Session 1, July 9
Decimal Expansion

First of all, welcome to *The Euclidean Algorithm and its Applications to Algebra and the Theory of Numbers.* In this course we will be working on various aspects of number theory, a branch of mathematics which is highly accessible yet deep in the scope of its results and applications. Major theorems of number theory are the keys behind encryption, curve fitting, and the proof of Fermat's Last Theorem, yet many branches of number theory can be explored by elementary school students.

One of the goals of this course is to seek and explain patterns in as many different situations as possible. Much of mathematics, especially mathematical proof, begins as observation. A pattern emerges, and the goal is to explain why the pattern exists, and to justify that the pattern will continue to exist. It is in this light that the majority of the time in this course, you will be working on problems, rather than watching a lecture. As you work the problems in this course, actively look for these patterns, and remember that behind almost every pattern is a reason, and a proof.

Also, the course is structured to give a varied experience to all. In other words, don't worry if you can't solve every problem in every session. Hopefully you will leave here with a greater understanding of number theory, but also many problems for further study.

The first session explores the writing of a fraction as a decimal.

1. Without a calculator, write the decimal representations of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, and $\frac{1}{10}$. Use long division.

2. Which of the decimals in Problem 1 were terminating?

3. If $\frac{1}{n}$ is a terminating decimal, what must be true about the number n? Use a calculator, if you like, to find more terminating decimals.

4. Given a terminating decimal $\frac{1}{n}$, explain how you could determine the length of the decimal without carrying out the long division. (i.e., using only information about the number n.)

5. If $\frac{1}{n}$ is a terminating decimal, what oculd you say about $\frac{2}{n}$? About $\frac{m}{n}$, if m is a counting number less than n?

Properties of terminating decimals are nice, but non-terminating decimals are much more interesting.

6. Using long division (preferred) or a calculator, find the decimal expansion for the fractions $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, $\frac{6}{7}$. How are the decimal expansions related?

7. Write the repeating digits of the decimal for $\frac{1}{7}$ in a circle to illustrate their order. Explain how the circle of digits can help you find the decimal expansions for $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, and $\frac{6}{7}$.

8. Repeat Problems 6 and 7 for the fractions $\frac{1}{13}$, $\frac{2}{13}$, $\ldots$, $\frac{12}{13}$. Are the decimal expansions still related? Is it still possible to write the digits of these expansions in circles?

9. Using long division (preferred) or a calculator, make a table showing the period for the decimal expansion of $\frac{1}{n}$ for every prime up to 47. Describe any patterns you notice in the table. Do you think every prime number larger than 5 will have a repeating decimal expansion of $\frac{1}{n}$? Why?

10. How can you tell, while doing long division on a fraction $\frac{1}{n}$, that its decimal is about to repeat? Is there repetition in the long division as well?

11. Using the patterns from 9, predict the period of the repeating decimal for $\frac{1}{107}$. How many possibilities are there?

Some decimals repeat, but not immediately. For example, $\frac{1}{22} = .0454545\ldots$ This is called an *eventually repeating* decimal.

12. Under what circumstances wil the decimal expansion of $\frac{1}{n}$ be eventually repeating?

13. How can you use the number n to determine the number of digits *before* $\frac{1}{n}$ starts to repeat? How can you use n to determine the number of digits in the repeating pattern?

14. Suppose $\frac{1}{n}$ is a repeating decimal. Can $\frac{2}{n}$ be eventually repeating? Can it be terminating? What about $\frac{m}{n}$, where m is less than n?

15. (Further exploration) Write the expansion of $\frac{1}{n}$, for various n, in base three. This will require you to carry out long division in base three. What properties change, and what properties stay the same? Which fractions terminate? What is the period of repeating decimals when they exist?