In Search of Algebra

James King

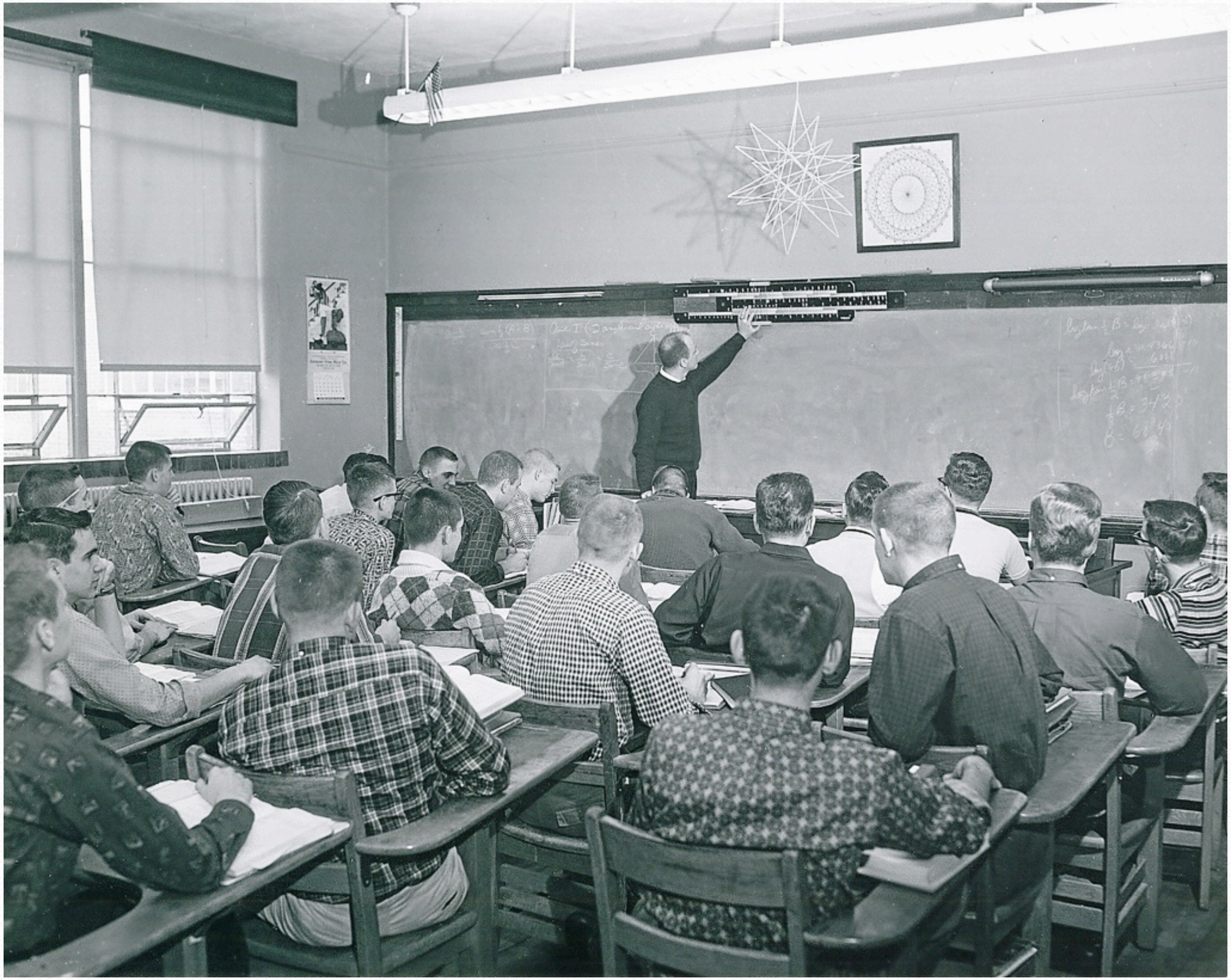
Tradition!

- One Definition of Tradition: **A specific custom or practice of long standing.** (World English Dictionary)
- A few years ago, I got involved in some writing and discussion of math curriculum, including Algebra. One popular contention was that schools should return to a “Traditional Algebra” course.
- This raised my curiosity. As a mathematician, my instinct is to ask for definitions. What do we mean by Traditional Algebra? Is this a well-defined concept? Is there a practice of long standing in American schools?
- So I decided to try to take a look.

Tradition?

There must some kind of tradition or common experience, since we can make jokes about it.

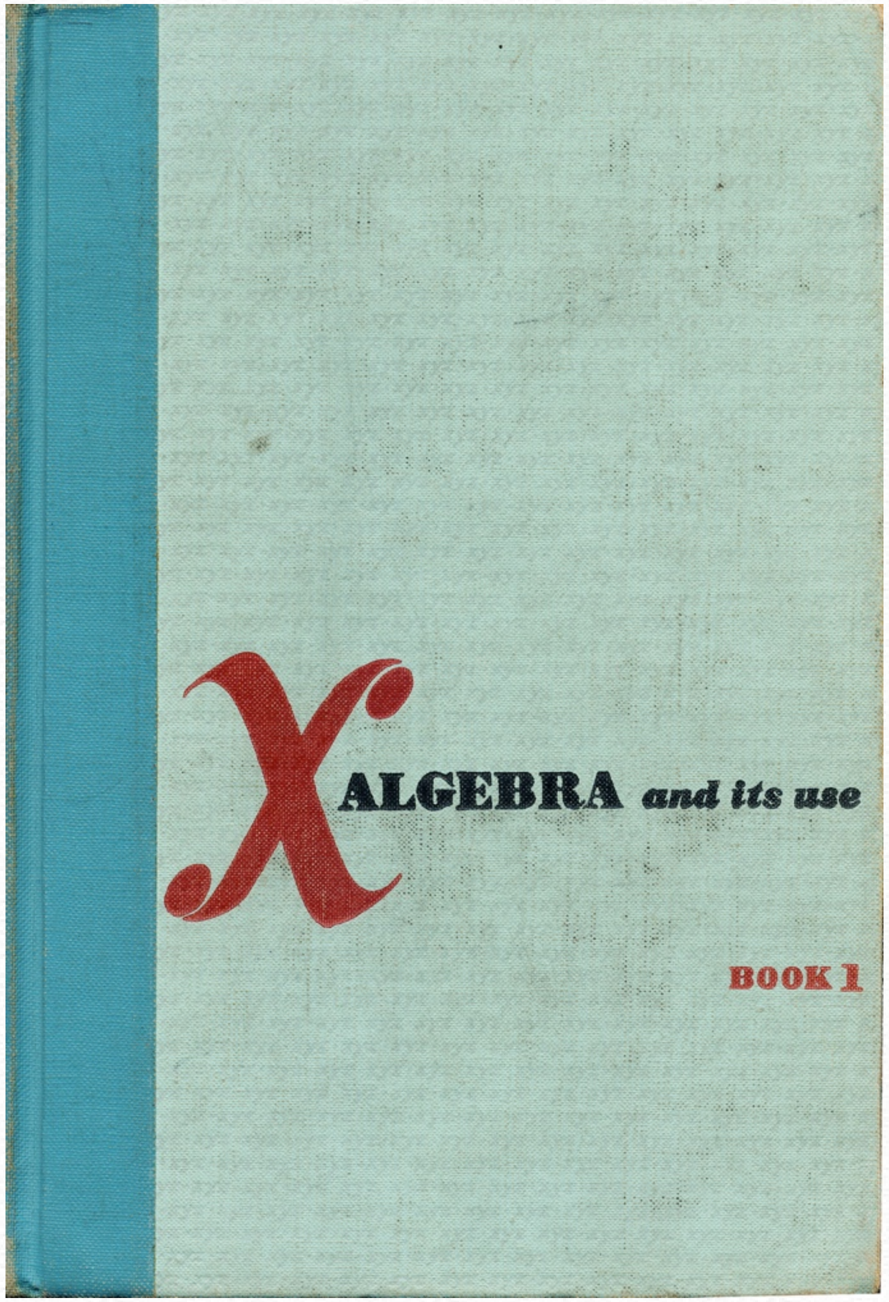
Tradition Part 1: My HS Math



Actually Senior Year, so not Algebra 1

My Algebra 1 Text

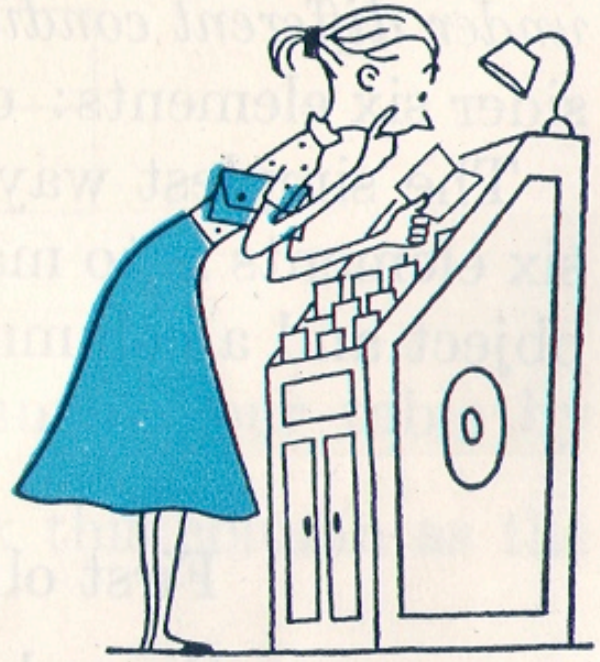
- I am old enough for my school math to have been really traditional, whatever that means. So I started with my own Algebra 1 book (grade 9, Isaac Litton Jr High School, Nashville, Tennessee).
- This book is copyright 1956 and was a new copy when I took algebra.
- Here are some problems.



Some ratio problems

25. Ted has mixed 20 lb. of cracked corn with 50 lb. of growing mash for his chickens, but his father says that $\frac{2}{5}$ of the total mixture should be corn. Should he add corn or mash, and how much?

