How Transformations Help us Think about Geometry

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CCSS, Rigid Motions, Dilations, etc.

- The topic of transformations in geometry has fascinated some of us for a long time, but it has become a subject of wide interest right now because of the Common Core.
- But transformations have been at the center of how mathematicians think about geometry for a long time.

Quick History

- The principle of superposition (picking up a figure and laying it on another) was implicit in geometry since ancient times, but it was not an official axiom or principle in Euclid.
- Then in the 19th Century, mathematicians discovered new geometries; and the concept of symmetries and permutations led to breakthroughs in algebra.

Felix Klein

- In 1872, Klein – at the University of Erlangen – proposed a new perspective on Geometry in a paper known since as the ***Erlangen Program***.

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Klein's Idea

- A geometry is a set of objects with the rules determined by its symmetries, i.e., its transformations. Two geometries may have the same objects but different transformations.
- The properties of the geometry are properties that are not changed by the transformations.

4 Kinds of Transformations: 4Geometries of the Plane

- Rigid motions: Plane geometry of congruent figures that we know and love
- Similarity transformations: The familiar geometry with similar figures, ratios, etc.
- Affine (matrix) transformations: Geometry of computer animation. Rectangles and parallelograms the same, ditto circles and ellipses
- Continuous (topological) transformations: Any loop is a circle. Any path is a segment.

Back to K12 Geometry

- We won't pursue the theoretical thread any further. But it is important to keep in mind that transformations are central to the math of the last 150 years, so this is not some new-fangled notion in the CCSS.
- So a committee of leading mathematicians and educators recommended that US schools follow international models and base geometry on rigid motions.
- – in 1923! (National Committee on Math Requirements)

From 1923 to 2013: Common Core Samples

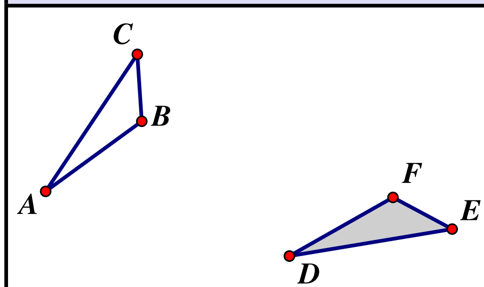
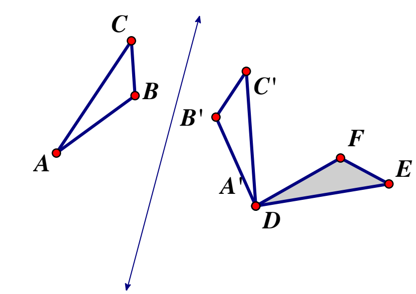
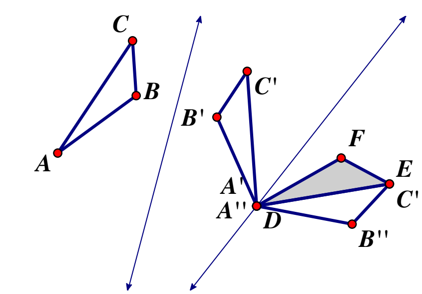
- Grade 8: *Verify experimentally the properties of rotations, reflections, and translations*
- Grade 8: *Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.*
- High School: *Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.*

SAS from Rigid Motions

- SAS: Given two triangles ABC and DEF so that angle ABC and angle DEF have equal measure, length AB = length DE, and length CB = length FE, then triangle ABC is congruent to triangle DEF.
- How do we prove this with rigid motions? Find a sequence of rigid motions that will take one triangle to the other given these assumptions.
- There is a choice of ways to do this. Start with a translation that takes B to E ... or start with a line reflection that takes B to E, or one could move A to B by a rotation. We will chose to use line reflections only. to keep our story simple.

