $W = \mathbb{P}^m$, $p = [1 : 0 : \cdots : 0]$, and $\{p\} \times H = (z_{0m} = 0) \cap (\{p\} \times W)$. Then $H = (y_m = 0) \subseteq W$ and hence $I = I(Z)$, the ideal of Z in the affine coordinate ring of Y, is generated by $\{z_{im} | i = 0, \dots, n\}$. Then by the definition of blowing up, $X = \operatorname{Proj} \oplus_{d \geq 0} I^d$ and $f^{-1}v \simeq \operatorname{Proj} \oplus_{d \geq 0} I^d / I^d \mathfrak{m}_v$.

Notice that $I^d / I^d \mathfrak{m}_v$ is a k-vector space generated by the degree d monomials in the variables $\{z_{im} | i = 0, \dots, n\}$. It follows that the graded ring $\oplus_{d \geq 0} I^d / I^d \mathfrak{m}_v$ is nothing else but $k[z_{im} | i = 0, \dots, n]$ and hence $f^{-1}v \simeq \mathbb{P}^n = V$, so the claim is proved in this case.

Next consider the case when $V \subseteq \mathbb{P}^n$ is arbitrary, but $W = \mathbb{P}^m$. In this case the calculation is similar, except that we have to account for the defining equations of V. They show up in the definition of the coordinate ring of Y in the following way: If a homogenous polynomial $g \in k[x_0, \dots, x_n]$ vanishes on V (i.e., $g \in I(V)_h$), then define $g_\gamma \in k[z_{ij}]$ for any $0 \leq \gamma \leq m$ by replacing x_α with $z_{\alpha\gamma}$ for each $0 \leq \alpha \leq n$. Then $\{g_\gamma | 0 \leq \gamma \leq m, g \in I(V)_h\}$ generates the ideal of Y in the affine coordinate ring of $C(\mathbb{P}^n \times \mathbb{P}^m)$. It follows that the above computation goes through the same way, except that the variables $\{z_{im} | i = 0, \dots, n\}$ on the exceptional $\mathbb{P}^n$ are subject to the equations $\{g_m | g \in I(V)_h\}$. However, this simply means that the exceptional locus of f, i.e., $f^{-1}v$, is cut out from $\mathbb{P}^n$ by these equations and hence it is isomorphic to V.

Finally, consider the general case. The way W changes the setup is the same as what we described for V. If a homogenous polynomial $h \in k[y_0, \dots, y_m]$ vanishes on W (i.e., $h \in I(W)_h$), then define $h_\alpha \in$

$k[z_{ij}]$ for any $0 \leq \alpha \leq n$ by replacing y_γ with $z_{\alpha\gamma}$ for each $0 \leq \gamma \leq m$. Then $\{h_\alpha | 0 \leq \alpha \leq n, h \in I(W)_h\}$ generates the ideal of Y in the affine coordinate ring of $C(V \times \mathbb{P}^m)$.

However, in this case, differently from the case of V, we do not get any additional equations. Indeed, we chose the coordinates so that $H = (y_m = 0)$ and hence $y_m \notin I(W)$, which means that we may choose the rest of the coordinates such that $[0 : \cdots : 0 : 1] \in W$. This implies that no polynomial in the ideal of W may have a monomial term that is a constant multiple of a power of y_m. It follows that, since $I = I(Z)$ is generated by the elements $\{z_{im} | i = 0, \dots, n\}$, any monomial term of any polynomial in the ideal of Y in the affine coordinate ring of $C(V \times \mathbb{P}^m)$ that lies in I^d for some $d > 0$, also lies in $I^d \mathfrak{m}_v$. Therefore these new equations do not change the ring $\oplus I^d / I^d \mathfrak{m}_v$ and so $f^{-1}v$ is still isomorphic to V. □

Notation 3.4. We will use the notation introduced in (3.3) for X, Y, Z, and f. We will also use $X_\mathbb{P}$, $Y_\mathbb{P}$, $Z_\mathbb{P}$, and $f_\mathbb{P} : X_\mathbb{P} \to Y_\mathbb{P}$ to denote the same objects in the case $W = \mathbb{P}^m$, i.e., $Y_\mathbb{P} = C(V \times \mathbb{P}^m)$, $Z_\mathbb{P} = C(V \times H)$ where $H \subset \mathbb{P}^m$ is such that $\{p\} \times H \subset \{p\} \times \mathbb{P}^m$ is a hyperplane section of $\{p\} \times \mathbb{P}^m$ in $\mathbb{P}^N$.

