Constructions with Compass Alone

James King, king@math.washington.edu
University of Washington, Dept. of Mathematics 354350, Seattle, WA 98195-4350
My website (with these materials): www.math.washington.edu/~king

In this session, we will carry out some interesting and perhaps surprising constructions with compasses alone. We will also talk about some of the math that we and students could see in these constructions. Finally, we will compare what can be constructed with compasses alone vs. straightedge and compass. The session will follow the outline below. If time begins to press, we may abbreviate some points. The materials from the session will be put on my website after this meeting.

You are seated at tables in groups. Please work with your neighbors and discuss together questions and answers

Outline of this session

1) Equilateral triangle construction and related constructions
2) The meaning of "construct"
3) Relations in two intersecting circles
4) Constructing a square with a compass alone
5) Midpoint construction without a straightedge
6) Inversion in a circle and division of an interval
7) Using line reflection to intersect a circle with a line -- without a straightedge. What does it mean and how to do it
8) Mohr-Mascheroni Theorem: what can one construct with compass alone?
9) Intersecting two lines without a straightedge (outline only)

NOTATION:

- AB means segment AB
- Line AB means the line through A and B
- Circle AB means the circle with center A through B
- Circle (A, BC) or Circle (A, r) means the circle with center A and radius BC or r.

1. Equilateral Triangle Construction

- Near the middle of a sheet of paper, draw and label the two points A and B as here.

- Then use your compass to draw the circle AB and circle BA. Label the intersection points C and D.

Questions

a) What can you say about triangles ABC and ABD? A reason?
b) What can you say about the quadrilateral ACBD? A reason?
c) If the distance AB = d, what is the distance CD?

- Now draw circle CD and circle DC. Label their intersections E and F.

Questions

d) What can you say about triangles CDE and CDF? A reason?
e) Are points E and F on line AB? Why?
f) What are the distances AE and AF?

- Use a straightedge to draw in lightly the line AB and the line CD. Let the point M be the intersection of AB and CD.
- Also draw in the sides of triangle ABC.

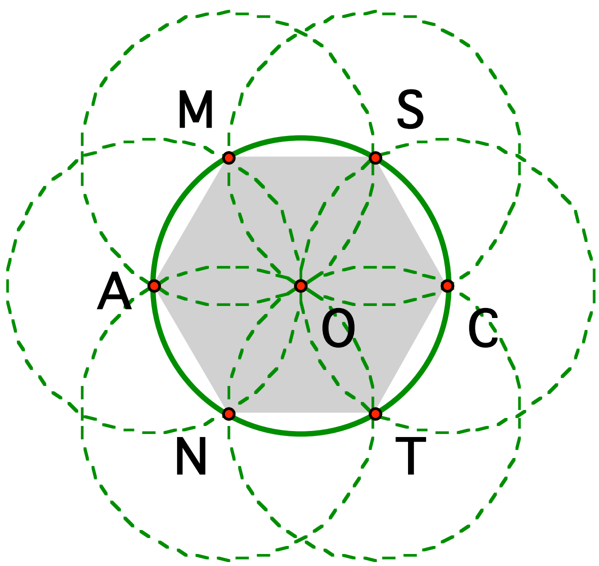
Questions

g) Let's think about what is construction. Which of the labeled points were constructed with compass alone? Which with straightedge and compass?
h) You have drawn point C and also segment AC in your figure. Have you **constructed** the points on the segment AC?

2. Constructing a square

Draw a circle OA. Our goal will be to construct a square ABCD inscribed in the circle. But we will first start out by extending our construction of equilateral triangles to the construction of an inscribed hexagon.

- Draw a circle OA with the compass.
- Then construct the intersections PQ of circle AO with circle OA (you can draw the whole circle or just the arcs needed for the intersections.
- Next construct the new intersection S of circle PA with circle OA (one of these is A already).
- Continue this process until you have formed the 6 vertices of a hexagon inscribed in OA.



Questions

i) If we set the distance OA to be 1 unit, what is the distance AS?
j) If points B and D were to be constructed on the circle so that ABCD is a square, what would the distance AB be?
k) If we have already constructed points at distances 1, 2 ,and the square root of 3, how can we construct the needed distance AB?

- Construct the intersection points X and Y of the circles AS and CM.
- Use these new points to construct B and D so that ABCD is a square.

3. Midpoint Construction with Compass Only

Given two points E and F, from Part 1 we know how to construct the midpoint of AB using intersecting lines. It is not obvious at all how to do this with compasses, but it is possible.

Start by constructing a point G on line EF as shown so that FE = FG. You should use only compasses and the constructions with equilateral triangles from Parts 1 and 2.

You will be given step-by-step instructions and then we will ponder why the construction works.

- Construct the intersection points P and Q of circles EF and GE.
- Construct circles PE and QE and intersect them. One point of intersection is E. Call the new point M.