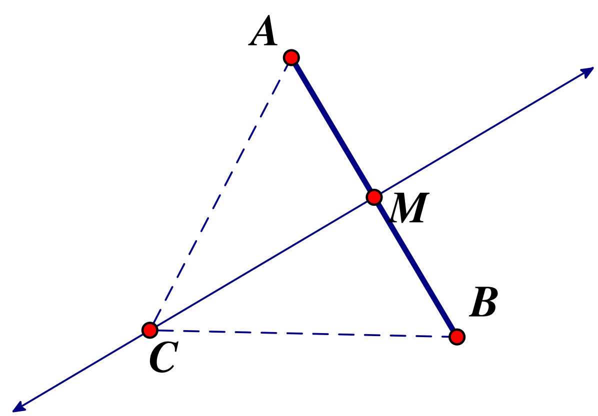
Executive Summary of the Proof of SAS

- Assume angle ABC = angle DEF; AB = DE; CB = FE.
- Here are the steps in a proof, but they are not a proof, since we need reasons why the steps work.
- The reasons will be explored on the next slide.

Step 1: Reflect A to D. ABC is reflected to A'B'C', with A' = D.	Step 2: Reflect B' to E in a line through D. A'B'C' is reflected to A''B''C'', with A'' = D and B'' = E. If C'' = F, stop.	Step 3: Reflect C'' to F in line DE. A''B''C'' is reflected to A'''B'''C''', with A''' = D, B''' = E, and C''' = E
		

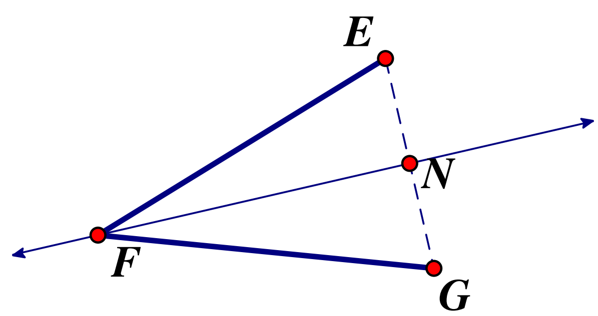
First steps (i)

