Cavalieri's Principle: Area and Volume

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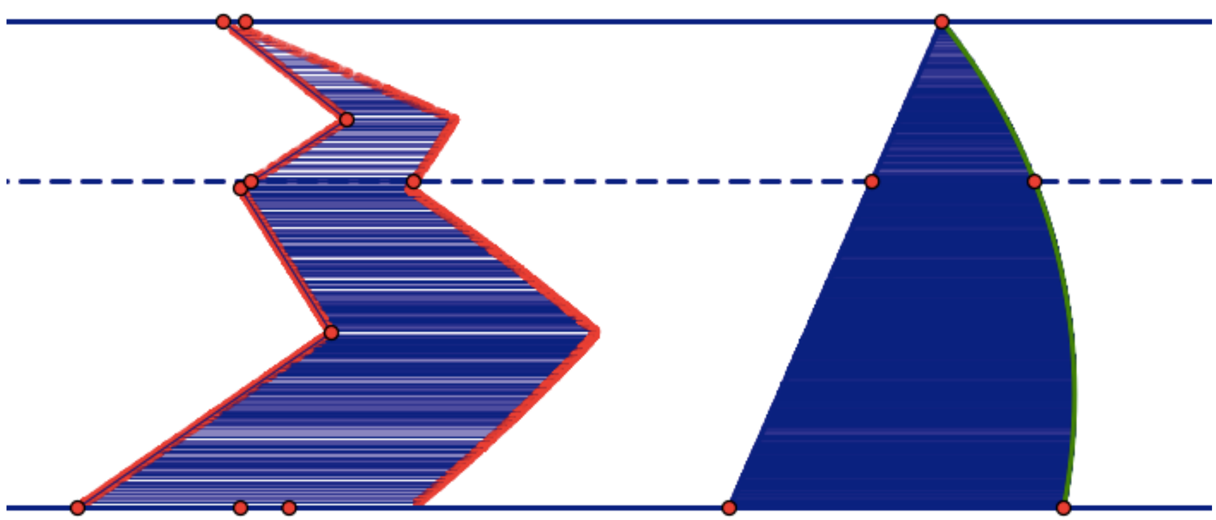
NWMI 2013

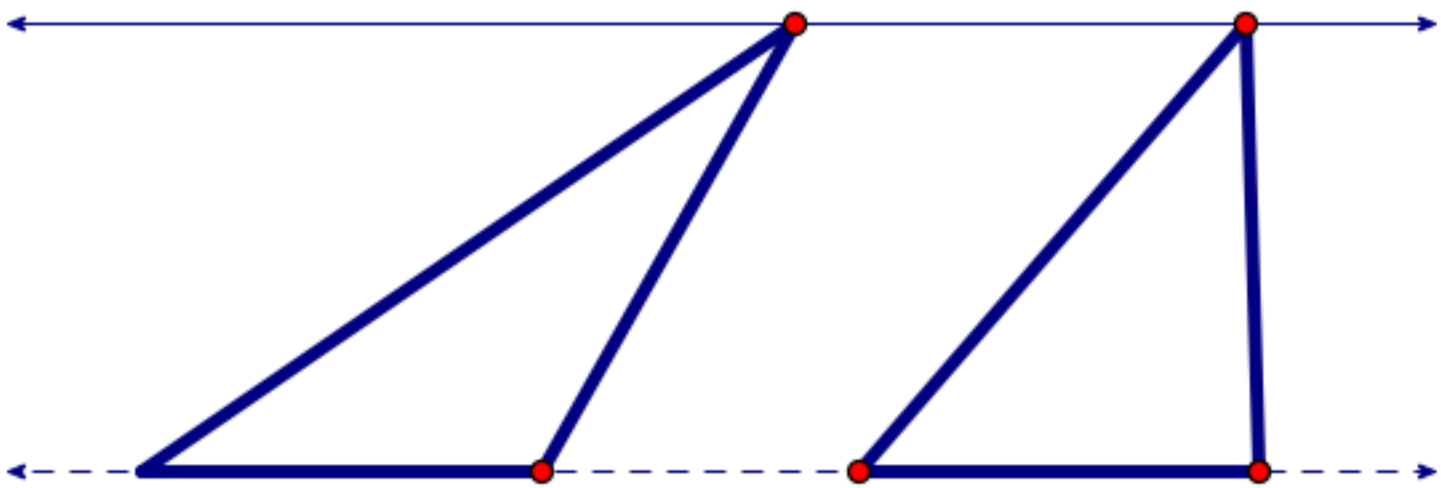
Cavalieri for Area

- 2-dimensional case: Suppose two regions in a plane are included between two parallel lines in that plane.
If every line parallel to these two lines intersects both regions in line segments of equal length, then the two regions have equal areas.



