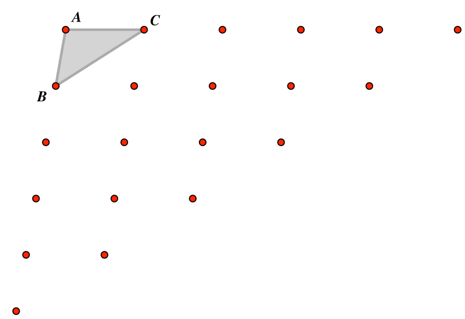
Cutting Up and Taping Together

James King
University of Washington
Email: king@math.washington.edu,   Website: www.math.washington.edu/~king

1. Scaling up a triangle

On your table take a sheet of paper with dotes and a shaded triangle, looking something like this (but the shape of your triangle may be different).



- Extend the sides AB and AC of triangle ABC to segments AB_2 and AC_2 to form a new triangle AB_2C_2 similar to ABC. The new sides are parallel to the corresponding sides of ABC and the lengths are doubled.
- Now draw more segments so that you have dissected AB_2C_2 into disjoint triangles all congruent to ABC. How many triangles are there in this dissection.
- Continue this by extending AB to AB_2 AB_3, AB_4, AB_5, so that the length of ABn is n times the length of AB. Then extend AC in the same manner to get segments AC_2 AC_3, AC_4, AC_5. Then again the triangles ABnCn are similar to ABC.
- Again connect dots with segments to dissect these triangles into triangles congruent to ABC. Then answer the following questions in the table below.
    1. How many triangles congruent to ABC are in your dissection for each case of n = 1 2, 3, 4, 5.
    2. How many of the triangles can be obtained from ABC by translation? (We call these "up" triangles.)
    3. How many of the triangles cannot be obtained from ABC by translation? (We call these "down" triangles.)

What special facts, relationships, formulas, about scaling and special numbers do you see in your table?

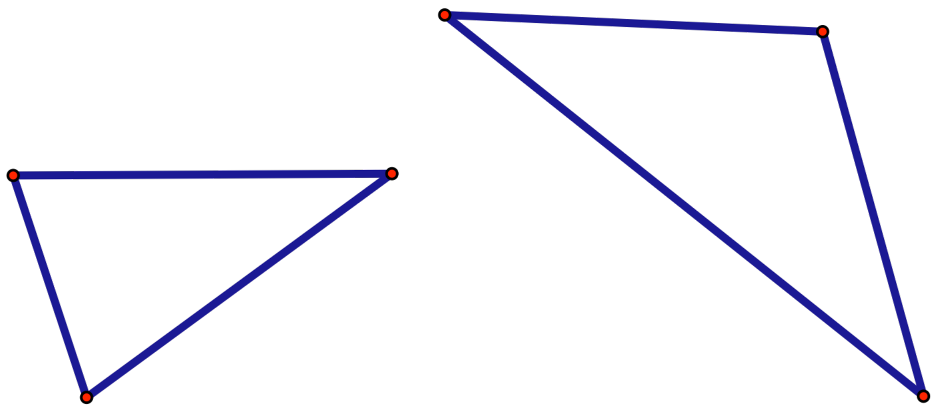
N	Total # triangles	#up triangles	#down triangles
2			
3			
4			
5			

Note: Rather than drawing as we have done today, this can also be done by taping togethercongruent triangles.

2. Merging Two Special Triangles to Scale Up

Cut out the triangles from you second sheet. (These may already be cut up for you.)

Now you have a set of Acute isosceles triangles that we will call As and Obtuse isosceles triangles that we will call Os.



- Put one A and one O together to form a new triangle A_1. Can you convince yourself that this new triangle is similar to A?
- If in fact the triangle A_1 is similar to A, use the relations that you see to deduce the **measure of each of the angles** of the triangles A and O.

Counting

Now combine some As and Os to form a triangle O_1 that is similar to O and is scaled up by the same scaling ratio as A to A_1. In other words, A_1 and O_1 should fit together just as A and O do. How many As and how many Os does it take to make O_1?