Corollary 3.5. *$f_\mathbb{P}$ is an isomorphism over $Y_\mathbb{P} \setminus \{v\}$ and the scheme theoretic preimage of v (whose support is the exceptional locus) via $f_\mathbb{P}$ is isomorphic to V:*

$$f_\mathbb{P}^{-1} v \simeq V.$$

Proof. This was proven as an intermediate step in, and is also straightforward from (3.3) by taking $W = \mathbb{P}^m$. □

Proposition 3.6. *Assume that V and W are both positive dimensional, $W \subseteq \mathbb{P}^m$ is a complete intersection, and the embedding $V \times \mathbb{P}^r \subset \mathbb{P}^N$ for any linear subvariety $\mathbb{P}^r \subseteq \mathbb{P}^m$ induced by the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ is projectively normal. Then X is Gorenstein.*

Proof. First note that the projective normality assumption implies that $Y_\mathbb{P} = C(V \times \mathbb{P}^m)$ is normal and hence we may consider divisors and their linear equivalence on it.

Let $H' \subset \mathbb{P}^m$ be an arbitrary hypersurface (different from H and not necessarily linear). Observe that $H' \sim d \cdot H$ with $d = \deg H'$, so $V \times H' \sim d \cdot (V \times H)$, and hence $C(V \times H') \sim d \cdot C(V \times H)$ as divisors on $Y_\mathbb{P}$.

Since $f_\mathbb{P}$ is a small morphism it follows that the strict transforms of these divisors on $X_\mathbb{P}$ are also linearly equivalent: $f_*^{-1} C(V \times H') \sim d \cdot f_*^{-1} C(V \times H)$ (where by abuse of notation we let $f = f_\mathbb{P}$). By the basic properties of blowing up, the (scheme-theoretic) pre-image of $C(V\times H)$ is a Cartier divisor on X which coincides with $f_*^{-1}C(V\times H)$ (as f is small). However, then $f_*^{-1}C(V\times H')$ is also a Cartier divisor and hence it is Gorenstein if and only if $X_{\mathbb{P}}$ is. Note that $f_*^{-1}C(V\times H')$ is nothing else but the blow up of $C(V\times H')$ along $C(V\times(H'\cap H))$.

By assumption W is a complete intersection, so applying the above argument for the intersection of the hypersurfaces cutting out W shows that X is Gorenstein if and only if $X_{\mathbb{P}}$ is Gorenstein. In other words, it is enough to prove the statement with the additional assumption that $W=\mathbb{P}^m$. In particular, we have $X=X_{\mathbb{P}}$, etc.

In this case the same argument as above shows that the statement holds for m if and only if it holds for $m-1$, so we only need to prove it for $m=1$. In that case $H\in\mathbb{P}^1$ is a single point. Choose another point $H'\in\mathbb{P}^1$. As above, $f_*^{-1}C(V\times H')$ is a Cartier divisor in X and it is the blow up of $C(V\times H')$ along the intersection $C(V\times H')\cap C(V\times H)$.

We claim that this intersection is just the vertex of $C(V)$.

To see this, view $Y=Y_{\mathbb{P}}=C(V\times\mathbb{P}^1)$ as a subscheme of $C(\mathbb{P}^n\times\mathbb{P}^1)$. Inside $C(\mathbb{P}^n\times\mathbb{P}^1)$ the cones $C(\mathbb{P}^n\times H)$ and $C(\mathbb{P}^n\times H')$ are just linear subspaces of dimension $n+1$ whose scheme theoretic intersection is the single reduced point v. Therefore we have that

$$C(V\times H')\cap C(V\times H)\subseteq C(\mathbb{P}^m\times H')\cap C(\mathbb{P}^m\times H)=\{v\}$$

proving the same for this intersection.