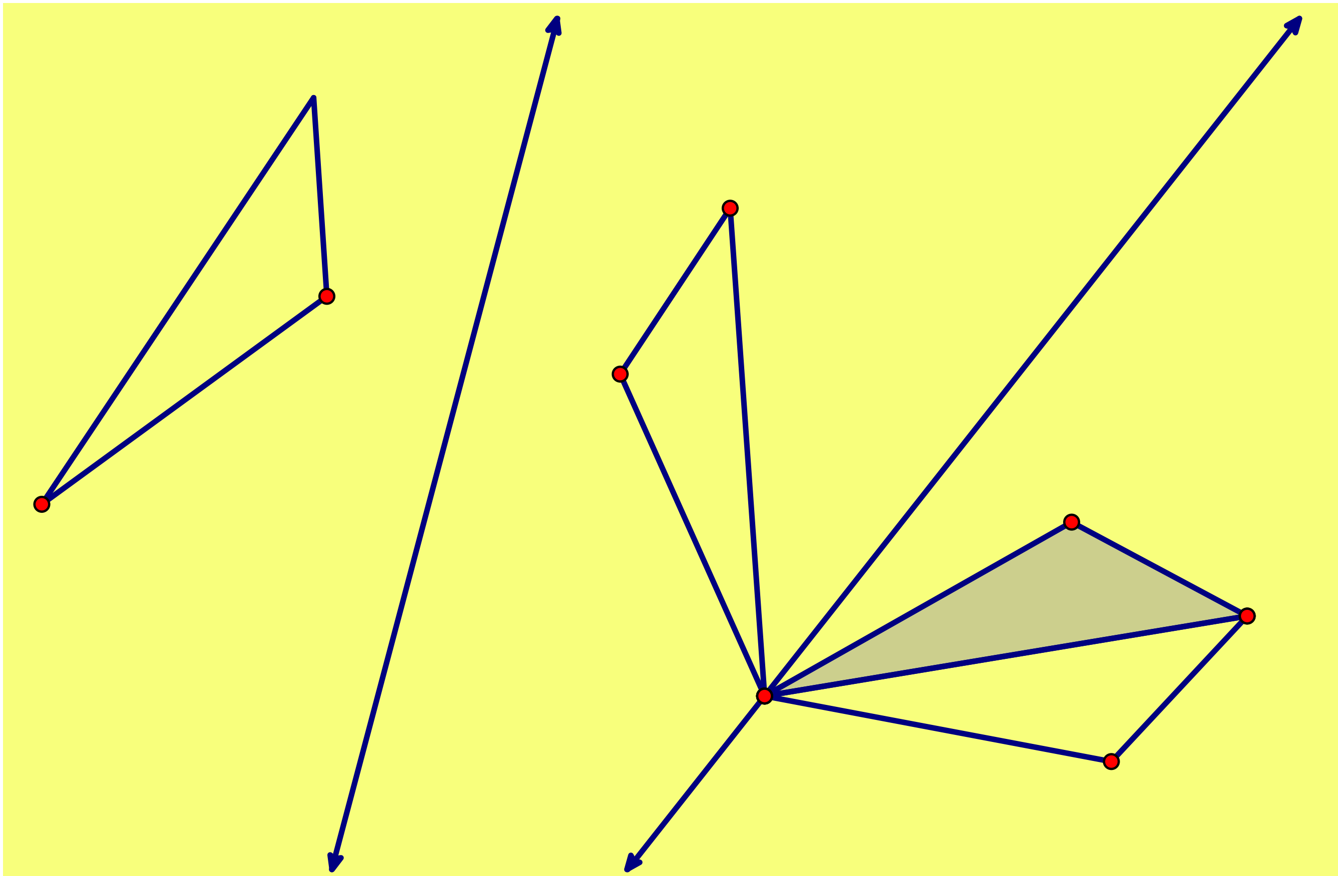
- We must assume a few properties of line reflections to have anything to work with. But we must not assume what we are proving. As a first step, we prove this. (But this needs to be based on axioms not spelled out here!)
- **Proposition**: If a point A is line reflected to point B, the line of reflection is the **perpendicular bisector** of segment AB.
- **Proof**. Let M be the intersection of AB with the mirror line. The reflection of M is M, so segment AM is reflected to BM, so these two segments are congruent and M is the midpoint of AB. Also, for any other point C on the reflection line, angle AMC is reflected to angle BMC, so these angles are congruent but also are supplementary, adding up to a straight angle, so the angles are right angles. QED?!
- **Corollary**: For any point C on the reflection line, the segment CA is reflected to CB, so the segments are congruent and the triangle ACB is isosceles.



First steps (ii)

- **Proposition**: If a segment FE is congruent to FG, then the angle bisector reflects point E to point G.
- **Proof.** Since line reflection preserves angle measure, the reflection in the bisector of the ray FE is ray FG. Let E’ be the reflection of point E. Since the segments FE’ and FG are congruent and lie on the same ray, the point E’ and point G are the same.QED?!
- **Corollary.** In this figure, since E is reflected to G, the triangle EFG is isosceles and the angle bisector of angle EFG is the perpendicular bisector of EG.





Switch to Sketchpad

Observation: Technology

- Considering this example, do you feel, as I do, that dynamic geometry software adds a lot to working with transformations?
- It may be that one reason transformations did not take hold earlier in high school is that transformations are much harder to draw and visualize with traditional tools. So the time is ripe now, with iPads and phones and laptops.
- In addition to being a great tool for studying transformations, the application of transformations in technology is also important. Geometry is not done on computers for designing airplanes or creating animated movies. And transformations are built in.

Observation: Some Open-endedness

- Multiple Solutions: This proof can be carried out in many ways, which makes it more interesting as something to think about. Try the same problem starting with a translation or even a rotation.
- Noticing: In looking at the example, one may notice that each time a triangle is reflected twice, it the second image is a rotation of the original triangle. This suggests a theorem that can be experimented with and ultimately proved: a sequence of two reflections in intersecting lines is a rotation with center at the point of intersection.
- Other questions: Given two congruent shapes that are not triangles, can one still move one to the other by three or fewer line reflections. If one starts with a translation, what happens next? Can one more rigid motion finish the job?