We will see that M is the midpoint of EF.

Questions

a) What isosceles triangles do you observe in the figure? What ones are similar?
b) Given that distance EF = 1, what are the sides of triangle EGP?
c) What are the sides of triangle EPM?
d) Why is M the midpoint?

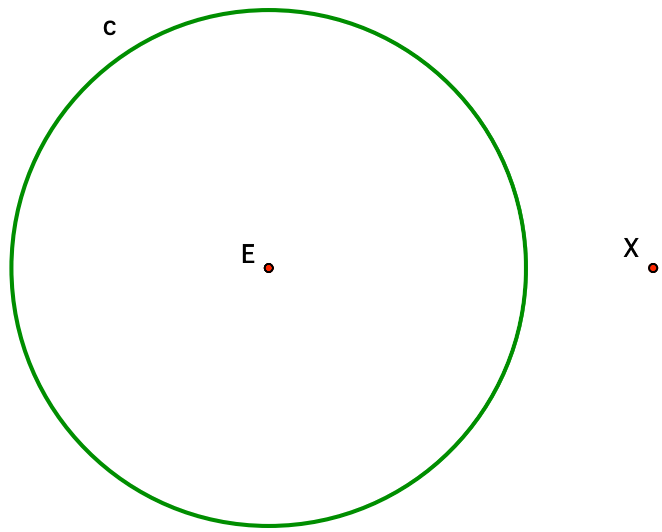
Some Additional Compass Construction Problems to Ponder

- Given points A and B and a positive integer n, construct a point C on ray AB so that AC = n AB.
- Given non-collinear points ABC, construct a point D so that ABCD is a parallelogram.
- Given points A and B, construct C and D so that ABCD is a square (this is not the same as the other square construction).

4. Inversion in a Circle

The midpoint construction on the previous page is a special case of a general transformation called inversion in a circle, which is a kind of reflection in a circle. On this page, we repeat the construction in another case and ask some general questions.

Draw a figure like this one: a circle c with center E and radius r. along with some point X outside the circle.



- Construct the intersection points P and Q of the circle XE with circle c.
- Continue as on the previous page to intersect circles PE and QE to construct a point Y.

Questions

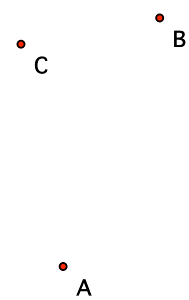
a) What isosceles triangles do you see in the figure? Which are similar?
b) If EX = x and the radius of c = r, what is the distance EY?
c) Special case of 2: If x happens to be = 2r, this is our previous construction. EY = (1/2)r
d) Special case of 3: If x = 3r, what is EY? Can we construct such an X with a compass?
e) Special case of positive integer n: If x = nr, what is EY? Can we construct such an X with a compass? How can we construct EY = (m/n)r for any rational m/n?

Definition: If P is any point and circle c has center E and radius r, we define the ***inversion*** of P in c as the point Q on ray EP so that EQ = r^2/EP.

5. Line reflection and intersection of a line and circle

Recall the definition of the reflection of a point in a line.

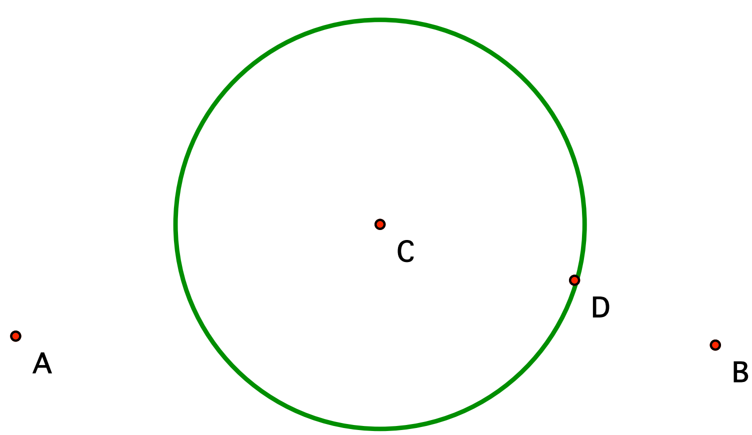
- Given a figure with the points A, B, C such as this, construct with compass only a point C1 so that C1 is the reflection of C in line AB.



- Add a point D to your figure and construct the reflection D1 of D in line AB.
- Now construct circle CD and circle C1 D1. Let the intersection point be P and Q.
- Explain why P and Q are the intersection points of circle CD with line AB.

Questions

a) How was it possible to construct points relative to a line without ever drawing the line?
b) In the construction of C1, what congruent triangles appear? How do you know they are congruent?
c) Explain why P and Q are the intersection points of circle CD with line AB as shown in the figure below.