Finally then $f_*^{-1}C(V\times H')$, the blow up of $C(V\times H')$ along the intersection $C(V\times H')\cap C(V\times H)$ is just the blow up of $C(V)$ at its vertex and hence it is smooth and in particular Gorenstein. This completes the proof. □

Lemma 3.7. *Let $V\subseteq\mathbb{P}^n$ and $W\subseteq\mathbb{P}^m$ be two normal complete intersection varieties of positive dimension. Assume that either $\dim V+\dim W>2$ or if $\dim V=\dim W=1$, then $n=m=2$. Then the embedding $V\times W\subset\mathbb{P}^N$ induced by the Segre embedding of $\mathbb{P}^n\times\mathbb{P}^m$ is projectively normal.*

Proof. It follows easily from the definition of the Segre embedding, that it is itself projectively normal and hence it is enough to prove that

$$H^0(\mathbb{P}^n\times\mathbb{P}^m,\mathscr{O}_{\mathbb{P}^N}(d)|_{\mathbb{P}^n\times\mathbb{P}^m})\to H^0(V\times W,\mathscr{O}_{\mathbb{P}^N}(d)|_{V\times W}) \tag{3.7.1}$$

is surjective for all $d\in\mathbb{N}$.

We prove this by induction on the combined number of hypersurfaces cutting out V and W. When this number is 0, then $V=\mathbb{P}^n$ and $W=\mathbb{P}^m$ so we are done.

Otherwise, assume that $\dim V \leq \dim W$ and if $\dim V = \dim W = 1$ then $\deg V = e \geq \deg W$. Let $V' \subseteq \mathbb{P}^n$ be a complete intersection variety of dimension $\dim V + 1$ such that $V = V' \cap H'$ where $H' \subset \mathbb{P}^n$ is a hypersurface of degree e. Then $V \times W \subset V' \times W$ is a Cartier divisor with ideal sheaf $\mathscr{I} \simeq \pi_1^* \mathscr{O}_{V'}(-e)$ where $\pi_1 : V' \times W \to V'$ is the projection to the first factor. It follows that for every $d \in \mathbb{N}$ there exists a short exact sequence,

$$0 \to \mathscr{O}_{\mathbb{P}^N}(d)|_{V' \times W} \otimes \pi_1^* \mathscr{O}_{V'}(-e) \to \mathscr{O}_{\mathbb{P}^N}(d)|_{V' \times W} \to \mathscr{O}_{\mathbb{P}^N}(d)|_{V \times W} \to 0,$$

and hence an induced exact sequence of cohomology

$$\begin{aligned} H^0(V' \times W, \mathscr{O}_{\mathbb{P}^N}(d)|_{V' \times W}) \to H^0(V \times W, \mathscr{O}_{\mathbb{P}^N}(d)|_{V \times W}) \to \\ \to H^1(V' \times W, \pi_1^* \mathscr{O}_{V'}(d-e) \otimes \pi_2^* \mathscr{O}_W(d)), \end{aligned}$$

where $\pi_2 : V' \times W \to W$ is the projection to the second factor.

Since by assumption V' is a complete intersection variety of dimension at least 2, it follows that $H^1(V', \mathscr{O}_{V'}(d-e)) = 0$.

If $\dim W > 1$, then it follows similarly that $H^1(W, \mathscr{O}_W(d)) = 0$.

If $\dim W = 1$, then since $0 < \dim V \leq \dim W$ we also have $\dim V = 1$. By assumption V and W are normal and hence regular, and in this case we assumed earlier that $\deg V = e \geq \deg W$. It follows that as long as $e > d$, then $H^0(V', \mathscr{O}_{V'}(d-e)) = 0$ and if $e \leq d$, then $d \geq \deg W$ and hence $H^1(W, \mathscr{O}_W(d)) = 0$.

In both cases we obtain that by the Künneth formula (cf. [EGAIII$_2$, (6.7.8)], [Kem93, 9.2.4]),

$$H^1(V' \times W, \pi_1^* \mathscr{O}_{V'}(d-e) \otimes \pi_2^* \mathscr{O}_W(d)) = 0,$$

and hence

$$H^0(V' \times W, \mathscr{O}_{\mathbb{P}^N}(d)|_{V' \times W}) \to H^0(V \times W, \mathscr{O}_{\mathbb{P}^N}(d)|_{V \times W})$$

is surjective. By induction we may assume that

$$H^0(\mathbb{P}^n \times \mathbb{P}^m, \mathscr{O}_{\mathbb{P}^N}(d)|_{\mathbb{P}^n \times \mathbb{P}^m}) \to H^0(V' \times W, \mathscr{O}_{\mathbb{P}^N}(d)|_{V' \times W})$$

is surjective, so it follows that the desired map in (3.7.1) is surjective as well and the statement is proven. □

Corollary 3.8. *Let $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$ be two positive dimensional normal complete intersection varieties and assume that if $\dim V = 1$, then $n = 2$. Then X is Gorenstein.*

Proof. Follows by combining (3.6) and (3.7). Note that in (3.6) the embedding $V \times W \hookrightarrow \mathbb{P}^N$ does not need to be projectively normal, only $V \times \mathbb{P}^r \hookrightarrow \mathbb{P}^N$ does, which indeed follows from (3.7). □

Example 3.9. Let k be an algebraically closed field. We will construct a birational projective morphism $f : X \to Y$ such that X is Gorenstein (and log canonical) and $R^1 f_* \omega_X \neq 0$.