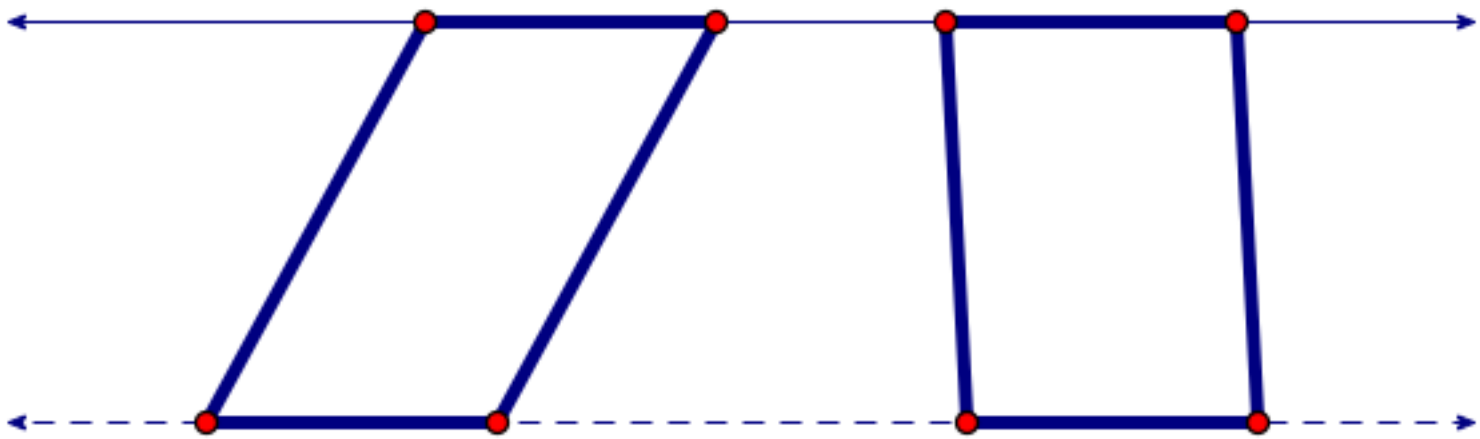
More Typical Examples

- Two triangles with congruent bases and the same height have the same area.