What's to Like?

- A solid definition of congruence
- A coherent development of geometry
- Experience with important math ideas
- More and better geometric intuition
- Connections with technology
- Connections with algebra

Solid Definition of Congruence

- The rigid motion definition is a clear, unambiguous concept. This gives meaning to congruence of any shapes, from polygons to ellipses and parabolas, to fractals with an easy extension to digital photos.
- This contrasts with the "definition" of congruence in many secondary texts: lots of intuition about cutting out and moving and "same size same shape" but no well-defined general concept, just tests for triangles and then ad hoc definitions for other shapes.
- Note that a rigid motion is not the same as superimposition of figures (cut out and move); rigid motions are defined for the whole plane, not just for points in the figure. The whole plane moves and nothing is cut out. This is sound mathematics that lays groundwork for more advanced math.

Worries and Concerns

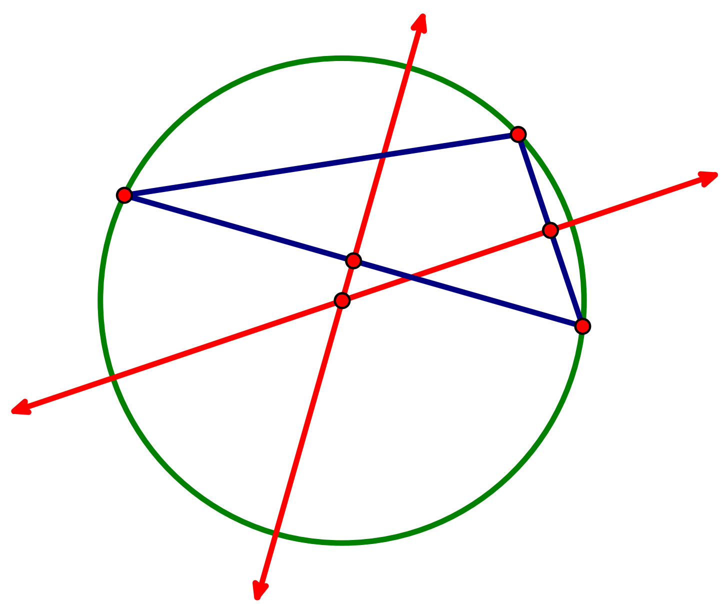
- Transformations, as in the past, may be added superficially but not treated as fundamental
- Belief that every proof must involve transformations. CCSS does not require "transformational geometry".
- Temptations to assume that anything about transformations that looks correct is correct, without making logical reasoning.
- The CCSS prescription for geometry is a new sequence of ideas. As yet there are few sources or textbooks that follow this train of logic. Most books about transformations assume a traditional geometry theorems (such as SAS) as a prerequisite.

“Transformational Geometry”

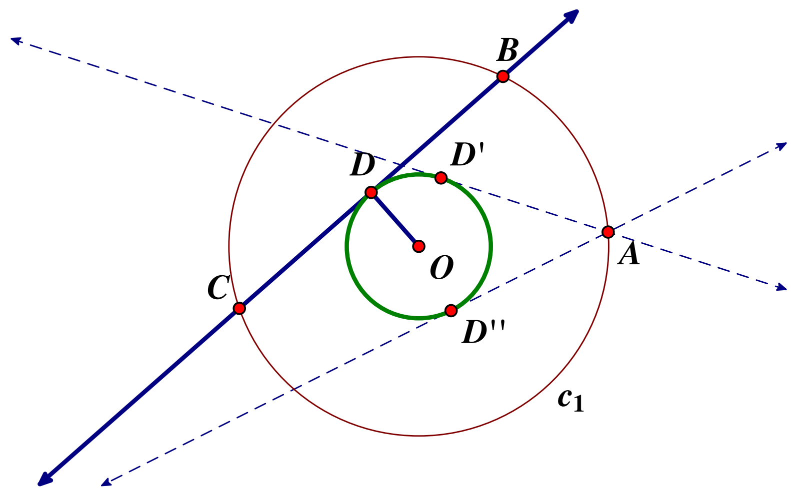
- CCSS proposes basing geometry on transformations, but this does not mean that every proof has to involve transformations. Once basic theorems are proved (such as SAS, etc.) all of the standard theorems can be proved. Some may be easier with transformations, but others are clearer with the proofs found in a traditional development. So it is not necessary to go overboard.