Let $E_1, E_2 \subseteq \mathbb{P}^2$ be two smooth projective cubic curves. Consider the construction in (3.2) with $V = E_1$, $W = E_2$. As in that construction let $f : X \to Y$ be the blow up of $Y = C(E_1 \times E_2)$ along $Z = C(E_1 \times H)$ where $H \subseteq E_2$ is a hyperplane section. The common vertex of Y and Z will still be denoted by $v \in Z \subset Y$. The map f is an isomorphism over $Y \setminus \{v\}$ and $f^{-1}v \simeq E_1$ by (3.3).

Proposition 3.10. *Both X and Y are smooth in codimension 1 with trivial canonical divisor and X is Gorenstein and hence Cohen-Macaulay.*

Proof. By construction $Y \setminus \{v\} \simeq X \setminus f^{-1}v$ is smooth, so the first statement follows. Furthermore, $Y \setminus \{v\} \simeq X \setminus f^{-1}v$ is an affine bundle over $E_1 \times E_2$, so by the choice of E_1 and E_2, the canonical divisor of $Y \setminus \{v\} \simeq X \setminus f^{-1}v$ is trivial. However, the complement of this set has codimension at least 2 in both X and Y and hence their canonical divisors are trivial as well. Since $E_1, E_2 \subset \mathbb{P}^2$ are hypersurfaces, X is Gorenstein by (3.8). $\square$

Let E denote $f^{-1}v$. So we have that $E \simeq E_1$ and there is a short exact sequence

$$0 \to \mathscr{I}_E \to \mathscr{O}_X \to \mathscr{O}_E \to 0.$$

Pushing this forward via f we obtain a homomorphism $\phi : R^1 f_* \mathscr{O}_X \to R^1 f_* \mathscr{O}_E$. Since the maximum dimension of any fiber of f is 1, we have $R^2 f_* \mathscr{I}_E = 0$. It follows that $R^1 f_* \omega_X = R^1 f_* \mathscr{O}_X \neq 0$, because $R^1 f_* \mathscr{O}_E \neq 0$ (it is a sheaf supported on v of length $h^1(\mathscr{O}_E) = 1$).

Example 3.11. Let k be an algebraically closed field of characteristic $p \neq 0$. Then there exists a birational morphism $f : X \to Y$ of varieties (defined over k) such that X is smooth of dimension 7 and $R^i f_* \omega_X \neq 0$. for some $i \in \{1, 2, 3, 4, 5\}$.

Let Z be a smooth 6-dimensional variety and L a very ample line bundle such that $H^1(Z, \omega_Z \otimes L) \neq 0$. (such varieties exist by [LR97]). By Serre vanishing $H^i(Z, \omega_Z \otimes L^j) = 0$ for all $i > 0$ and $j \gg 0$. Let m be the largest positive integer such that $H^i(Z, \omega_Z \otimes L^m) \neq 0$ for some $i > 0$.

After replacing L by L^m we may assume that there exists a $q > 0$ such that $H^q(Z, \omega_Z \otimes L) \neq 0$, but $H^i(Z, \omega_Z \otimes L^j) = 0$ for all $i > 0$ and $j \geq 2$. Note that $q < 6$, because $H^6(Z, \omega_Z \otimes L)$ is dual to $H^0(Z, L^{-1}) = 0$.

Let Y be the cone over the embedding of Z given by L, $f : X \to Y$ the blow up of the vertex $v \in Y$, and $E = f^{-1}v$ the exceptional divisor of f. Note that $E \simeq Z$ and $\omega_E(-jE) \simeq \omega_Z \otimes L^j$ for any j.

For $j \geq 1$ consider the short exact sequence

$$0 \to \omega_X(-jE) \to \omega_X(-(j-1)E) \to \omega_E(-jE) \to 0.$$

Claim 3.11.1. $R^i f_* \omega_X(-E) = 0$ for all $i > 0$ and $R^i f_* \omega_X = 0$ for all $i > 0$, such that $H^i(Z, \omega_Z \otimes L) = 0$.