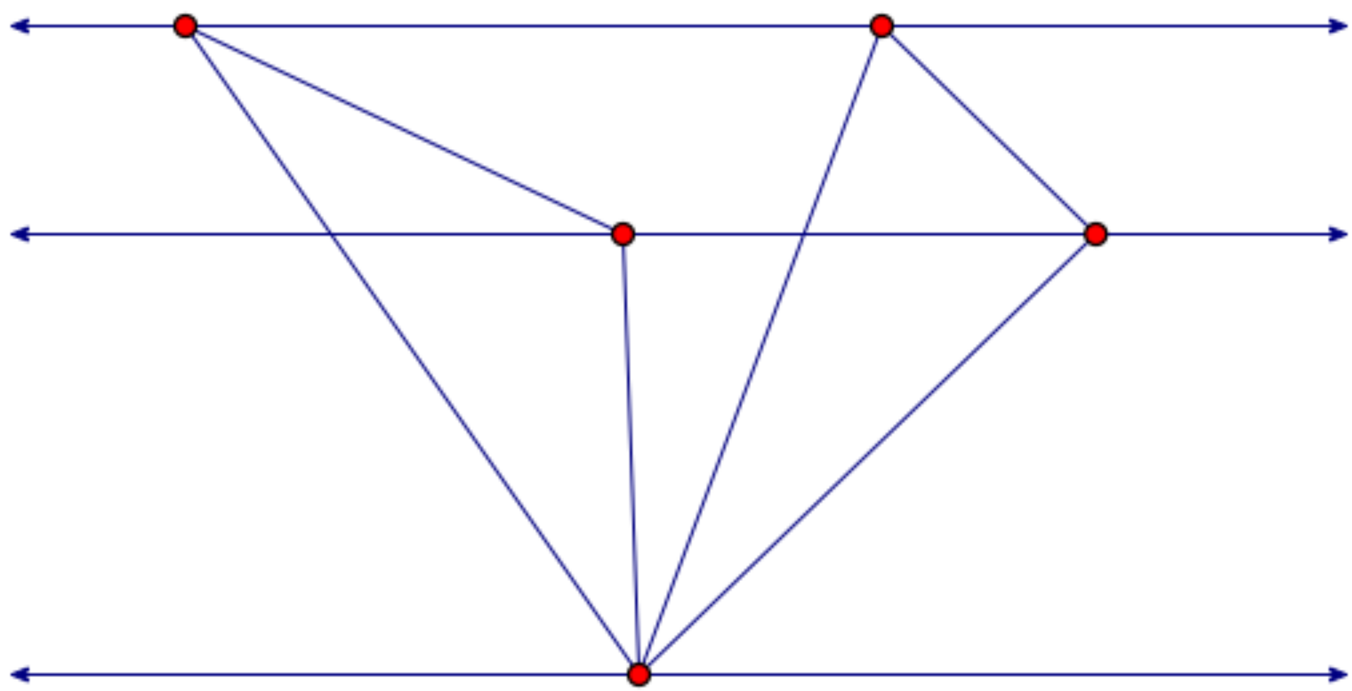


Parallelograms

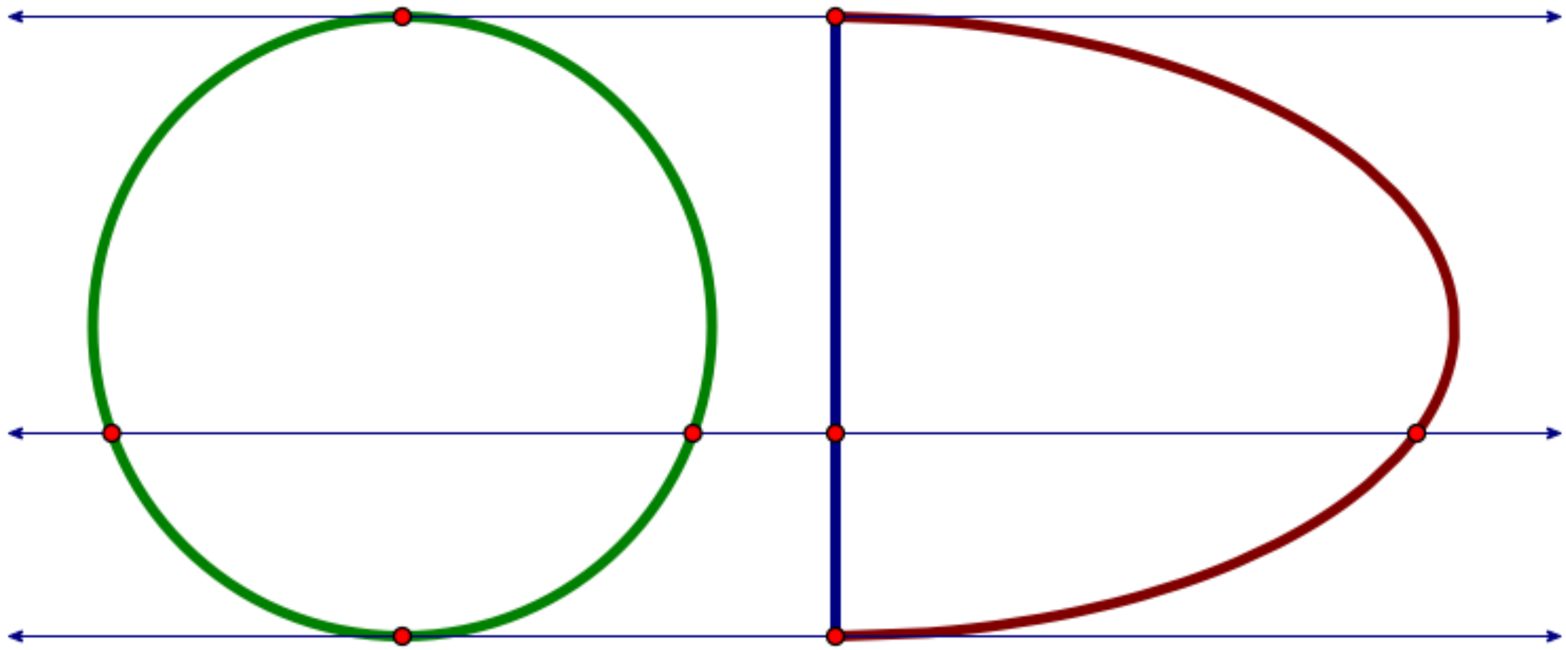
- Parallelograms with congruent bases and the same height.