26. The denominator of a fraction is 3 more than the numerator. If 5 is added to both parts, the resulting fraction is equivalent to $\frac{4}{5}$. Find the numerator and denominator of the original fraction.



27. Jane's mother gave her \$2.45 with which to buy Easter cards. If she got some 10-cent cards, $\frac{2}{3}$ as many 5-cent cards, and $\frac{1}{5}$ as many 15-cent cards, how many of each kind did she get?

Linear Equations

EQUATIONS WITH NEGATIVE ROOTS

As you already know, negative numbers can be used to represent the value of some things or a decrease in the value of other quantities. It is, therefore, quite possible for an equation, either simple or more difficult, to contain negative numbers and to have a negative root.

Such equations can be solved in the same way as the equations in Exercise 38. No additional rules or steps are needed, but a little extra care in checking signs in each step may be necessary to prevent errors.

EXAMPLES

$$\begin{aligned} 3x + 5 &= x - 13 \\ 3x - x + 5 &= -13 \\ 3x - x &= -13 - 5 \\ 2x &= -18 \\ x &= -9 \end{aligned}$$

(Signs different, quotients negative.)

$$\begin{aligned} x - 3 &= 5x + 13 \\ -3 &= 5x - x + 13 \\ -13 - 3 &= 5x - x \\ -16 &= 4x \\ -4 &= x \end{aligned}$$

$$\begin{aligned} -2x + 8 &= 15 - x \\ -2x + x &= 15 - 8 \\ -x &= 7 \\ x &= -7 \end{aligned}$$

(Find value of x, not $-x$. Coefficient of -1 is understood.)

And got lots of practice

EXERCISE 39

Solve and check:

1. $2x = x - 21$ **2.** $3d = 18 + 5d$

3. $7b - 48 = 9b$ **4.** $4a - 9a = 25$

5. $12x + 49 = 25$ **6.** $13c - 6 = 7c + 36$

7. $5a - 11 = 7 - a$ **8.** $10y - 8 = 14y + 24$

9. $2d - 21 = 7d + 4$ **10.** $27a = 45 + 12a$

11. $6y + 7 = 5y - 8$ **12.** $15w - 40 = w + 2$

13. $4 - 3a = 6a - 17$ **14.** $25x = 15x - 90$

15. $2x - 2 = 8x + 4$ **16.** $2b + 15 = -4b + 18$

17. $5n + 7 = 3 - 3n$ **18.** $9 - 3x = 7x - 1$

19. $4a - 5 = 3a + 7$ **20.** $6w + 8 - 2w = 18$

21. $15 + 5a = 57 - 2a$ **22.** $7n + 4 = 2n - 6$

23. $11x = 8 - 7x - 44$ **24.** $4x - 15 = 3x + 1$

25. $3x - 36 = 24 - 2x$ **26.** $15 - 4c = 27 - c$

27. $13a + 3 = 4a - 27$ **28.** $5w - 7 = 11w + 17$

29. $6a - 7 = 29 + 4a$ **30.** $-a - 12 = 4a - 2$

31. $6x + 1 - 12x = 25$ **32.** $2x + 10 = 22 - x$

33. $8n + 2.6 = 5n - 1$ **34.** $12 - 5b = 3b - 24$

35. $2a - 13 = 4 + 7a$ **36.** $4c + 2 = 4.8 - 3c$

37. $5x - 36 = 3x - 54$ **38.** $12x - 8 = 6x + 7$

39. $12 - 6b = -3b + 11$ **40.** $2x - 10 = 56 + 8x$

41. $7x + 9 = 17 + 11x$ **42.** $x + 44 = -5x + 86$

43. $6x + 18 = 9 - 4x$ **44.** $5x - 19 = x - 27$

45. $11y - 0.5 = 3y - 2.1$ **46.** $6y + 32 = 7y - 4$

Plus, how to compute square roots by hand

be used very much because it can be considerably shortened and generalized to include irrational numbers also, but you should be thoroughly familiar with it before you attempt the arithmetical process based on this principle.

EXERCISE 138

Find the following roots by the equation method. Show all of your work.

1. $\sqrt{324}$	2. $\sqrt{529}$	3. $\sqrt{729}$	4. $\sqrt{961}$
5. $\sqrt{1296}$	6. $\sqrt{1089}$	7. $\sqrt{1936}$	8. $\sqrt{1764}$
9. $\sqrt{2304}$	10. $\sqrt{3025}$	11. $\sqrt{3249}$	12. $\sqrt{2401}$
13. $\sqrt{4356}$	14. $\sqrt{5184}$	15. $\sqrt{8649}$	16. $\sqrt{3844}$
17. $\sqrt{5476}$	18. $\sqrt{9409}$	19. $\sqrt{7225}$	20. $\sqrt{6561}$
21. $\sqrt{5776}$	22. $\sqrt{4624}$	23. $\sqrt{7921}$	24. $\sqrt{2916}$
25. $\sqrt{9801}$			

Square root by arithmetical method. A comparison of the two solutions shown below will demonstrate how the arithmetical method of finding square root is only an outline form of the basic steps in the equation process for determining roots. Study this illustration so that you understand the reason for each step in the arithmetical procedure.

Let $(B + u) = \sqrt{676}$

676 between 100 and 10,000, so 2-digit root,
676 between 400 and 900, so, $B = 20$.

$$400 + 2(20)u + u^2 = 676 \qquad \begin{array}{r} 2\ \ 6 \\ \sqrt{6'76} \end{array}$$

$$2(20)u + u^2 = 276 \qquad \underline{4}$$

or

$$u[2(20) + u] = 276 \qquad 2(20) + 6 \quad 2\ 76$$

$$u^2 + 40u - 276 = 0 \qquad 46 \quad \underline{2\ 76}$$

$$(u + 46)(u - 6) = 0$$

$$u = 6$$

$$B + u = 26$$

Therefore, the complete process for finding a numerical square root may be summarized in the following rules. Each rule is specifically illustrated in the corresponding step of the example below.

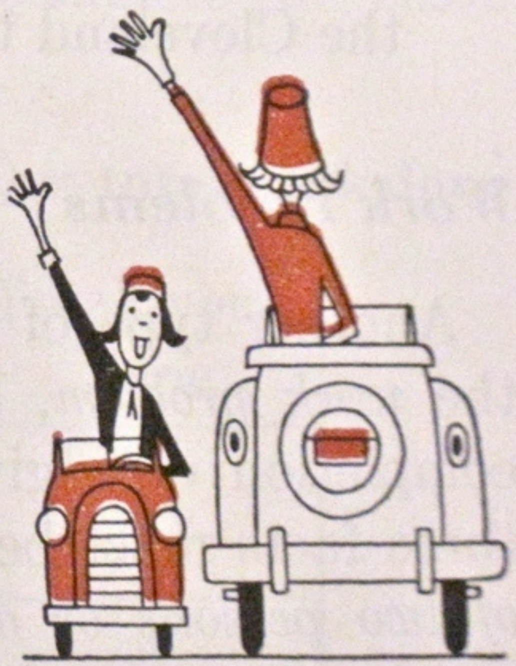
To find the square root of any numerical quantity:

1. Separate the digits in the radicand into groups of two digits each, counting in each direction from the decimal point.
2. Determine the largest square contained in the last group on the left and put its square root directly above this group as the first digit of the root.
3. Put the square of this digit under the last group in the radicand.
4. Subtract and bring down beside this remainder the next group in the radicand.
5. For trial divisor to determine the next digit in the root, use twice the base number already found (twice the root already found times 10).
6. Determine by inspection how many times this trial divisor will divide into the dividend in Step 4. Write this quotient as second digit in the root and add it to the trial divisor.
7. Multiply the resulting total divisor by the digit added to the root in Step 6, write the product under the dividend in Step 4, and subtract. (Since this product contains the square of the second digit as well as the product of this digit and the trial divisor, it may be larger than the dividend. If so, replace the digit by the next smaller one and remember when estimating it in Step 6 to allow for its square.)
8. If more numbers remain in the radicand, bring down each group with the preceding remainder and repeat Steps 5–7 for each.
9. If the required root of a given radicand does not have an exact value, add zeros in groups of two on the right of the decimal and continue repeating Steps 5–7 until the root is determined to the required number of decimal places.

And the Calvin Problem!

8. A serious accident happened on the highway between Glennville and Johnstown, which are 42 mi. apart. If the patrol car starting from Glennville averaged 70 mph. and an ambulance called from Johnstown at the same time averaged 56 mph. and both arrived at the same time, how long did it take the patrolman to reach the accident?

9. Two hours after Mrs. Kane, averaging 45 mph., leaves Bridgeport for Portland, 261 mi. away, her sister starts from Portland to Bridgeport over the same route, traveling at 50 mph. How far will Mrs. Kane travel before the two sisters meet?



10. Two busses leave Nashville at the same time, traveling in opposite directions. If one averages 8 mph. more than the other for 6 hrs., they will be 528 mi. apart. What is the rate of each?

Seriously, what do we see?

- The introduction to the book says

  > ***[The text] has made balance and equation its keynote. Since the equation is the basic concept of mathematics, and particularly of algebra, the authors have introduced it at the very start. Throughout the text the first emphasis is on understanding. The "why" of every concept .. is then followed by a complete explanation of the "how".***

- What does it include: **equations, polynomials, radicals, fractions, ratios and proportion.**

- **It has almost no graphing.** There is one page of graphing of systems of equations and then one parabola on almost the last page of the book.

- The word ***function*** does not appear in the text. Nor does ***logarithm***, **compound interest, or sequences or series.**

Tradition Part 2: The 19th Century

This Algebra text by Charles Davies was one of the earliest American math texts. It (along with another book on Geometry) went through many editions over many decades.