A Traditional Example

- To prove that for any triangle, there is a circle passing through all 3 vertices, the proof based on intersecting perpendicular bisectors of the sides is just fine.



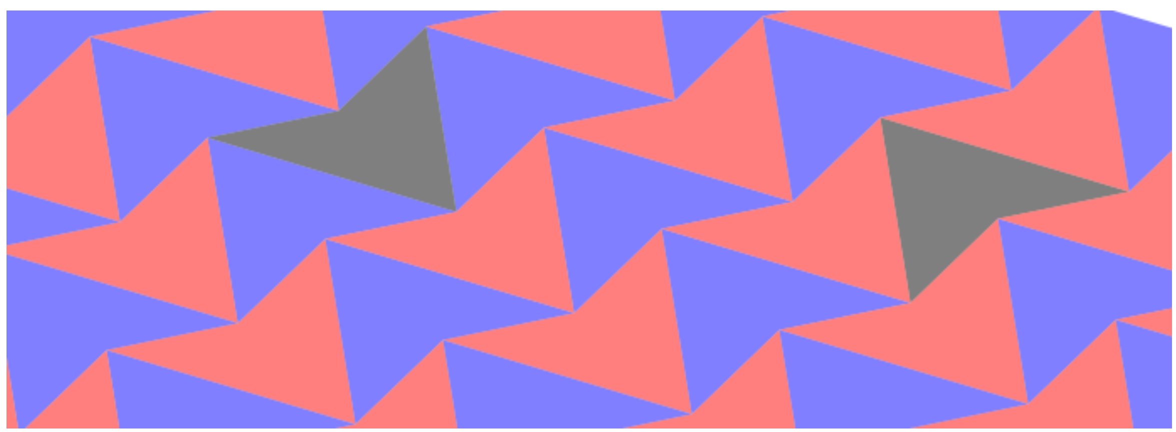
A Rotational Wrinkle



- Given a circle with center O and a point A outside the circle, a challenging construction problem is to construct tangents to the circle through A. There is a pretty, but somewhat sophisticated construction that uses the circle with diameter O.
- But thinking about transformations, there is another proof. Draw the circle with center O through A. Construct any tangent to the circle and intersect it with the new circle at a points B and C. Rotate point B to A with center O. This will also rotate the tangent to to a tangent through A. Rotate C to A to get the other tangent.