Even if there is no common base



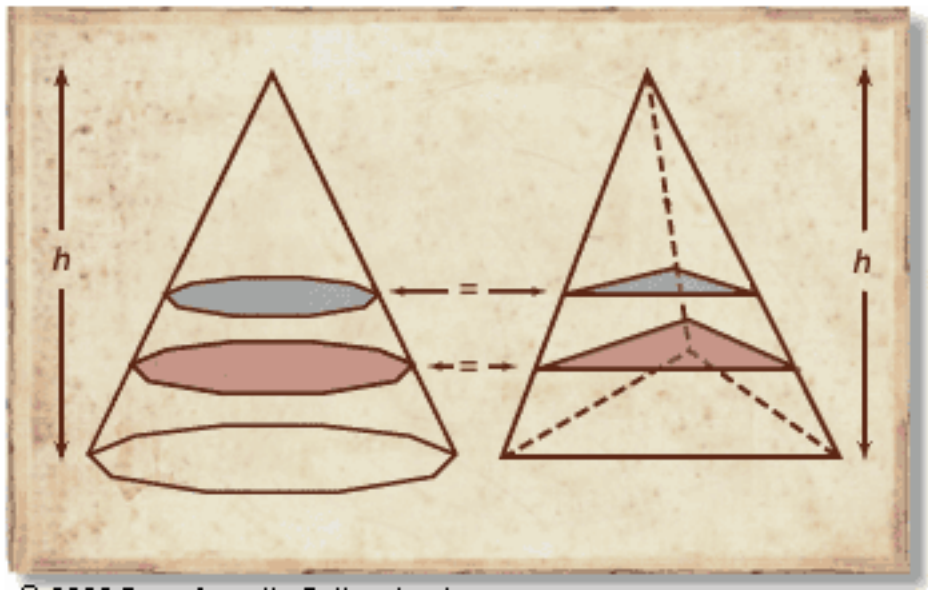
Other Plane Shapes



Cavalieri for Volume

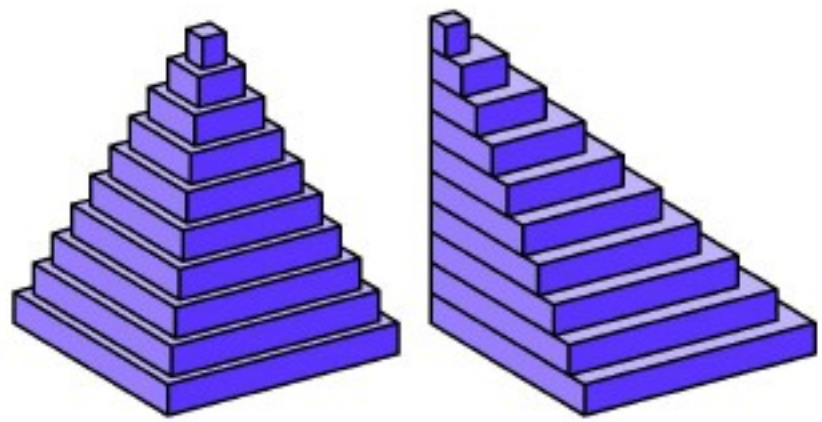
- 3-dimensional case: Suppose two regions in three-space (solids) are included between two parallel planes. **If every plane parallel to these two planes intersects both regions in cross-sections of equal area, then the two regions have equal volumes.**

Two Pyramids with equal base area and equal height



Why are all the areas of slices equal if the bases are equal?