Proof of Claim. As $-E$ is f-ample we have, by Serre vanishing again, that $R^i f_* \omega_X(-jE) = 0$ for all $i > 0$ and some $j > 0$. If either $j > 1$ or $j = 1$ and $H^i(Z, \omega_Z \otimes L) = 0$, then $R^i f_* \omega_E(-jE) = H^i(Z, \omega_Z \otimes L^j) = 0$ by the choice of L. Therefore, the exact sequence

$$0 = R^i f_* \omega_X(-jE) \to R^i f_* \omega_X(-(j-1)E) \to R^i f_* \omega_E(-jE) = 0$$

gives that $R^i f_* \omega_X(-(j-1)E) = 0$. The claim follows by induction. $\square$

From the above claim it follows that

$$0 = R^q f_* \omega_X(-E) \to R^q f_* \omega_X \to R^q f_* \omega_E(-E) \to R^{q+1} f_* \omega_X(-E) = 0$$

Since $R^q f_* \omega_E(-E) = H^q(Z, \omega_Z \otimes L) \neq 0$, we obtain that $R^q f_* \omega_X \neq 0$ as claimed. $\square$

Remark 3.12. The above example is certainly well known (see for example [CR11b, 4.7.2]) and one can easily construct examples in dimension ≥ 3 (using for example the results of [Ray78] and [Muk79]). We have chosen to include the above example because of its elementary nature.

Proposition 3.13. *There exists a variety T and a generically finite projective separable morphism to an abelian variety $\lambda : T \to A$ defined over an algebraically closed field k such that:*

- *If $\operatorname{char} k = 0$, then T is Gorenstein (and hence Cohen-Macaulay) with a single isolated log canonical singularity, and $R^1\lambda_*\omega_T \neq 0$, and*
- *If $\operatorname{char} k = p > 0$, then T is smooth, and $R^i\lambda_*\omega_T \neq 0$ for some $i > 0$.*

Proof. First assume that $\operatorname{char} k = 0$ and let $f : X \to Y$ be as in (3.9). We may assume that X and Y are projective. Let $X' \to X$ and $Y' \to Y$ be birational morphisms that are isomorphisms near $f^{-1}(v)$ and v respectively such that there is a birational morphism $f' : X' \to Y'$ and a generically finite morphism $g : Y' \to \mathbb{P}^n$. We let $v' \in Y'$ be the inverse image of $v \in Y$ and $p \in \mathbb{P}^n$ its image. We may assume that there is an open subset $\mathbb{P}^n_0 \subset \mathbb{P}^n$ such that $g|_{Y'_0}$ is finite where $Y'_0 = g^{-1}(\mathbb{P}^n_0)$. Note that if we let X'_0 be the inverse image of Y'_0 and $g' = g \circ f'$, then we have $R^i g'_* \omega_{X'_0} = g_* R^i f'_* \omega_{X'_0}$.

Let A be an n-dimensional abelian variety, $A' \to A$ a birational morphism of smooth varieties and $A' \to \mathbb{P}^n$ a generically finite morphism. We may assume that there are points $a' \in A'$ and $a \in A$ such that

$(A', a') \to (A, a)$ is locally an isomorphism and $(A', a') \to (\mathbb{P}^n, p)$ is locally étale.

Let U be the normalization of the main component of $X' \times_{\mathbb{P}^n} A'$ and $h : U \to X'$ the corresponding morphism. We let $E \subset (f' \circ h)^{-1}(v') \subset U$ be the component corresponding to $(v', a') \in Y' \times_{\mathbb{P}^n} A'$. Then, the morphism $(U, E) \to (Y' \times_{\mathbb{P}^n} A', (v', a')) \to (A, a)$ is étale locally (on the base) isomorphic to $(X, f^{-1}(v)) \to (Y, v) \to (\mathbb{P}^n, p)$.

Let $\nu : T \to U$ be a birational morphism such that ν is an isomorphism over a neigborhood of $E \subset U$ and $T \setminus \nu^{-1}(E)$ is smooth. Let $\lambda : T \to A$ be the induced morphism. It is clear from what we have observed above that $\lambda(E)$ is one of the components of the support of $R^1\lambda_*\omega_T \neq 0$ and T has the required singularities.