ELEMENTS

OF

ALGEBRA:

TRANSLATED FROM THE FRENCH OF

M. BOURDON.

REVISED AND ADAPTED TO THE COURSES OF MATHEMATICAL INSTRUCTION IN THE UNITED STATES;

BY CHARLES DAVIES,

PROFESSOR OF MATHEMATICS IN THE MILITARY ACADEMY

AND

AUTHOR OF THE COMMON SCHOOL ARITHMETIC, ELEMENTS OF DESCRIPTIVE GEOMETRY, SURVEYING, AND A TREATISE ON SHADOWS AND PERSPECTIVE.

NEW-YORK:

PUBLISHED BY WILEY & LONG,

No. 161 Broadway.

STEREOTYPED BY A. CHANDLER.

1835.

Davies includes lots of work with symbolic expressions

CASE III.

72. To reduce a fraction to an entire or mixed quantity.

RULE.

Divide the numerator by the denominator for the entire part, and place the remainder, if any, over the denominator for the fractional part.

EXAMPLES.

1. Reduce $\frac{ax+a^2}{x}$ to a mixed quantity.

$$\frac{ax+a^2}{x}=a+\frac{a^2}{x} \quad \textit{Ans.}$$

2. Reduce $\frac{ax-x^2}{x}$ to an entire or mixed quantity.

Ans. $a-x$.

3. Reduce $\frac{ab-2a^2}{b}$ to a mixed quantity.

Ans. $a-\frac{2a^2}{b}$.

And some familiar ratio problems

18. The hour and minute hands of a clock are exactly together at 12 o'clock ; when are they next together ?

Ans. $1\,h.\ 5\frac{5}{11}\,min.$

19. A man and his wife usually drank out a cask of beer in 12 days ; but when the man was from home, it lasted the woman 30 days ; how many days would the man alone be in drinking it ?

Ans. 20 days.

20. If A and B together can perform a piece of work in 8 days, A and C together in 9 days, and B and C in 10 days : how many days would it take each person to perform the same work alone ?

Ans. A $14\frac{34}{49}$ days, B $17\frac{23}{41}$, and C $23\frac{7}{31}$.

21. A laborer can do a certain work expressed by a, in a time expressed by b ; a second laborer, the work c in a time d ; a third, the work e, in a time f. It is required to find the time it would take the three laborers, working together, to perform the work g.

Ans. $$x=\frac{bdfg}{adf+bcf+bde}.$$

Application.

$a=27$; $b=4$ | $c=35$; $d=6$ | $e=40$; $f=12$ | $g=191$;
x will be found equal to 12.

But some might not be allowed in school nowadays, e.g, #19.

This textbook by Robinson is copyrighted 1848

UNIVERSITY EDITION.

A

THEORETICAL AND PRACTICAL

TREATISE ON

ALGEBRA;

IN WHICH THE EXCELLENCIES OF THE DEMONSTRATIVE METHODS OF THE FRENCH, ARE COMBINED WITH THE MORE PRACTICAL OPERATIONS OF THE ENGLISH; AND CONCISE SOLUTIONS POINTED OUT AND PARTICULARLY INCULCATED.

DESIGNED FOR SCHOOLS, COLLEGES AND PRIVATE STUDENTS

BY HORATIO N. ROBINSON, A.M.

FIFTH EDITION.

CINCINNATI:
PUBLISHED BY JACOB ERNST,
NO. 183 MAIN STREET.
1848.

From Robinson's Introduction

> Some apology may appear requisite for offering a new book to the public on the science of Algebra ...

But the intrinsic merits of a book are not alone sufficient to secure its adoption, ... In addition to merit, it must be adapted to the general standard of scientific instruction given in our higher schools; **it must conform in a measure to the taste of the nation, and correspond with the general spirit of the age in which it is brought forth.**

The elaborate and diffusive style of the French, as applied to this science, can never be more than *theoretically* popular among the English; and the severe, brief, and practical methods of the English are almost intolerable to the French. ...

In this country, our authors and teachers have generally adopted one or the other of these schools, and thus have brought among us difference of opinion, drawn from these different standards of measure for true excellence.

Very many of the French methods of treating algebraic science are not to be disregarded or set aside. First principles, theories and demonstrations, are the essence of all true science, and the French are very elaborate in these. **Yet no effort of individuals, and no influence of a few institutions of learning, can change the taste of the American people, and make them assimilate to the French, any more than they can make the entire people assume French vivacity, and adopt French manners.**

Several works, modified from the French, have had, and now have considerable popularity, but they do not naturally suit American pupils. They are not sufficiently practical to be unquestionably popular; and excellent as they are, they fail to inspire that enthusiastic spirit, which works of a more practical and English character are known to do.

At the other extreme are several English books, almost wholly practical, with little more than arbitrary rules laid down. **Such books may in time make good *resolvers* of problems, but they certainly fail in most instances to make scientific algebraists.**

Robinson's Algebra of 1850

These topics look familiar

CONTENTS.

	Page.
Introduction	9
SECTION I.	
Addition	12
Subtraction	16
Multiplication	19
Division	26
Negative Exponents	27
Algebraic Fractions	34
Greatest Common Divisor	37
Least Common Multiple	42
Addition of Fractions	44
Subtraction of Fractions	46
Multiplication of Fractions	47
Division of Fractions	49
SECTION II.	
Equation of one unknown Quantity	52
Question producing Simple Equations	62
Equations of Two unknown Quantities	70
Equations of Three or more unknown Quantities	76
Problems producing Simple Equations of Two or more unknown quantities	87
Interpretation of negative values in the Solution of Problems	94
Demonstration of Theorems	97
Problem of the Couriers	98
Application of the Problem of the Couriers	102

The last topics do not look like our Algebra course

SECTION III.

INVOLUTION	106
Some application of the Binomial Theorem	108
Evolution	113
Cube root of Compound quantities	121
Cube root of Numerals	123
Brief method of Approximation to the Cube root of Numbers	124
Exponential Quantities and Surds	128
PURE EQUATIONS	133
Binomial Surds	141
Problem producing pure Equations	147
Problem of the Lights	151
Application of the Problem	152

Calvin and Hobbes in1850

Many other theorems are demonstrable by algebra, but we de fer them for the present, as some of them involve quadratic equations, which have not yet been investigated; and we close the subject of simple equations by the following quite general problem in relation to *space*, *time* and *motion*.

To present it at first, in the most simple and practical manner, let us suppose

Two couriers, A *and* B, 100 *miles asunder on the same road set out to meet each other*, A *going* 6 *miles per hour and* B 4. *How many hours must elapse before they meet, and how far will each travel?*

A general solution for rate problems

(Art. 59.) Let us now make the problem general.

Two couriers, A *and* B, d *miles asunder on the same road, set out to meet each other;* A *going* a *miles per hour,* B *going* b *miles per hour. How many hours must elapse before they meet, and how far will each travel?*

Taking the same notation as in the particular case,

Let $x=$ A's distance, $y=$ B's, and $t=$ the time.

Then $x+y=d$ (1) $x=at$ $y=bt$ (2)

Therefore $(a+b)t=d$ Or $t=\frac{d}{a+b}$ (3)

And $x=at=\frac{ad}{a+b}$ $y=bt=\frac{bd}{a+b}$ (4)

If $a=b$, then $x=\frac{1}{2}d$ and $y=\frac{1}{2}d$. A result perfectly obvious, the rates being equal. Each courier must pass over one half the distance before meeting.

If $a=0$ $x=\frac{0\times d}{0+b}=0$ and $y=\frac{bd}{b}=d$. That is, one will be at rest, and the other will pass over the whole distance.

Robinson, p. 102

An application to clocks

1. *The hour and minute hands of a clock are together at 12 o'clock. When are they next together?*

By the equation, $t=\frac{d}{a-b}$ this problem and all others like it are already resolved. All we have to do is to determine the values of d, a, and b.

There are 12 spaces (hours) round the dial plate of a clock; hence d may represent 12. a and b are the relative motions of the hands. a moves 12 spaces or entirely round the dial plate while b moves one space. Hence $a=12$, $b=1$, and $a-b=11$.

Consequently, $t=\frac{12}{11}=1$h. 5m. $27\frac{3}{11}$s.

Again. We may demand what time the hour and minute hands of a clock are together between 3 and 4.

From 12 o'clock to past 3 o'clock there are 3 revolutions to pass over in place of one, and the solution is therefore $t=\frac{3\times 12}{11}$ and so on for any other hour.

Robinson, p. 103

The Rest of the Contents

Robinson, 1850

SECTION IV.

	Page.
Quadratic Equations	157
Particular mode of completing a Square (Art. 99)	160
Special Artifices in resolving Quadratics (Art. 106)	169
Quadratic Equations containing two or more unknown quantities	175
Questions producing Quadratic Equations	183

SECTION V.

Arithmetical Progression	189
Geometrical Progression	195
Harmonical Proportion	199
Problems in Progression and Harmonical Proportion	200
Geometrical Proportion	205

SECTION VI.

Binomial Theorem—its Demonstration, &c.	213
" " its General Application	219
Infinite Series	221
Summation of Series	225
Revision of a Series	231
Exponential Equations and Logarithms	233
Application of Logarithms	244
Compound Interest	246
Annuities	247

SECTION VII.

General Theory of Equations	251
Binomial Equations	260
Newton's Method of Divisors (Art. 167)	264
Equal Roots	267
Transformation of Equations (by substitution)	270
Transformation by Division	275
Synthetic Division	278
General Properties of Equations	286
Sturm's Theorem	297
Newton's Method of Approximation	305
Horner's " "	307
Solution of Equations	311
Application of Equations to the Extraction of Roots	322

APPENDIX.

Specific Gravity	325
Maxima and Minima	327

Problems from the Equations Chapter

7. A jockey has two horses, and two saddles which are worth 15 and 10 dollars, respectively. Now if the better saddle be put on the better horse, the value of the better horse and saddle would be worth $\frac{4}{3}$ of the other horse and saddle. But if the better saddle be put on the poorer horse, and the poorer saddle on the better horse, the value of the better horse and saddle is worth once and $\frac{2}{13}$ the value of the other. Required the worth of each horse? *Ans.* 65 and 50 dollars.