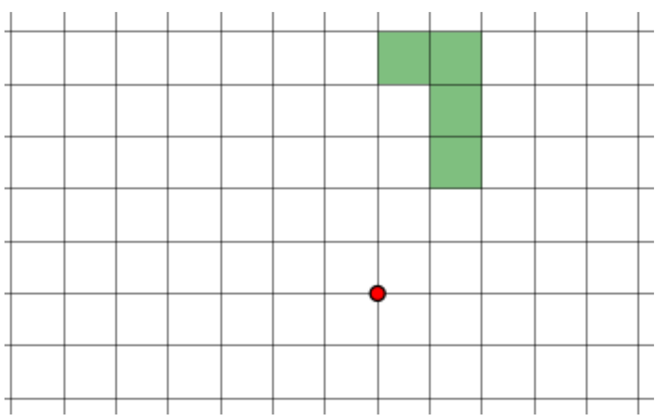
The math in symmetry

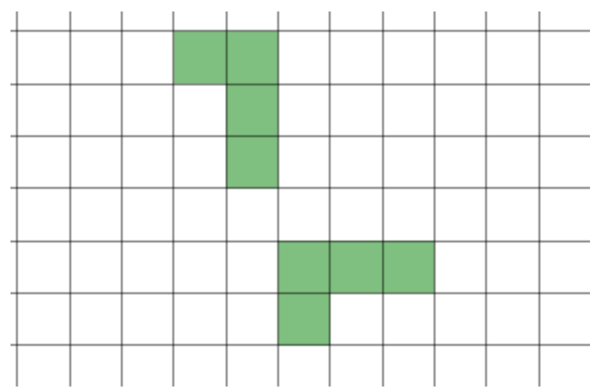
- Symmetric patterns can be approached just as art projects, but there is a lot of math that can be mined. For example, in this figure, what rigid motion will move a red quadrilateral to a blue one? What rigid motion will move one gray quadrilateral to the other?
- Also, notice that there is a theorem here: Any quadrilateral will tessellate the plane.



The Coordinate Plane

- Besides technology, one way to experiment with transformations is on the coordinate plane. This connects the transformations to formulas in the coordinate plane.
- Even rather simple questions provide a challenge and (hopefully) insight.
- Examples: In the left figure, rotate the shape by 90 degrees with the center point shown. In the right figure, find the center and angle of rotation that takes one shape to the other.



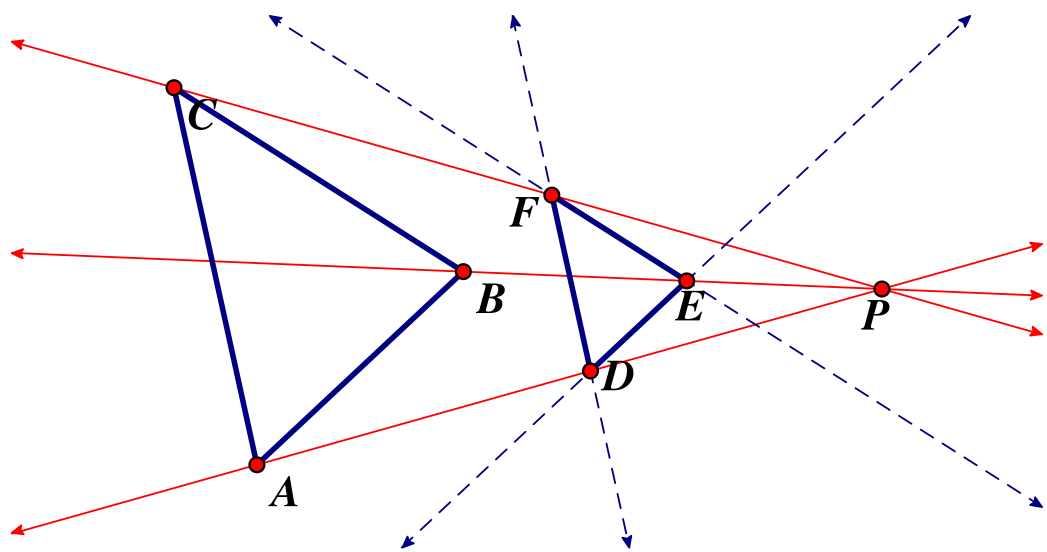


Dilations and Similarity

- I have given short shrift to using dilations - combined with rigid motions– to define similarity. This is the companion piece to the congruence story, and has many of the same features.
- But similarity deserves its own long talk, for which there is no time today.
- But here are some things to think about.