Assume now that the $\operatorname{char} k = p > 0$ and let $f : X \to Y$ be a birational morphism of varieties such that X is smooth and $R^i f_*\omega_X \neq 0$ for some $i > 0$. This i will be fixed for the rest of the proof. The existence of such morphisms is well-known (see (3.12)) and an explicit example in dimension 7 is given in (3.11). Further let A be an abelian variety of the same dimension as X and Y and set $n = \dim A = \dim X = \dim Y$. There are embeddings $Y \subset \mathbb{P}^{m_1}$, $A \subset \mathbb{P}^{m_2}$ and $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2} \subset \mathbb{P}^M$. Let H be a very ample divisor on $\mathbb{P}^M$ and $U \subset Y \times A$ the intersection of n general members $H_1, \ldots, H_n \in |H|$ with $Y \times A$. By choice the induced maps $h : U \to Y$ and $a : U \to A$ are generically finite, U intersects $v \times A$ transversely so that $V = U \cap (v \times A)$ is a finite set of reduced points and $U \setminus V$ is smooth by Bertini's theorem (cf. [Har77, II.8.18] and its proof). It follows that any singular point $u \in U$ is a point in V and (U, u) is locally isomorphic to (Y, v). We claim that a is finite in a neighborhood of $u \in U$. Consider any contracted curve i.e. any curve $C \subset U \cap (Y \times a(u))$. We must show that $u \notin C$. Let $\nu : T \to U$ be the blow up of U along V and $\tilde{C}$ the strict transform of C on T. We let $\mu : \mathrm{Bl}_V\mathbb{P}^M \to \mathbb{P}^M$, $E = \mu^{-1}(u) \cong \mathbb{P}^{M-1}$ and we denote $h_i = \mu_*^{-1}H_i|_E$ the corresponding hyperplanes. To verify the claim it suffices to check that $\nu^{-1}(u) \cap \tilde{C} = \emptyset$. But this is now clear as $\nu^{-1}(u) \cong Z \subset \mathbb{P}^{M-1}$ and the h_i are general hyperplanes so that $Z \cap h_1 \cap \ldots \cap h_n = \emptyset$ as Z is $(n-1)$-dimensional.

Let $\lambda = a \circ \nu : T \to A$ be the induced morphism. By construction the support of the sheaf $R^i\nu_*\omega_T$ is V. Since a is finite on a neighborhood of $u \in U$, it follows that $0 \neq a_*R^i\nu_*\omega_T \subset R^i\lambda_*\omega_T$ and hence $R^i\lambda_*\omega_T \neq 0$ for the same $i > 0$. □

4. Main result

Proposition 4.1. *Assume that $\lambda : X \to A$ is generically finite on to its image where X is a projective Cohen Macaulay variety and A is an abelian variety. If $\mathrm{char}(k) = p > 0$, then we assume that there is an ample line bundle L on A whose degree is not divisible by p. If $R^i\pi_{\widehat{A}*}\mathscr{L} = 0$ for all $i < n$, then $R^i\lambda_*\omega_X = 0$ for all $i > 0$.*

Proof. By Theorem A of [PP11], $R^i\Phi(\mathscr{O}_X) = R^i\pi_{\widehat{A}*}\mathscr{L} = 0$ for all $i < n$, is equivalent to

$$H^i(X, \omega_X \otimes R^g\Psi(L^\vee)) = 0 \qquad \forall\ i > 0,$$

where L is sufficiently ample on $\widehat{A}$ and $R^g\Psi(L^\vee) = \lambda^*\widehat{L^\vee}$ (cf. (2.2)). It is easy to see that this is in turn equivalent to

$$H^i(X, \omega_X \otimes \lambda^*(\widehat{t^*_{\widehat{a}}L^\vee})) = 0 \qquad \forall\ i > 0,\ \forall\ \widehat{a} \in \widehat{A},$$

where L is sufficiently ample on $\widehat{A}$. By [Muk81, 3.1], we have $\widehat{t^*_{\widehat{a}}L^\vee} = \widehat{L^\vee} \otimes P_{-\widehat{a}}$ and hence $H^i(X, \omega_X \otimes \lambda^*(\widehat{L^\vee} \otimes P_{-\widehat{a}})) = 0$. Thus, by cohomology and base change, we have that

$$\mathbf{R}\widehat{S}(\mathbf{R}\lambda_*\omega_X \otimes \widehat{L^\vee}) =^{(2.1.2)} \mathbf{R}\Phi(\omega_X \otimes \lambda^*\widehat{L^\vee}) = R^0\Phi(\omega_X \otimes \lambda^*\widehat{L^\vee}).$$

In particular $\mathbf{R}\lambda_*\omega_X \otimes \widehat{L^\vee}$ is WIT-0.