8. A merchant finds that if he mixes sherry and brandy in quantities which are in proportion of 2 to 1, he can sell the mixture at 78 shillings *per dozen;* but if the proportion be 7 to 2 he can sell it at 79 shillings *per dozen.* Required the price per dozen of the sherry and of the brandy?

Ans. Sherry, 81*s*. Brandy, 72*s*.

In the solution of this question, put $a=78$. Then $a+1=79$.

9. Two persons, *A* and *B*, can perform a piece of work in 16 days. They work together for four days, when *A* being called off, *B* is left to finish it, which he does in 36 days. In what time would each do it separately?

Ans. *A* in 24 days; *B* in 48 days.

10. What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$? *Ans.* $\frac{4}{5}$.

11. Two men wishing to purchase a house together, valued at 240 (a) dollars; says *A* to *B*, if you will lend me $\frac{2}{3}$ of your money I can purchase the house alone; but says *B* to *A*, if you lend me $\frac{3}{4}$ of yours, I can purchase the house. How much money had each of them? *Ans.* *A* had \$160. *B* \$120.

Robinson, p. 89

Note the Ratio notation and also the problem at the bottom

CHAPTER V.

COMPOUND INTEREST.

(Art. 153.) Logarithms are of great utility in resolving some questions in relation to compound interest and annuities; but for a full understanding of the subject, the pupil must pass through the following investigation:

Let p represent any principal, and r the interest of a unit of this principal for one year. Then $1+r$ would be the amount of \$1, or £1. Put $\mathcal{A}=1+r$.

Now as two dollars will amount to twice as much as one dollar, three dollars to three times as much as one dollar, &c.

Therefore, $1 : \mathcal{A} :: \mathcal{A} : \mathcal{A}^2$ = the amount in 2 years,

And $1 : \mathcal{A} :: \mathcal{A}^2 : \mathcal{A}^3$ = the amount in 3 years,

&c. &c.

Therefore, $\mathcal{A}^n$ is the amount of one dollar or one unit of the principal in n years, and p times this sum will be the amount for p dollars. Let a represent this amount; then we have this general equation,

$$p\mathcal{A}^n=a.$$

In questions where n, the number of years, is an unknown term, or very large, the aid of logarithms is very essential to a quick and easy solution.

For example, *what time is required for any sum of money to double itself, at three per cent. compound interest?*

Here $a=2p$, and $\mathcal{A}=(1.03)$, and the general equation becomes

$$p(1.03)^n=2p$$

Or $(1.03)^n=2.$ Taking the logarithms

$$n \text{ log. } (1.03)=\text{log. } 2, \quad \text{or} \quad n=\frac{\text{log. } 2}{(\text{log. } 1.03)}=\frac{.30103}{.012837}=23.45$$

years nearly.

2. A bottle of wine that originally cost 20 cents was put away for two hundred years: what would it be worth at the end of that time, allowing 5 per cent. compound interest?

Robinson, p 246

A text from 1881

THE

COMPLETE ALGEBRA.

FOR HIGH SCHOOLS, PREPARATORY SCHOOLS, AND ACADEMIES.

BY

EDWARD OLNEY,

Professor of Mathematics in the University of Michigau.

NEW EDITION.

NEW YORK

SHELDON & COMPANY,

8 MURRAY STREET.

Early Problems in Olney

32. Two farms belong to A and B. A has twenty acres less than B. If twice A's number of acres be taken from three times B's, the remainder will be one hundred. How many acres has each?

33. One number is seven less than another, and if three times the less be taken from four times the greater, the remainder will be six times the difference between the two numbers. What are the numbers?

34. Anna is four years younger than Mary. If twice Anna's age be taken from five times Mary's, the remainder will be thirty-five years. What is the age of each?

35. One number is ten less than another. If three times the less be taken from five times the greater, the remainder will be seven times the difference of the two numbers. What are the numbers?

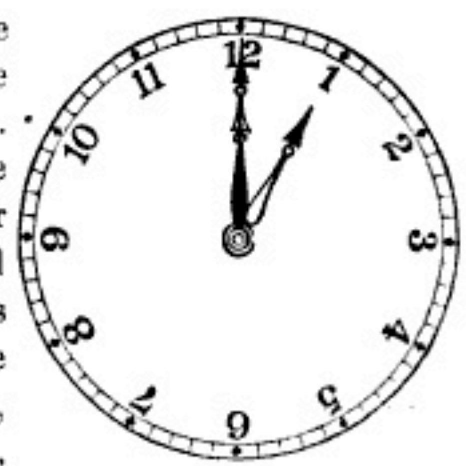
Proportion Problems

21. A privateer espies a merchantman 10 miles to leeward at 11.45 A. M., and, there being a good breeze, bears down upon her at 11 miles per hour, while the merchantman can only make 8 miles per hour in her attempt to escape. After 2 hours chase the topsail of the privateer being carried away, she can only make 17 miles while the merchantman makes 15. At what time will the privateer overtake the merchantman? *Ans.*, At 5.30 P. M.

22. A hare, 50 of her leaps before a greyhound, takes 4 leaps to the greyhound's 3; but 2 of the greyhound's leaps are as much as 3 of the hare's. How many leaps must the greyhound take to overtake the hare?

Suggestion. Let $3x =$ the number of the hound's leaps,
whence $4x =$ " " " hare's "
in the same time. Then $2:3::3x:4x+50$. $\therefore x = 100$; and the hound takes 300 leaps.

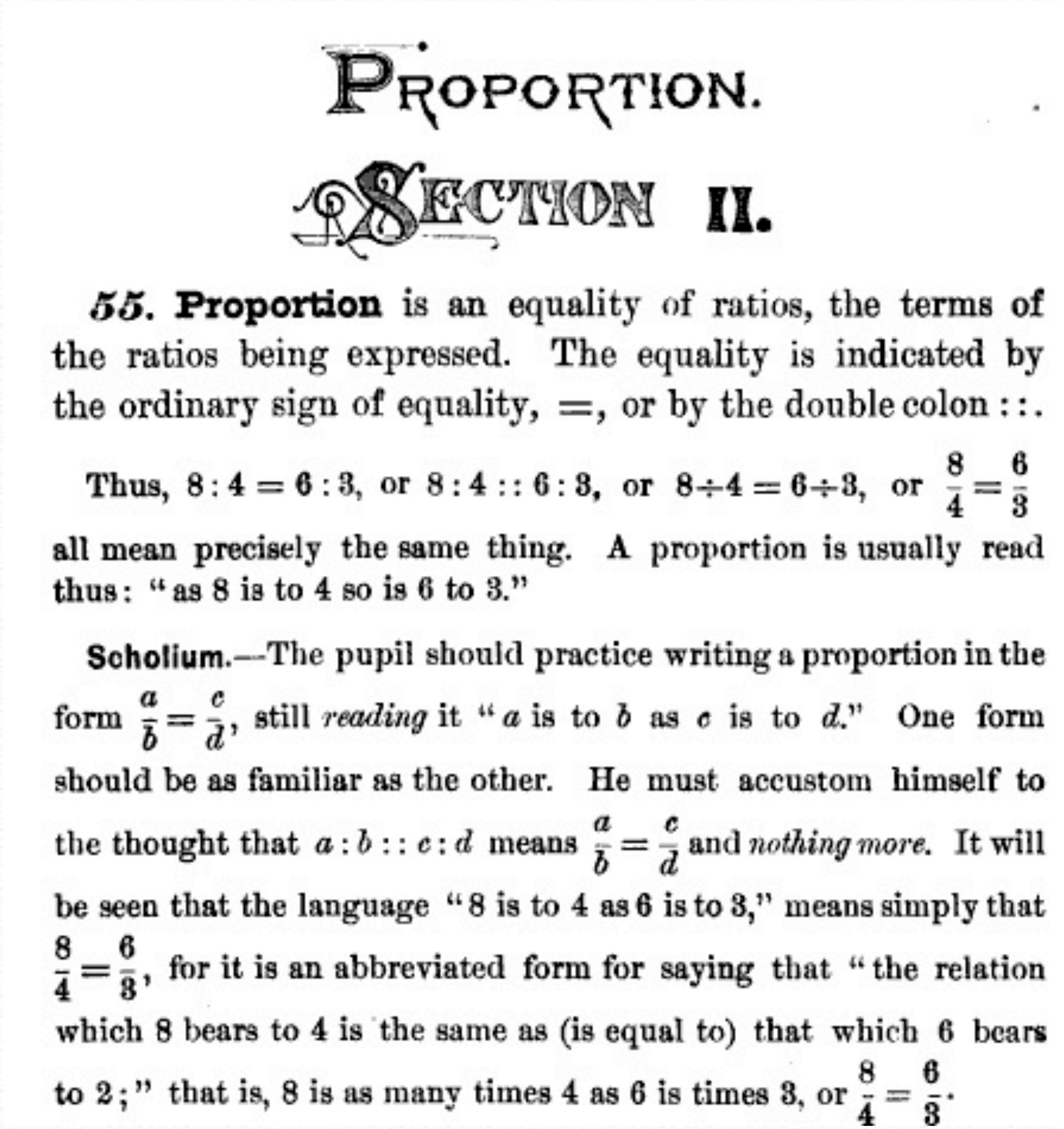
23. The hour and minute hands of a clock are exactly together at 12 M. When are they next together?



Suggestion.—Measuring the distance around the dial by the hour spaces, the whole distance around is 12 spaces. Now, when the hour hand gets to 1, the minute hand has gone clear around, or over 12 spaces. But as the hour hand has gone *one* space, the minute hand has *gained* only 11 spaces. Now as the minute hand must *gain* an entire round, or 12 spaces, to overtake the hour hand, we have the question: If the minute hand gains 11 spaces in 1 hour, how long will it take to gain 12 spaces? $\therefore 11:12::1 \text{ hour} : x$ hours; and $x = 1\frac{1}{11}$ hours, or 1 hour $5\frac{5}{11}$ minutes.