Key to why the volumes are equal

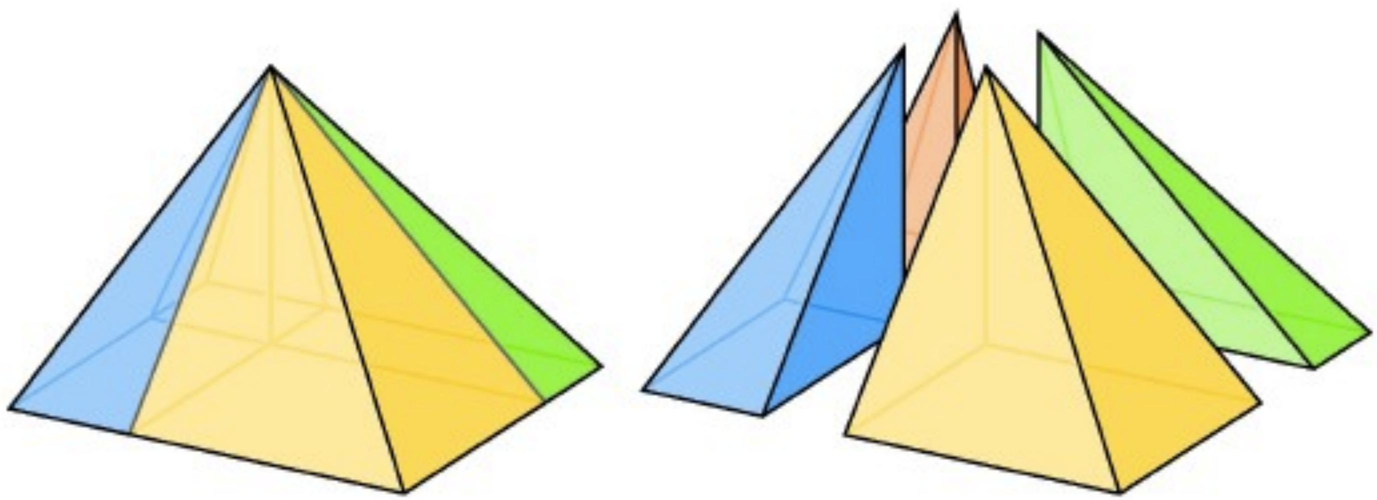


The volumes are approximately the sum of the volumes of the slabs of equal volume. The exact volume is a limit.

Decks of cards and stacks of CDs or pennies provide examples of equal volume



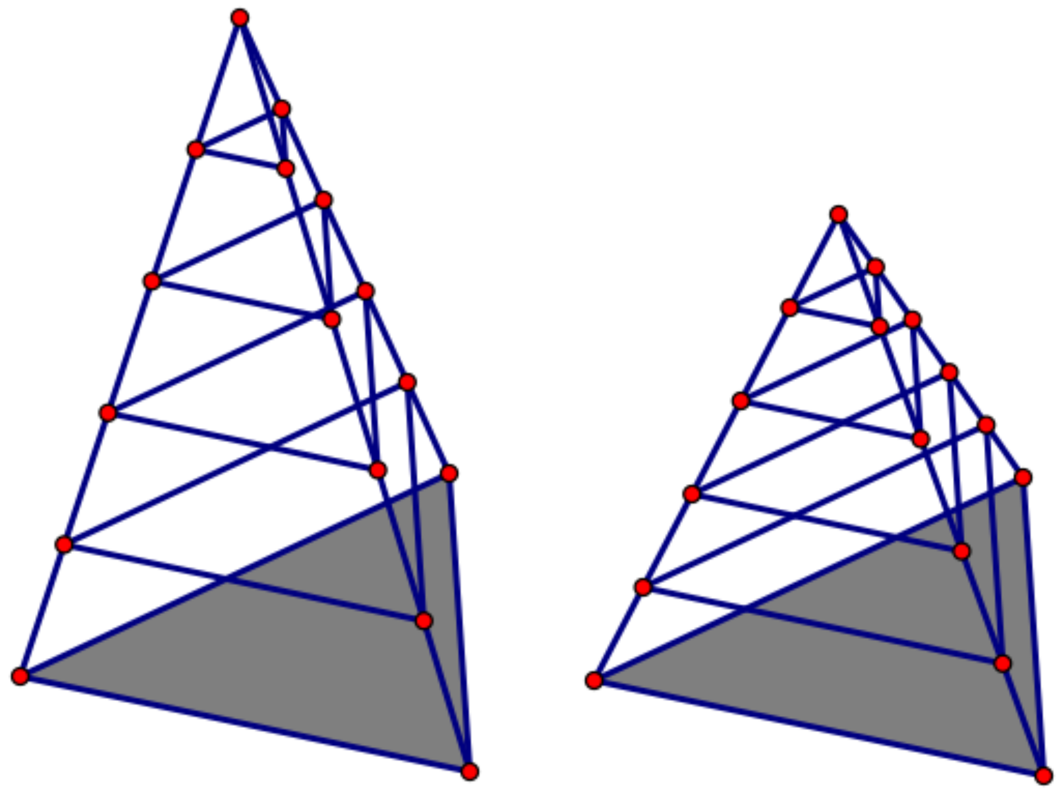
We have seen that for at least one pyramid inside a cube the volume = (1/3) base area x height