Now we will continue building up bigger A's and O's by exactly the same method. Form an A_2 from A_1 and O_1 as A_1 was formed from A and O. And the same for O_2.

Now again, we will make a table of the number of A's and O's to make each shape. Look for patterns. Can you figure out how to compute the numbers without making the shape? What is the rule?

N	#As	#Os	
A			
A1			
A2			
A3			
A4			
A5			

N	#As	#Os	
O			
O1			
O2			
O3			
O4			
O5			

In the last column, write the ratio of O/A in each row (if you have a calculator). Does this ratio appear to approach a limit?

The Ratios of the Sides in the Triangles

- Study the figure of A_1, which is made of one A and one O. You have already found the angles from this figure. Use the fact that A and A_1 are similar and that all 3 triangles are isosceles to find the ratio of the long side to the short side of A.

Regular Polygon

Form a regular polygon P from the smallest possible number of As and Os. What is the shape of this polygon P?

Then form larger polygons P_1, P_2, etc. from the larger A_1, A_2, etc. Again, make a table of the number of As and Os that make up P_n. What is the pattern here?

N	#As	#Os	
P			
P_1			
P_2			
P_3			
P_4			
P_5			

3. Silver Rectangles and Similarity

Take a regular sheet of 8.5x11 inch paper and fold the long sides in half to form two smaller "half" rectangles.

Is the half rectangle similar to the original? How can you tell visually? How can you tell numerically?

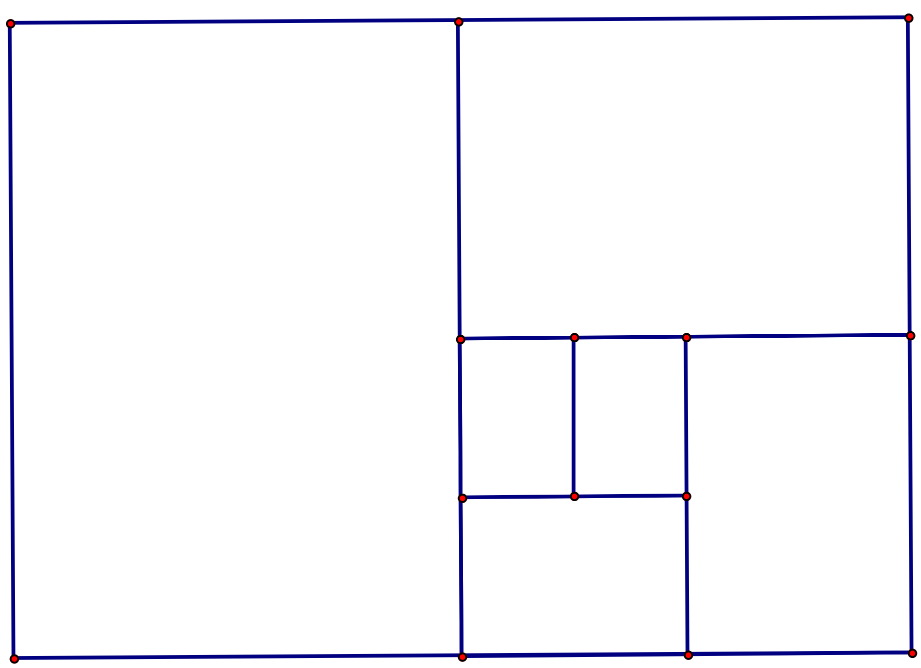
You also have a "skinnier" sheet of paper that is also 11 inches long. Fold this paper in the same way into half rectangles.

- Is the half rectangle similar to the original? How can you tell visually**? How can you tell numerically?**
- A rectangle with this property that half the rectangle is similar to the whole is called a silver rectangle. If you want to cut down a regular sheet of paper to a silver sheet, so that the length is still 11 inches, what is the width?
- Suppose you go into a store to buy printer paper and find that they are selling this silver shape of paper. Where are you?