Claim 4.2. For any ample line bundle M on A, we have that

$$H^i(X, \omega_X \otimes \lambda^*(\widehat{L^\vee} \otimes M \otimes P_{-\widehat{a}})) = 0 \qquad \forall\ i > 0,\ \forall\ \widehat{a} \in \widehat{A}.$$

Proof. We follow the argument in [PP03, 2.9]. For any $P = P_{-\widehat{a}}$, we have

$$H^i(X, \omega_X \otimes \lambda^*(\widehat{L^\vee} \otimes M \otimes P)) = R^i\Gamma(X, \omega_X \otimes \lambda^*(\widehat{L^\vee} \otimes M \otimes P)) =^{\text{P.F.}}$$

$$R^i\Gamma(A, \mathbf{R}\lambda_*\omega_X \otimes \widehat{L^\vee} \otimes M \otimes P) = \mathrm{Ext}^i_{D(A)}((M \otimes P)^\vee, \mathbf{R}\lambda_*\omega_X \otimes \widehat{L^\vee}) =^{(2.1.1)}$$

$$\mathrm{Ext}^{i+g}_{D(\widehat{A})}(R^g\widehat{S}((M \otimes P)^\vee), R^0\Phi(\omega_X \otimes \lambda^*\widehat{L^\vee})) =$$

$$H^{i+g}(\widehat{A}, R^0\Phi(\omega_X \otimes \lambda^*\widehat{L^\vee}) \otimes R^g\widehat{S}((M \otimes P)^\vee)^\vee) = 0 \qquad i > 0.$$

(The third equality follows as $M \otimes P$ is free, the fifth follows since $R^g\widehat{S}(M \otimes P)^\vee$ is free and the last one since $i + g > g = \dim \widehat{A}$.) $\square$

Let $\phi_L : \widehat{A} \to A$ be the isogeny induced by $\phi_L(\widehat{x}) = t^*_{\widehat{x}}L \otimes L^\vee$, then $\phi_L^*\widehat{L^\vee} = L^{\oplus h^0(L)}$. We may assume that the characteristic does not divide the degree of L so that ϕ_L is separable. Let $X' = X \times_A \widehat{A}$, $\phi : X' \to X$ and $\lambda' : X' \to \widehat{A}$ the induced morphisms. Note that $\phi_*\mathscr{O}_{X'} = \lambda^*(\phi_{L*}\mathscr{O}_{\widehat{A}}) = \lambda^*(\oplus P_{\alpha_i})$ where the α_i are the elements in

$K \subset \widehat{A}$, the kernel of the induced homomorphism $\phi_L : \widehat{A} \to A$. By the above equation and flat base change

$$H^i(X', \omega_{X'} \otimes \lambda'^* \phi_L^*(\widehat{L^\vee} \otimes M)) = \bigoplus_{\alpha \in K} H^i(X, \omega_X \otimes \lambda^*(\widehat{L^\vee} \otimes M \otimes P_\alpha)) = 0$$

for all $i > 0$. But then $H^i(X', \omega_{X'} \otimes \lambda'^*(L \otimes \phi_L^* M)) = 0$ for all $i > 0$. Note that if M is sufficiently ample on A then so is $L \otimes \phi_L^* M$ on $\widehat{A}$. It follows by an easy (and standard) spectral sequence argument that $R^i\lambda'_*\omega_{X'} = 0$ for $i > 0$. Since ω_X is a summand of $\phi_*\omega_{X'} = \mathbf{R}\phi_*\omega_{X'}$, and $\mathbf{R}\lambda_*\mathbf{R}\phi_*\omega_{X'} = \mathbf{R}\phi_{L*}\mathbf{R}\lambda'_*\omega_{X'}$, it follows that $R^i\lambda_*\omega_X$ is a summand of $R^i\lambda_*\phi_*\omega_{X'} = \phi_{L*}R^i\lambda'_*\omega_{X'}$ and hence $R^i\lambda_*\omega_X = 0$ for all $i > 0$. $\square$

Proof of (1.3). Immediate from (3.13) and (4.1). $\square$

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