6. Mohr-Mascheroni Theorem

We have seen that many of the points that we normally construct with a straightedge and compass can in fact be constructed without the straightedge.

So a **natural question** is this: which points can be constructed using the compass alone and which ones need the straightedge as well?

The answer is given by the Mohr-Mascheroni Theorem, which states that **any points that can be constructed with straightedge and compass and be constructed with compass alone.** This seems very surprising at first consideration, but we are already part way to proving this theorem.

The proof consists of noting that straightedge and compass construction is based on

a) Intersecting two lines
b) Intersecting a line and a circle
c) Intersecting two circles

We can certainly do (c) with a compass. We have also seen how to do (b) except is the special (and also difficult and important) special case of when the center of the circle lies on the line. We will outline point (a) below.

In practice, instead of laboriously duplicating the straightedge constructions, one can sometimes find clever ways to do compass constructions more simply, as we did in the midpoint construction using inversion.

History of the theorem

The Italian mathematician Lorenzo Mascheroni proved this theorem and published it in 1797. However, in 1928 the Danish mathematician Hjelmslev discovered an old book written by G. Mohr and published in 1672 in Amsterdam. It was written in Latin with the (translated) title of The Danish Euclid. In this book was a complete proof of the same theorem. So now the theorem is called the Mohr-Mascheroni theorem and compass-only constructions are Mohr-Mascheroni constructions.

References

The website cut-the-knot has proofs, discussion and print bibliography. For example this page:
http://www.cut-the-knot.org/do_you_know/compass.shtml

Role of constructions in geometry teaching

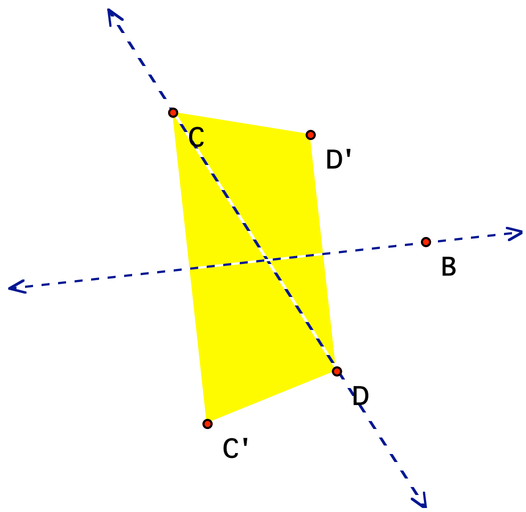
- Use of geometry concepts reinforced
- Visualization and visual reasoning
- Construction logic close to proof logic
- Problem solving
- History and culture

7. Intersection of two lines and Mohr-Mascheroni Theorem

Intersection of two lines

The other tricky basic construction with compass only is the intersection of line AB with line CD. There will not be time to carry this out during the session, so at the end of these sheets we will given a reference on the web. But it is possible to give a quick outline of how to intersect two lines.

Step one: Reflect C and D across line AB to get points C' and D'. Then either CC'D'D or CC'DD' is a trapezoid and the intersection of line AB with line CD is the intersection of line CD with line C'D'



Step two: The ratio of lengths CC' and DD' can be used to locate and construct on line CD the intersection point of CC' and DD'. (The construction of proportional segments can also be done with compasses by construction proportional chords).
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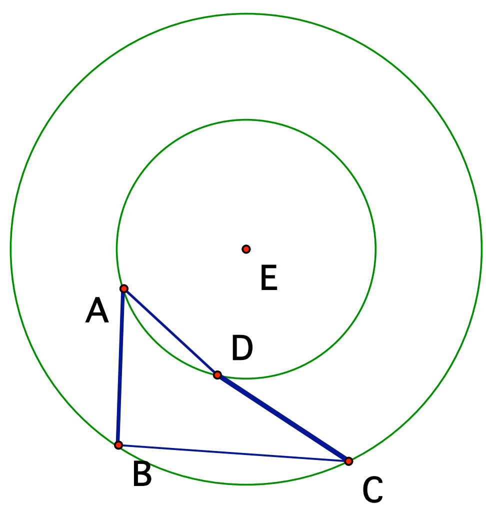
Proportion problem: Given 3 lengths a, b, c find x so that b/a = x/c.



Very short outline of proportion method: (We assume c < b, or else rewrite the problem as c/a=x/b.)

- Pick a point O and construct circles OA and OB with radius a and b. Construct a point C on circle OB so that the chord BC has length c.
- Let AB = u. Then construct the circle with center C and radius u to get a point D on circle EA so that CD = u. There are two such points but D can be chosen so that the rotation with center E that takes ray EA to EB also takes ED to EC (this takes a bit of reasoning).
- With this choice of D, the triangles OCD and OEF are similar and so EF is the required x.