How can one show the formula for all pyramids?

Scaling the Heights

- If two pyramids have the same base but different heights, then the **volumes are in the same proportion as the heights**. Imagine two stacks of cards with the same number of cards but one is thicker than the other.

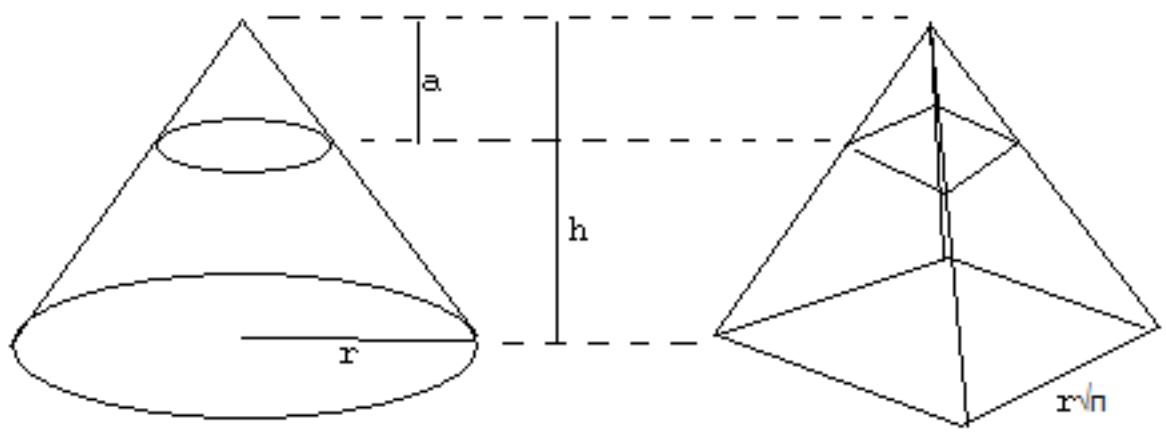


Changing the Base

- If two pyramids have bases of different shape but the same area, one can break up the bases (approximately) into congruent small squares whose area sum is approximately the area of the base, so if one knows the formula for square bases, the formula is true for all bases.
- Thus Volume = (1/3) area of base x height holds for all "pyramids".

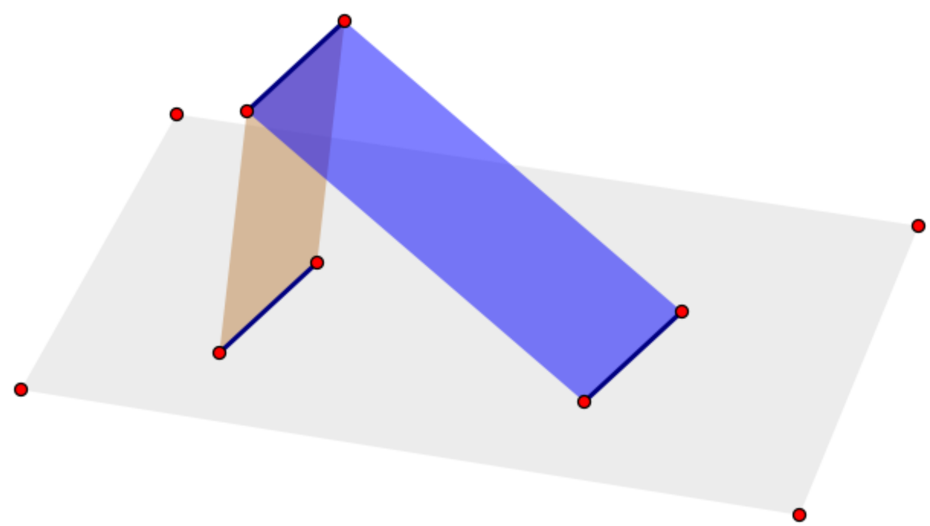
What's a pyramid?