Olney, p. 310

Old ratio notation and new


PROPORTION.

SECTION II.

55. **Proportion** is an equality of ratios, the terms of the ratios being expressed. The equality is indicated by the ordinary sign of equality, $=$, or by the double colon $::$.

Thus, $8:4=6:3$, **or** $8:4::6:3$, **or** $8 \div 4=6 \div 3$, **or** $\frac{8}{4}=\frac{6}{3}$ **all mean precisely the same thing. A proportion is usually read thus: "as 8 is to 4 so is 6 to 3."**

Scholium.—The pupil should practice writing a proportion in the form $\frac{a}{b}=\frac{c}{d}$, still *reading* it "a is to b as c is to d." One form should be as familiar as the other. He must accustom himself to the thought that $a:b::c:d$ means $\frac{a}{b}=\frac{c}{d}$ and *nothing more*. It will be seen that the language "8 is to 4 as 6 is to 3," means simply that $\frac{8}{4}=\frac{6}{3}$, for it is an abbreviated form for saying that "the relation which 8 bears to 4 is the same as (is equal to) that which 6 bears to 2;" that is, 8 is as many times 4 as 6 is times 3, or $\frac{8}{4}=\frac{6}{3}$.

Tuesday, January 7, 14

Observations on the 19th Century

- Quite extensive treatment of formal algebraic expressions.
- Explanations how to calculate roots by hand and how to compute tables.
- Discussion of special series is quite thorough.
- Colon notation for ratios much in evidence.
- Logarithms and their tables are used.
- Some problems seem very familiar and contemporary, though perhaps employing antique cultural references.
- No matrices or determinants (these were 19th century research math).
- No graphs. No functions. No data tables.

Tradition Part 3: Early 20th Century

This textbook is copyright 1910 but was reprinted in 1917.

FIRST COURSE IN ALGEBRA

BY

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Chapter "Identities and Equations of Condition"

27. The length of a rectangle is 5 feet more than four times its width. Its perimeter is 90 feet. Find the dimensions.

28. At Pittsburg on June 21 the day is 6 hours and 6 minutes longer than the night. How long is the night? the day?

29. A's age is twice B's, and C is 7 years older than A. The sum of their ages is 67 years. Find the age of each.

30. A's age is three times B's, and C is 10 years older than B. Five years hence the sum of their ages will be 60 years. Find the age of each now.

Hawkes, et al., p. 52

Some Algebra Problems with Big Numbers

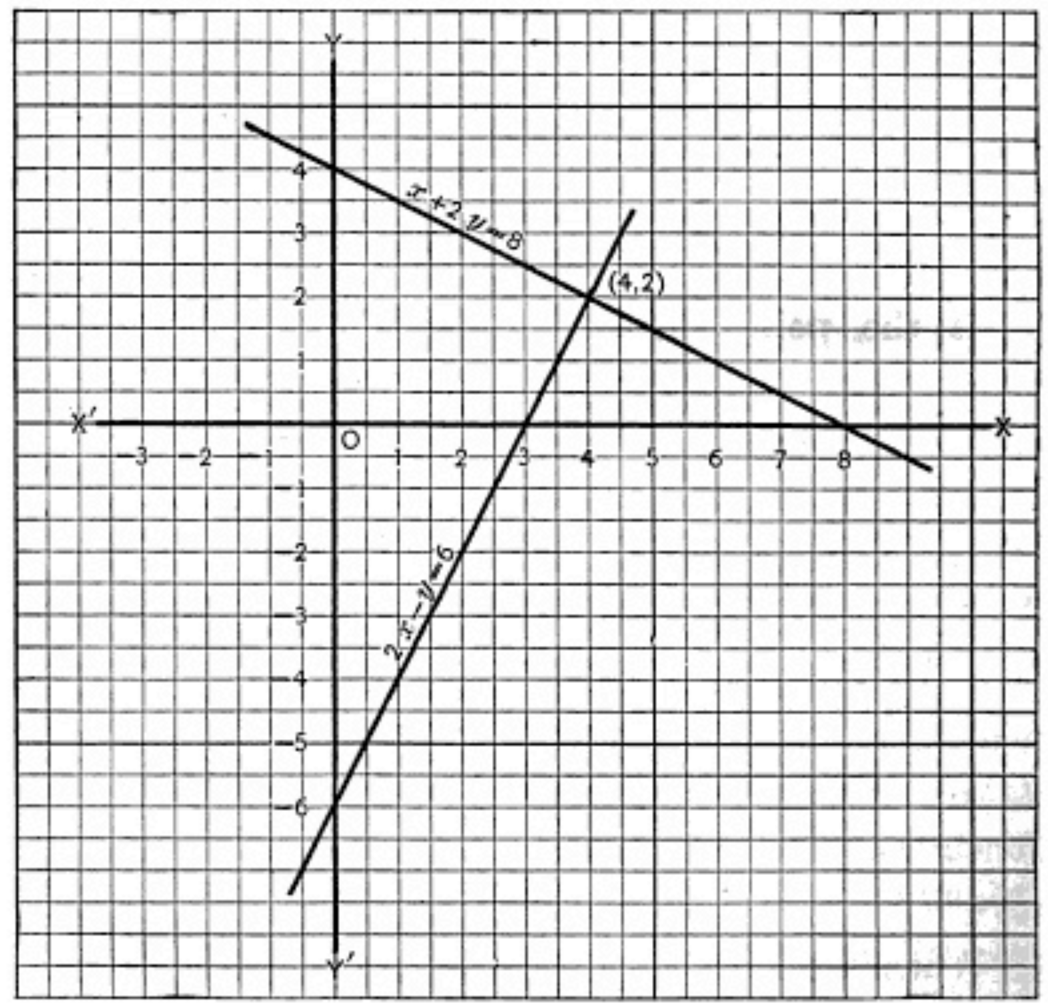
41. The number of United States troops engaged in the Civil War was 15,621 less than nine times the number engaged in the War of the Revolution, which was 266,841 less than the number engaged in the War of 1812. If the total number of United States troops engaged in the three wars was 3,658,811, find the number engaged in each.

42. St. Peter's Cathedral (Rome) has a capacity 29,000 greater than that of St. Paul's Cathedral (London), and 17,000 and 22,000 greater respectively than the Cathedral at Milan and St. Paul's church (Rome). The combined capacity of all is 148,000. Find the capacity of each.

Hawkes, et al., p. 54

Graphs appear!

81. Graphical solution of linear equations in two variables. If we construct the graphs of the two equations $x + 2y = 8$ and



$2x - y = 6$ as indicated in the adjacent figure, it is seen that for the point of intersection of the graphs x is 4 and y is 2.

Hawkes, et al., p. 198

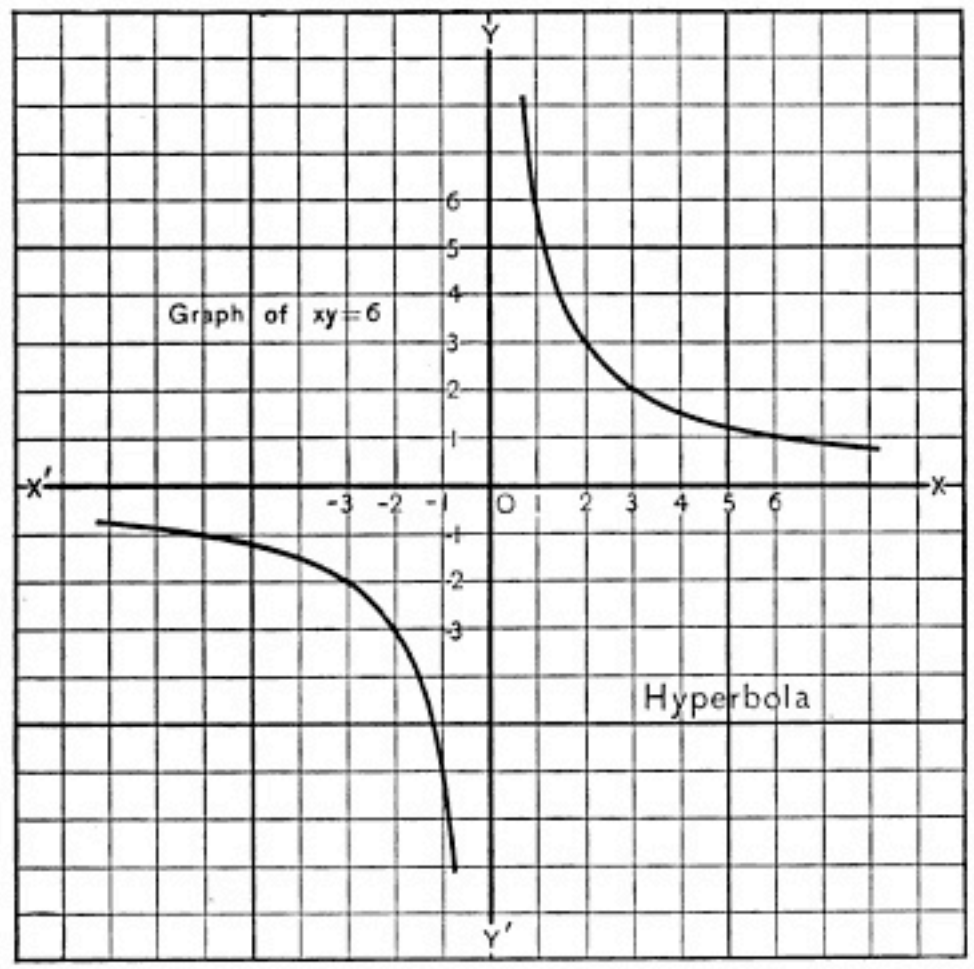
Even graphs of quadratics

2. Graph the equation $xy = 6$.

Solution: Solving for y in terms of x, $y = \frac{6}{x}$.

