Session 2, July 10
Base Conversion using the Division Algorithm

Some groups yesterday observed that their work was going on "in base 7" when working on the $\frac{1}{7}$ problem. This is mostly true; the possible remainders, from 0 to 6, represent the digits of base 7. A significant difference is that a number "in base 7" can have several digits, while the possible remainders in the $\frac{1}{7}$ problem would all be single-digit numbers in base 7.

For example, in base 7, $5 + 4 = 12$ (read as one seven and two ones). In the remainder system, $5 + 4 = 2$. We will be exploring different bases today, and remainder systems tomorrow. So, some of those loose ends we found yesterday probably won't get discussed until tomorrow.

The division algorithm, used repeatedly, can make the conversion from a base-ten number to another base. Here's how to convert the number 1234 to base seven:

$$\begin{array}{rcccc} 1234 & = & 7 \cdot 176 & + & 2 \\ 176 & = & 7 \cdot 25 & + & 1 \\ 25 & = & 7 \cdot 3 & + & 4 \\ 3 & = & 7 \cdot 0 & + & 3 \end{array}$$

Note that at each stage, the quotient becomes the next divident, and the divisor is always the new base. By reading the remainders in revserse, we can read the number in the new base: $1234 = (3412)_7$, a notation that means that 1234 is written as 3412 in base seven. The "4" of this number actually represents $4(7^2) = 196$ in base ten. Each digit has a particular place value.

1. Show, using place value, that $(3412)_7$ is equal to 1234 in base ten.

2. As practice, compute these values in base seven *without leaving it*: $3412 + 326$; $3412 - 346$; 3412×346; $3412 \div 22$ (find quotient and remainder, both in base seven).

In bases higher than ten, additional letters are used to represent the single digits for "10", "11", etc. Hexadecimal computer code is written in base 16, with the letters A through F representing 10 through 15.

3. Use the division algorithm to convert the base-ten number 1234 to base two, to base three, to base ten, to base sixteen.

Why ask to convert to base ten in Problem 3? Because it can help you see why the algorithm works!

4. Use the results from Problems 1 and 3 to explain why the algorithm works. In particular, why do we read the remainders in reverse? Why does the remainder "4" in our example represent $4(7^2)$?

5. Convert the base-ten number 65403 to base two, to base four, to base sixteen. Do you notice anything unusual going on?

6. Convert the base-seven number $(34652)_7$ to base two, to base three, to base sixteen. Do this without converting to base ten first; in other words, carry out the algorithm using base-seven arithmetic. You may find your work in Problem 2 helpful.

7. Convert the base-sixteen number $(7F09)_{16}$ to base four.

As in Session 1, long division can be carried out to find the expansion of a fraction as a decimal. In other bases, although it is a misnomer to call this expansion a decimal, this is still the commonly used term.

8. Write the base-ten numbers 3, 9, 27, 243, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{7}$, $\frac{1}{9}$ as decimals in base two and in base three. You may find your work from yesterday helpful here. Don't forget that the fraction "$\frac{1}{7}$" in base two will not involve writing the numeral "7".

9. Why are some of the fractions in Problem 8 easier to write in one base or the other?

10. Find the period of repetition for each repeating decimal you found in Problem 8. Is any of this related to yesterday's work?

11. Write the base-ten fraction $\frac{1}{7}$ as a decimal in each base between two and ten, using long division in the new base. Do the expansions have anything in common? How can you tell when a decimal expansion is about to repeat? As an extension, write the prime fractions from Session 1 in each base between two and ten, looking for similiarities and differences in the expansions.

Yesterday, a participant suggested that any rational number (a number in the form $\frac{p}{q}$, where p and q are integers, and q is nonzero) must have a terminating or repeating decimal.

12. Can you explain why, using what you've learned about remainders?

Is the converse true? That is, can every terminating or repeating decimal be written as a rational number? Here's a method used by many algebra texts for converting repeating decimals into fractions. Suppose we want to convert $.\overline{324}$ to a fraction. Let $x = .\overline{324}$. Then

$$\begin{aligned} x &= .\overline{324} \\ 1000x &= 324.\overline{324} \end{aligned}$$

Subtract the top equation from the bottom to obtain

$$\begin{aligned} 999x &= 324 \\ x &= \frac{324}{999} \end{aligned}$$

This may or may not result in a simplified fraction, but it will result in a rational number.

13. Try this method with the following decimals, especially the last one (which is an "eventually repeating" decimal): $.\overline{356}$, $.\overline{53}$, $.\overline{2}$, $.\overline{027}$, $.01\overline{356}$.

14. Does this explain why every repeating decimal expansion represents a rational number (a fraction in the form $\frac{m}{n}$)? Explain why or why not.

15. Will every terminating decimal expansion represent a rational number? Explain why or why not.

16. Describe rules which will determine if a rational number has a terminating, repeating, or eventually repeating decimal expansion in base ten. How do these rules change for other bases?