- Cones count as pyramids in the previous discussion. The formula is the same for the same reasons.



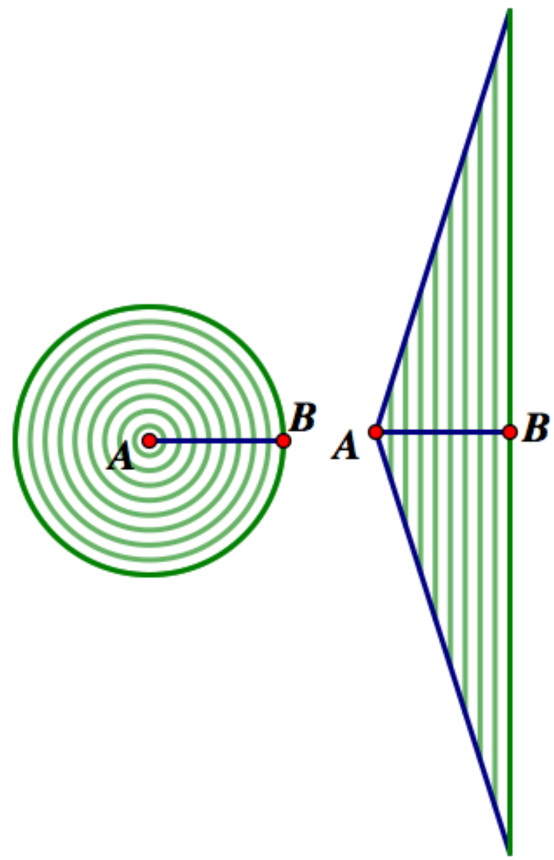
When Cavalieri is *NOT* true

- Do *NOT* try to apply Cavalieri to areas of planes or surfaces in 3-space. This is just not true. Think of the example of a two slanting rectangles of the same height but different slope.



Circle Area = (1/2) radius x circumference

- Instead of an arrangement of straight sticks, one can view a circle as a collection of concentric rings. When straightened, the area formula for a triangle becomes the area formula for a circle.