Assigning values to x as indicated in the following table, we then compute the corresponding values of y.

$x =$	-6	-5	-4	-3	-2	-1	$-\frac{3}{4}$	$\frac{3}{4}$	1	2	3	4	5	6	8
$y =$	-1	$-\frac{6}{5}$	$-\frac{3}{2}$	-2	-3	-6	-8	8	6	3	2	$\frac{3}{2}$	$\frac{6}{5}$	1	$\frac{3}{4}$



Proceeding as before with the numbers in the table, we obtain the two-branched curve of the above figure. The curve does not

A 1915 Text

"The interest and value of the subject are emphasized by showing (a) how algebra grows out of arithmetic, (6) how much easier it is to solve problems by algebra than by arithmetic, and (c) by applying these algebraic solutions to countless problems of every-day experience, as well as to those taken from other studies in the school."

ELEMENTARY ALGEBRA

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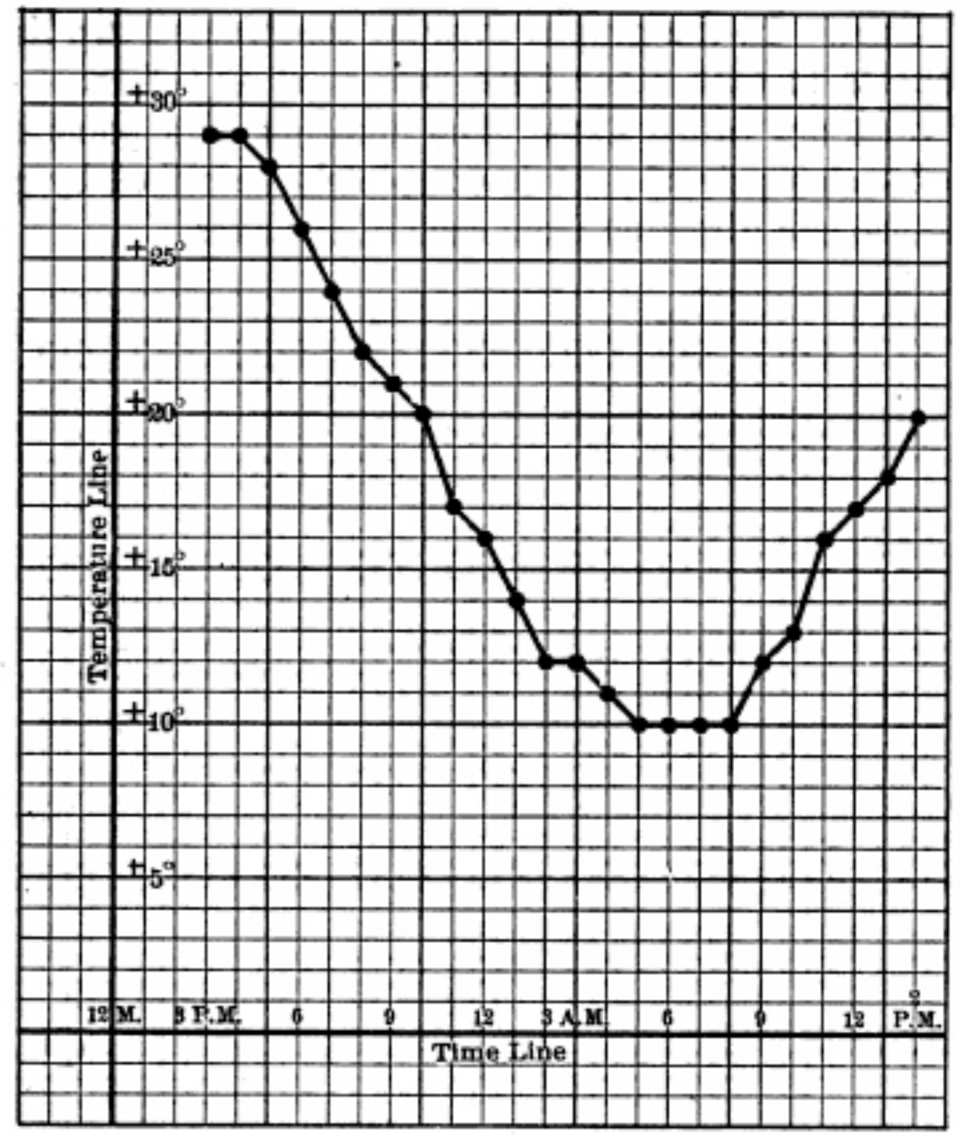
Graphic Representation of Statistics

GRAPHIC REPRESENTATION*

159. Graphic Representation of Statistics. A graphic representation of the temperatures recorded on a certain day is shown on the next page. The readings were as follows:

3 P.M.	29°	9 P.M.	21°	3 A.M.	12°	9 A.M.	12°
4 P.M.	29°	10 P.M.	20°	4 A.M.	11°	10 A.M.	13°
5 P.M.	28°	11 P.M.	17°	5 A.M.	10°	11 A.M.	16°
6 P.M.	26°	12 M'T.	16°	6 A.M.	10°	12 Noon	17°
7 P.M.	24°	1 A.M.	14°	7 A.M.	10°	1 P.M.	18°
8 P.M.	22°	2 A.M.	12°	8 A.M.	10°	2 P.M.	20°

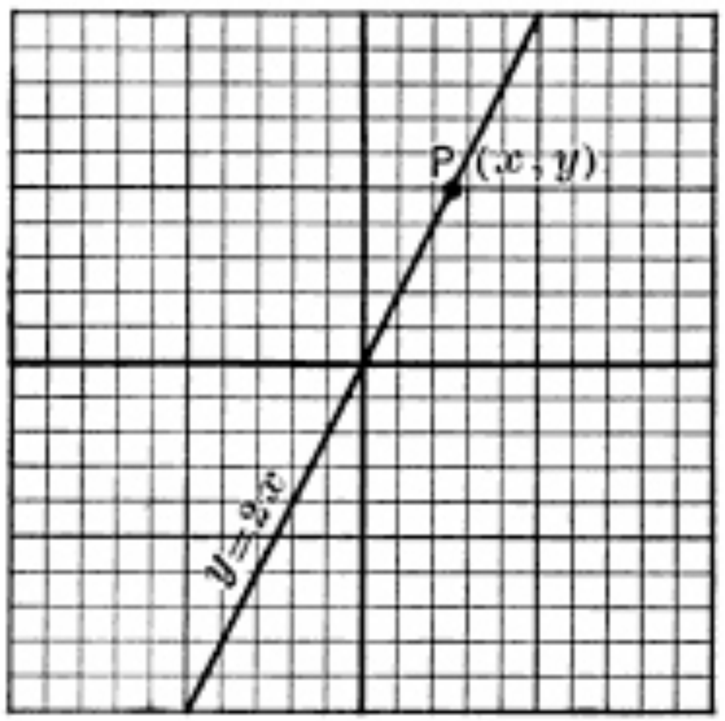


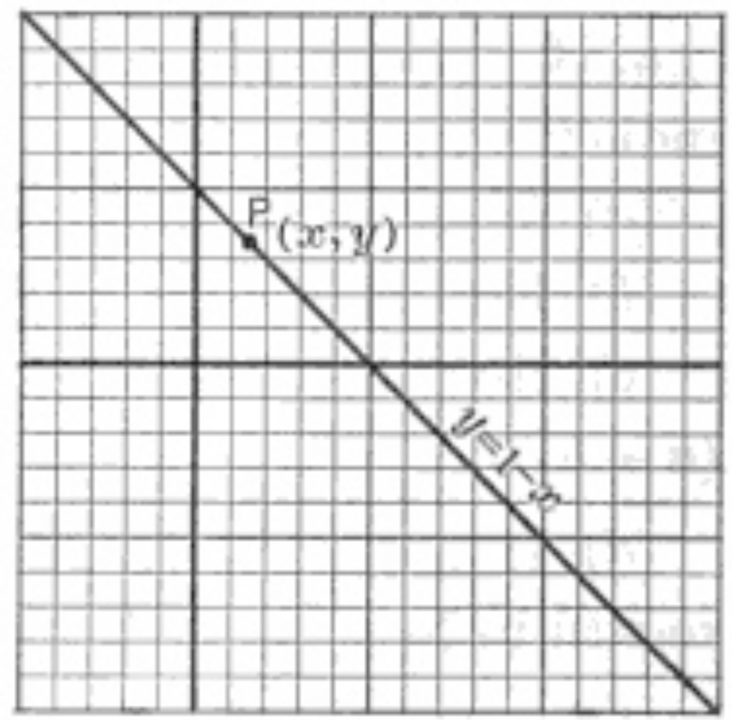
Functions mentioned!

Variable. An algebraic expression whose value changes in any one problem or discussion is called a **variable.**

E.g. As P moves along the line $y = 2x$, both x and y are variables.