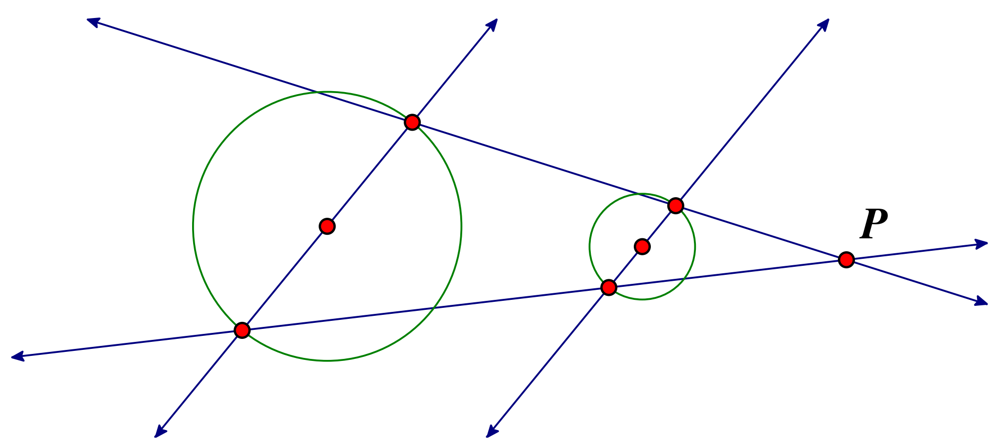
Two Triangles with Parallel Sides

- Draw any triangle ABC. The draw 3 lines parallel to the sides of ABC, forming a new triangle DEF. Draw 3 lines, one through a vertex of ABC and the other through the corresponding vertex of the new triangle. The three lines will concur at a point P, which is the center of a dilation that takes ABC to DEF (and thus the triangles are similar).



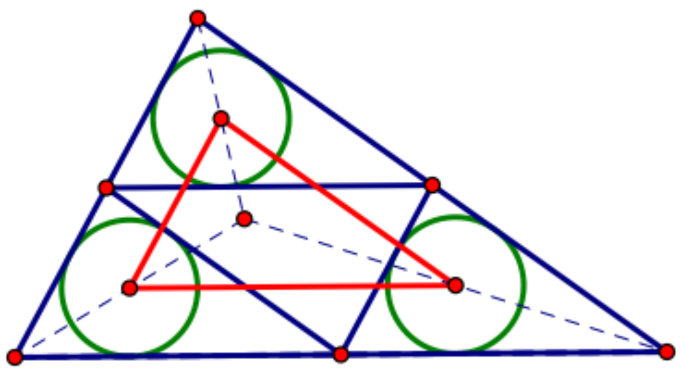
Two circles

- Here's a question: In a standard geometry course, how do you prove that any two circles are similar? (In other words, is the definition of similar powerful enough to apply to this case, or is it simple swept under the rug?)
- Here is a figure that shows how two circles of different radius are always related by a dilation.



Fun with a triangle

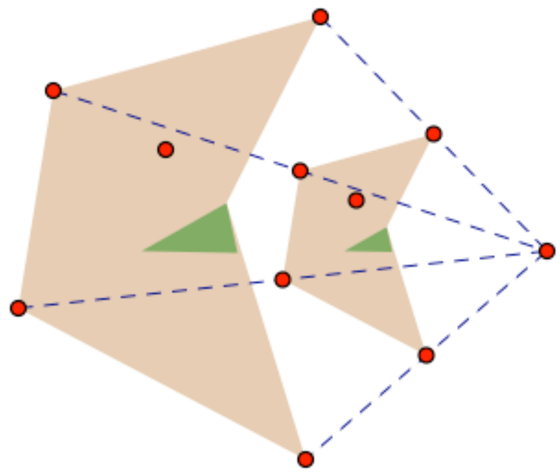
- The vertices of the red triangle are the incenters of the corner midpoint triangles. Find dilations in the figure that show the red triangle is congruent to the midpoint triangle of the large triangle.



A Few Resources

- Illustrative Math Project (on the web, in development)
- Richard Brown, *Transformational Geometry*, Dale Seymour (out of print)
- H. H. Wu, "Teaching Geometry According to the Common Core Standards", http://math.berkeley.edu/~wu/Progressions_Geometry.pdf
- My website (eventually, for this presentation and others) http://www.math.washington.edu/~king
- NCTM publications
- William Barker and Roger Howe, *Continuous Symmetry from Euclid to Klein*, American Mathematics Society (this is a college-level geometry text focused on transformations)

Thanks



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