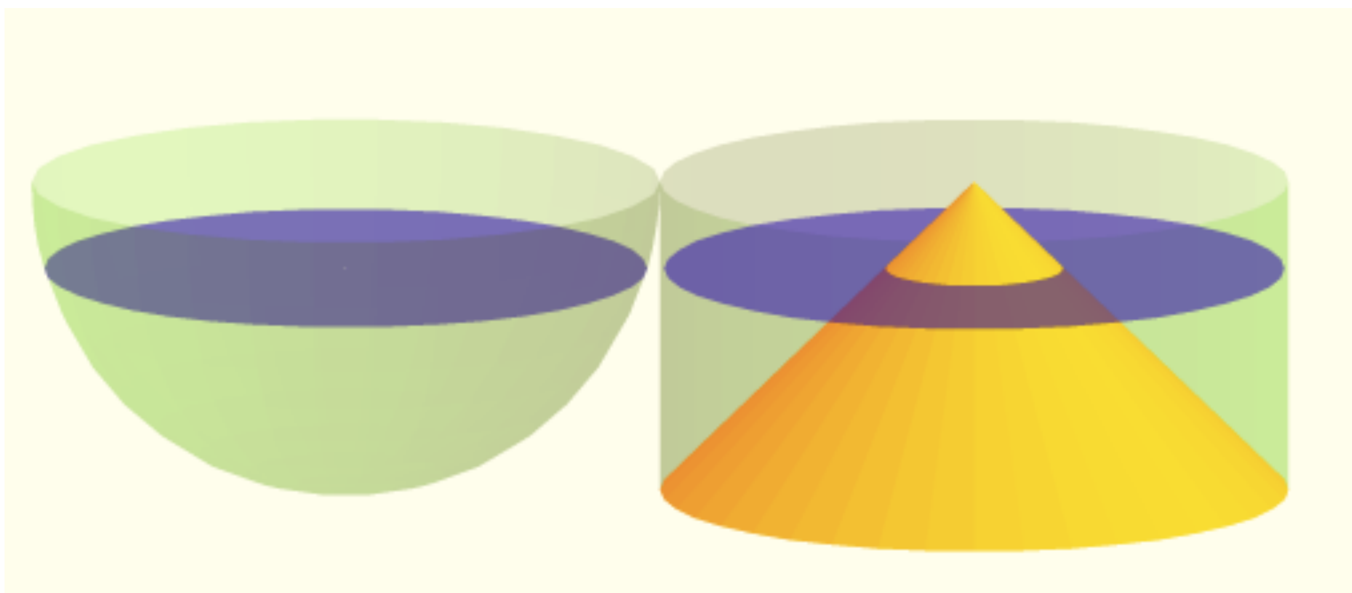


Sphere Volume = (1/3) radius x spherical area

- The sphere can be cut (approximately) into pyramids. The "base area" = $4\pi r^2$.

Even cooler proof of sphere volume

- Use the Pythagorean Theorem to see that these slices have the same area.



Application: modeling in stereology

Explore by topic:

- Home
- Introduction
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  - Symbols Used in Stereology
- The Optical Fractionator
  - Tissue Preparation for Stereology
  - Selecting the XY fraction of the section
  - Criteria for counting cells
- Sampling
  - Fractionator Principle
  - SRS Preferential Sections
  - VUR Sections
  - IUR Sections
  - Faver Sections
- Counting Frame
  - Counting Rules
  - Disector
  - Optical Disector
  - Edge Effect
  - Optical Disector vs. Unbiased Brick
- Coefficient of Error
  - CE and Biological Variability

Cavalieri Estimator

The Cavalieri estimator is used to estimate volume. If you follow the rules you will arrive at an unbiased estimate of volume. The precision of sampling should also be considered. How many sections and how many points per section are needed? There are several ways to think about this (see precision), and come up with a starting point for a pilot study. The following is an example of a pilot study but the number of sections and number of points will depend on your material.

Begin by preparing a set of sections. The Cavalieri estimator is prone to overprojection if the set of slices are thick. The sections are sampled using the fractionator principle. A possible starting point for a pilot study might be 10 to 15 sections. Suppose that the tissue is sectioned into 80 sections. If every 6th section is used, then around 13 sections are used. The fraction of sections used is the section sampling fraction or ssf.

The Cavalieri estimator is usually performed using a point grid. The points need to be appropriately spaced. The points are usually spaced equally both across and down. With computer-assisted stereology it is very easy to mark points so don't worry if you think you will be doing too much work because the points are too close together. Make the points dense enough to cover any bays, peninsulas and isthmuses in the tissue. Because the points don't cost much work, the parameter to adjust to affect precision is the section interval.

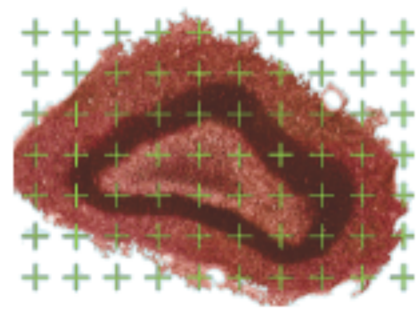

Figure 1. Point grid overlaid onto tissue.

Toss the point probe randomly onto each section. Count how many points hit the region of interest. Keep a record of how many points hit each section. The results are used to compute the volume and the CE.

Thanks, Sr. Cavalieri … & Zu Geng

Bonaventura Cavalieri(1598-1647) was an italian mathematician. He was a precursor of infinitesimal calculus. Cavalieri, Kepler and other mathematicians, who lived during the century preceding Newton and Leibniz, invented and used intuitive infinitesimal methods to solve area and volume problems.

Twenty years after the publication of Kepler's *Stereometria Doliorum*, Cavalieri wrote a very popular book: *Geometria indivisibilibus* (1635). In this book, the Italian mathematician used what is now known as Cavalieri's Principle.

Zu Geng, born about 450, was a chinese mathematician who used what is now know as the Principle of Liu Hui and Zu Geng to calculate the volume of a sphere. Liu-Zu theory is equivalent to Cavalieri's Principle.