Function. When two variables are connected by a fixed relation so that the value of one depends on the value of the other, then *one variable is said to be a function of the other.*



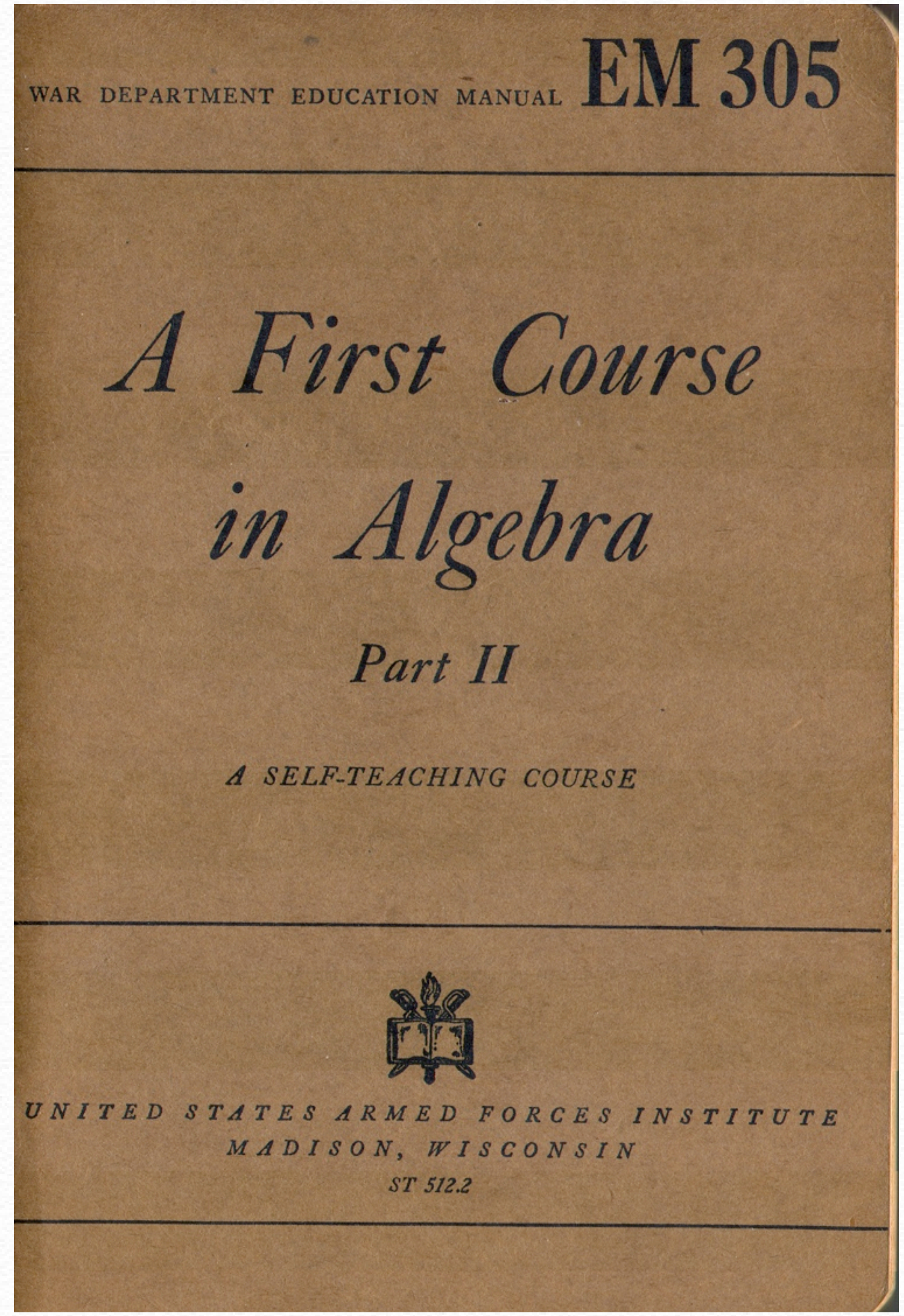


Math Reform in the Teens and Twenties

- **There was an effort in math education reform in the early part of the 20th Century.**
- **One of the leaders, was E. H. Moore of the University of Chicago, one of the very top mathematicians of his day.**
- **Moore wrote a paper in which he elaborated on the use of graph paper and the function concept in secondary school instruction.**
- **He advocated learning in "math labs" and promoted the idea of an integrated math curriculum.**
- **In the end, some of these ideas (the math labs for example) fell by the wayside as impractical when the number of students in high schools increased tremendously during the early part of the century. A high school education changed from the education attained by a small elite, to the education pursued by a large population of students, including many recent immigrants.**
- **However, the function concept appears in high school at this time, and is adopted in some textbooks (though not in MY high school Algebra 1 book!).**

Algebra goes to War

During the mobilization for war, there was great interest in upgrading the math level of soldiers



Interpolation was still an important applied tool

- An important challenge in the old days was interpolating from tables, such as tables of logarithms
- or trig tables or log trig tables.

★193. Interpolation in a five-place table.

Example 1. Find log 25.637 from a five-place table.

Solution. By the principle of proportional parts, log 25.637 is $\frac{7}{10}$ of the way from log 25.630 to log 25.640.

$$\begin{array}{ll} \textit{From table:} & \log 25.630 = 1.40875 \\ & \log 25.637 = \quad ? \\ \textit{From table:} & \log 25.640 = 1.40892 \end{array} \Bigg] 17 \qquad \begin{array}{c} \textbf{Tabular difference is} \\ .40892 - .40875 = .00017. \\ .7(17) = 11.9, \text{ or } 12. \end{array}$$

$$\log 25.637 = 1.40875 + .7(.00017) = 1.40875 + .00012 = 1.40887.$$

Example 2. Find N if $\log N = 2.40971$.

Solution. The mantissa is .40971, which numerically is between the consecutive entries .40960 and .40976 in the five-place table; these mantissas correspond to 2568 and 2569. Since .40971 is $\frac{11}{16}$ of the way from .40960 to .40976, we assume that the significant part of N is $\frac{11}{16}$ of the way from 25680 to 25690.

$$16\left[\begin{array}{l} 11\left[\begin{array}{l} .40960, \textit{ mantissa for } 25680 \\ .40971, \textit{ mantissa for } \quad ? \end{array}\right] x \\ \quad .40976, \textit{ mantissa for } 25690 \end{array}\right] 10 \qquad \begin{array}{r} \frac{11}{16} = .7, \textit{ to nearest tenth.} \\ x = .7(10) = 7. \\ 25680 + 7 = 25687. \end{array}$$

Hence, .40971 is the mantissa for 25687 and $N = 256.87$.

NOTE 1. In interpolating (*from one entry to the next larger one*) in a table of mantissas, if there is equal reason for choosing either of two successive digits, for uniformity **we agree to select the even digit.**

EXERCISE 108

Find the four-place logarithm of each number from Table II.

1. 1826. **4.** 12.67. **7.** .3013. **10.** .03147. **13.** 90,090.

The Post-Sputnik New Math

CONTENTS

Chapter

1. SETS AND THE NUMBER LINE 1
   1- 1. Sets and Subsets 1
   1- 2. The Number Line 7
   1- 3. Addition and Multiplication on the Number Line 14

2. NUMERALS AND VARIABLES 19
   2- 1. Numerals and Numerical Phrases 19
   2- 2. Some Properties of Addition and Multiplication 23
   2- 3. The Distributive Property 29
   2- 4. Variables 34

3. SENTENCES AND PROPERTIES OF OPERATIONS 41
   3- 1. Sentences, True and False 41
   3- 2. Open Sentences 42
   3- 3. Truth Sets of Open Sentences 45
   3- 4. Graphs of Truth Sets 47
   3- 5. Sentences Involving Inequalities 49
   3- 6. Open Sentences Involving Inequalities . . . 50
   3- 7. Sentences With More Than One Clause 52
   3- 8. Graphs of Truth Sets of Compound Open Sentences 54
   3- 9. Summary of Open Sentences 56
   3-10. Identity Elements 57
   3-11. Closure 60
   3-12. Associative and Commutative Properties of Addition and Multiplication 61
   3-13. The Distributive Property 66
   3-14. Summary: Properties of Operations on Numbers of Arithmetic 71
   Review Problems 73

4. OPEN SENTENCES AND ENGLISH SENTENCES 77
   4- 1. Open Phrases and English Phrases 77
   4- 2. Open Sentences and English Sentences . . . 82
   4- 3. Open Sentences Involving Inequalities . . . 86
   Review Problems 90

Chapter

5. THE REAL NUMBERS 97
   5- 1. The Real Number Line 97
   5- 2. Order on the Real Number Line 101
   5- 3. Opposites 108
   5- 4. Absolute Value 113
   5- 5. Summary 118
   Review Problems 119

6. PROPERTIES OF ADDITION 121
   6- 1. Addition of Real Numbers 121
   6- 2. Definition of Addition 124
   6- 3. Properties of Addition 129
   6- 4. The Addition Property of Equality 132
   6- 5. The Additive Inverse 135
   6- 6. Summary 141
   Review Problems 142

7. PROPERTIES OF MULTIPLICATION 145
   7- 1. Multiplication of Real Numbers 145
   7- 2. Properties of Multiplication 150
   7- 3. Use of the Multiplication Properties . . . 156
   7- 4. Further Use of the Multiplication Properties 159
   7- 5. Multiplicative Inverse 162
   7- 6. Multiplication Property of Equality 165
   7- 7. Solutions of Equations 167
   7- 8. Reciprocals 172
   7- 9. The Two Basic Operations and the Inverse of a Number Under These Operations 180
   Review Problems 182

8. PROPERTIES OF ORDER 185
   8- 1. The Order Relation for Real Numbers 185
   8- 2. Addition Property of Order 187
   8- 3. Multiplication Property of Order 195
   8- 4. The Fundamental Properties of Real Numbers. 200
   Review Problems 206

9. SUBTRACTION AND DIVISION FOR REAL NUMBERS 209
   9- 1. Definition of Subtraction 209
   9- 2. Properties of Subtraction 212
   9- 3. Subtraction in Terms of Distance 219
   9- 4. Division 223
   9- 5. Common Names 229
   9- 6. Fractions 233
   9- 7. Summary 241
   Review Problems 242

INDEX . . . . following page 246

The New Math brought in new content, perspectives and terminology (some stayed around and some not)

The second question remains: Where are some of the points on the number line which do not correspond to rational numbers? It will be proved in a later chapter that, for example, the real number $\sqrt{2}$ is an irrational number. Let us locate the points with coordinates $\sqrt{2}$ and $^{-}\sqrt{2}$, respectively.