Silver Spirals

Take a silver rectangle R and cut it in half to make two smaller rectangles, then cut one of the smaller ones in half and continue as shown

The rectangles spiral in to a limit point. Can you locate this point by geometry or numerically?



Silver in 3D

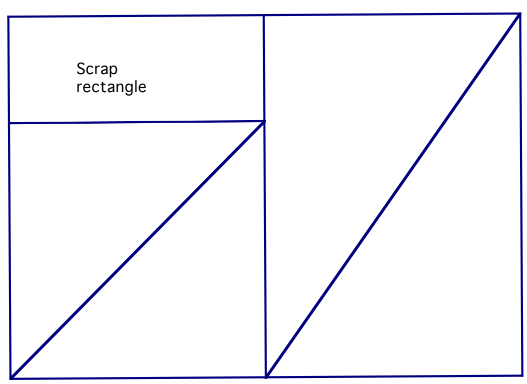
There are two interesting, related shapes that can be constructed from Silver Rectangles. Suppose the length of the silver rectangle = 2s. Then the shapes we need are the silver rectangle itself and also right triangles obtained by cutting a silver rectangle along a diagonal. Also the

Silver Prism

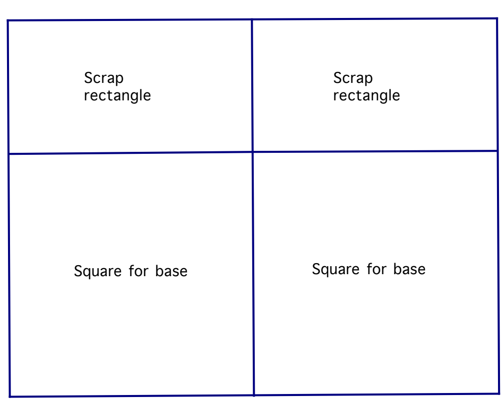
Suppose the width of a silver rectangle = w. Then cut out two squarea of side w and two isosceles right triangles with sides = w. Assemle a prism from these pieces. Can you see a connection with a familiar polyhedron?

Silver Pyramid

Cut up a silver rectangle into 4 right triangles (plus a scrap rectangle) as shown in this diagram (using folding to produce the cut lines). Two of the triangles are isosceles and two are obtaine by cutting a silver rectangle along its diagonal. If the length of the large silver rectangle in the figure = 2s, what are the other dimensions?



Also, cut one extra square of side = s from an 8.5 x 11 sheet as shown.



Now with the square as the base, attach the 4 triangles to form a square-based pyramid.

See how you can combine 3 of these pyramids to form a familiar polyhedron

Answers to some of the questions

- The number of triangles congruent to ABC in triangle ABnCn is n^2. The number of "up" triangles is n(n+1)/2 and the number of "down" triangles is n(n-1)/2

- The ratio long-side/short-side in the A triangles is the golden ratio
  $\phi = (1 + 5^{1/2})/2$.
  An important property is that $\phi^2 = \phi + 1$. The ratio of sides in the O triangles is the same.

- The regular polygon associated wth the golden ratio is the pentagon.

- The ratio of length/width in a silver rectangle is the square root of 2. Thus if a rectangle is 11 inches in length, the width will be approximately 7.79 inches. The paper in the workshop was down US letter paper cut down to this dimension. It was not A4 paper, which can be purchased in the US in various places.

- Printer paper sheets that are silver rectangles are the standard papers in Europe and many other countries. The standard printer size is called A4 paper. But this is also A5, A3, A2, etc.

- The right triangle obstained by cutting a silver rectangle along a diagonal has sides proportional to the square roots of 1, 2, 3.

- The prism formed from the silver rectangle is half a cube, formed by cutting a cube along a diagonal plane.

- The pyramid formed from the silver rectangle is one-third a cube. Each pyramid is formed by joining a given vertex to the poinots of one of the square sides not adjacent to the vertex.