13-1. Equivalent Open Sentences

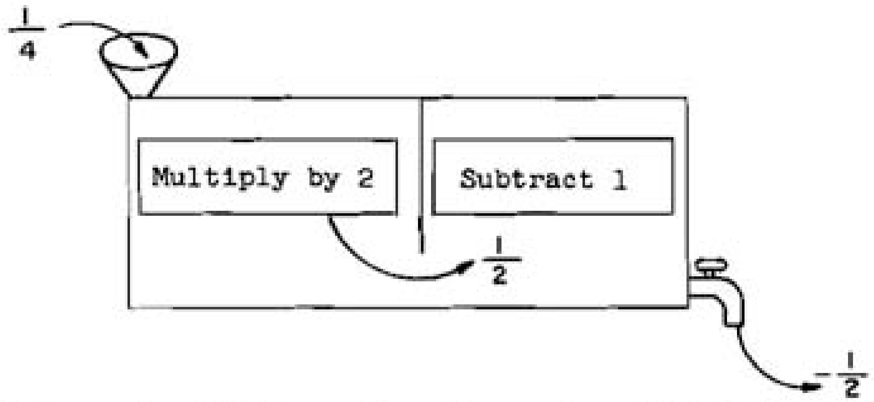
Throughout this course we have been solving open sentences, that is, finding their truth sets. At first we guessed values of the variable which made the sentence true, always checking to verify the truth of the sentence. Later we learned that certain operations, when applied to the members of a sentence, yielded other sentences with exactly the same truth set as the original sentence. We say that:

> Two sentences are equivalent if they have the same truth set.

School Mathematics Study Group, Algebra

Multiple representations of functions and graphing of inequalities appeared, and also matrices and vectors

(b) Imagine a special computing machine which accepts any positive real number, multiplies it by 2 , subtracts 1 from this, and gives out the result.



15-2. Systems of Inequalities

In 15-1 we defined a system of equations as a compound open sentence in which two equations are joined by the connective "and". We also introduced a notation for this. Carrying the idea over to inequalities, let us consider systems like the following:

(a) $\begin{cases} x + 2y - 4 > 0 \\ 2x - y - 3 > 0 \end{cases}$ (b) $\begin{cases} 3x - 2y - 5 = 0 \\ x + 3y - 9 \leq 0 \end{cases}$

(c) What would the graph of $x + 2y - 4 > 0$ be? You recall that we first draw the graph of

$$x + 2y - 4 = 0,$$

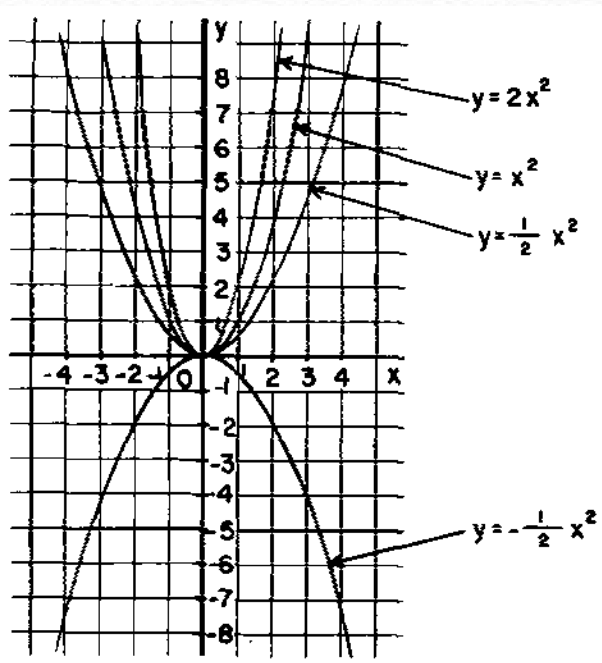
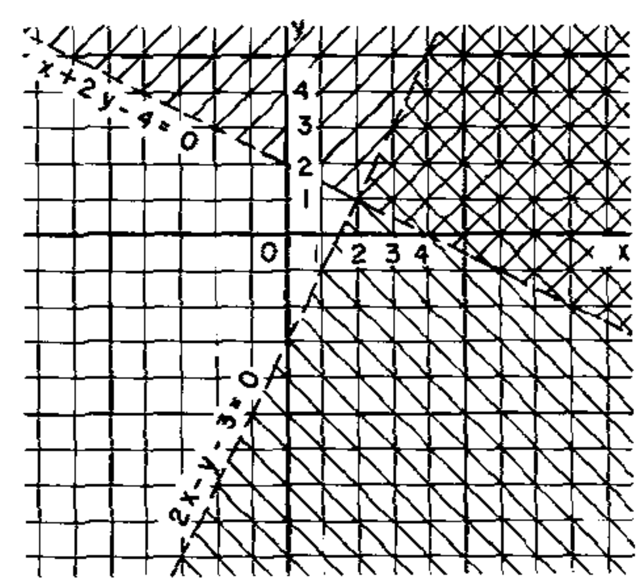


Figure 2.



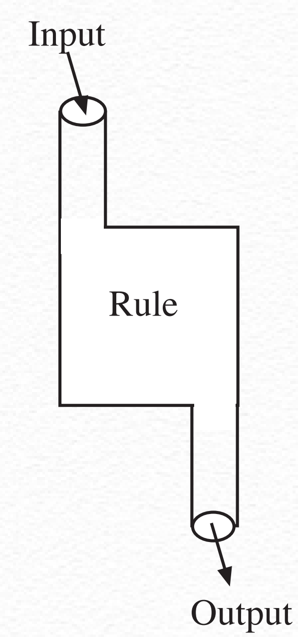
School Mathematics Study Group, Algebra

New Math and Aftermath

- The New Math did not succeed in its ambitious program and was succeeded by other textbooks, including a back to basics movement.
- But despite the apparent failure of the New Math, some concepts and practices introduced then are still very much with us in all our textbooks, whether or not they are called “reform” or “traditional”. For example, functions are everywhere in algebra today, as in this 21st century textbook.

1.3 What is a Function? What is not a Function?

Some people like to think of a function as a machine. You supply an input, the machine follows a rule, and then gives you back an output.



The function
is the machine.

The rule is what
it does to the input to
get the output.

Let’s look at an example. “Add 2 to the input” is a rule that defines a function.

Technology

- By my choice of periods for the textbooks, we have not seen the impact of technology that is so important today.

- Some skills are not taught at all: using and interpolating tables of logarithms and trig functions, algorithms for taking square roots.

- Other topics have come to the fore because of graphing technology. When one can easily graph a quadratic or other polynomial, then the use of graphical interpretation becomes accessible. Even “traditional” textbooks use graphing in a way that was never seen before the 1970s.

- This it not to mention Computer Algebra Systems, which could also cause major changes as they become factors in school math.

- And I have also ignored the internet, even though I found the scans of many of the old textbooks using Google!

Algebra the Math vs Algebra the Course

- As a university mathematician, I am also aware that we use the word Algebra also to refer to an area of mathematics. This has also changed, but not in the same way as the Algebra Course.

- Starting in the late 19th century, mathematicians started to manipulate other math objects, such as permutations, using an analog of multiplication.

- From this, in the early 20th century, Abstract Algebra was born. This focuses on the properties of abstract symbolic systems with operations and axioms: for example, groups, fields and vector spaces.

- So the secondary algebra course from this perspective looks more and more like analysis or pre-calculus, while the university Algebra courses tend to be about abstract structures even when they are computational and applied (no one bothers to say Abstract Algebra).

Example: A snippet from a book on Algebraic Geometry

We will consider some applications of standard bases in this section. The multiplicity, and Milnor and Tjurina number computations we introduced in §2 can be carried out in an algorithmic fashion using standard bases. We begin by using Theorem (4.3) to prove Proposition (2.11), which asserts that if I is a zero-dimensional ideal of $k[x_1, \ldots, x_n]$ such that $0 \in \mathbf{V}(I)$, then the multiplicity of 0 is

$$
\begin{aligned}
&\dim k[x_1, \ldots, x_n]_{\langle x_1,\ldots,x_n\rangle}/Ik[x_1, \ldots, x_n]_{\langle x_1,\ldots,x_n\rangle} \\
(5.1) \qquad &= \dim k[[x_1, \ldots, x_n]]/Ik[[x_1, \ldots, x_n]] \\
&= \dim k\{x_1, \ldots, x_n\}/Ik\{x_1, \ldots, x_n\},
\end{aligned}
$$

where the last equality assumes $k = \mathbb{R}$ or $\mathbb{C}$. The proof begins with the observation that by Theorem (2.2), we know that

$$\dim k[x_1, \ldots, x_n]_{\langle x_1,\ldots,x_n\rangle}/Ik[x_1, \ldots, x_n]_{\langle x_1,\ldots,x_n\rangle} < \infty.$$

A computational but abstract algebra result from Group Theory

As well as five other families of simple groups, there are 26 “oddities”—one-off simple groups that don’t fit into any family. The biggest one of these “oddities” has order

808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000,000.

The classification of all simple finite groups was accomplished in the middle and later 20th century, being finished in 1980. It was one of the great achievements of modern algebra.114

Ancient Traditions

- Many ideas of algebra go back to the civilization of Mesopotamia, and also to India and China.

- The first great book of algebra is the book of Muhammed ibn Musa Al-Kwarizmi, a Persian working in Baghdad 1200 years ago. It presented the systematic solution of linear and quadratic equations.

- This book was translated and introduced into Europe in the Renaissance.

- Hence the arabic words “algebra” and “algorithm” came to English and other Western languages.

Conclusion

- What I have observed from looking at textbooks is that there is a tradition in algebra of some topics and problems that persist over decades and even centuries.

- But on the other hand, it is also true that our textbooks change in substantial ways from decade to decade (not to mention difference in perspective from individual authors). So our algebra courses do evolve over the years, whether we are looking at conservative or radical approaches to the subject.

- No modern textbook really looks like a textbook of fifty years, nor should it. Though it is interesting and a bit instructive to study old textbooks and see how -- for a very long time -- teachers have struggled to find the best way to explain algebra to their students.

Conclusion: Another Perspective