Session 7: The Joy of Sets

86. Spend 15 minutes revisiting the Simplex lock problem from Monday. Use what you've learned to try and make some more progress toward a solution, or toward a different method if you've already found one.

87. Find the first five powers of 99, and explain what is happening using the Binomial Theorem or Pascal's Triangle... which are basically the same thing.

    Recall the Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

88. What's the sum of the numbers in the 8th row of Pascal's Triangle?

89. Each number in Pascal's Triangle is the sum of the two numbers above it. Use this to explain why the sum of the numbers in a row of Pascal's Triangle is a power of 2.

90. Use the Binomial Theorem to prove that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

    This is the Sigma notation way of saying "row sums of Pascal's Triangle." Remember that the Binomial Theorem is true for all choices for x and y.

A *set* is a grouping of things, like $\{1,2,3\}$. A *subset* is a grouping of things that may or may not contain each thing of the original set: $\{1,3\}$ is a subset (and it's the same subset as $\{3,1\}$). One subset contains all the elements, and one subset contains none of them: the notation, $\{\}$, is called the *empty set*.

91. (a) How many subsets of $\{1,2,3\}$ have exactly two elements?
    (b) How many subsets of $\{1,2,3\}$ are there?
    (c) How many subsets of $\{1,2,4,8,16\}$ have exactly three elements?

92. Let $S = \{a_1, \ldots, a_n\}$ be an n-element set.
    (a) How many 3-element subsets does S have? (Assume $n \geq 3$.)
    (b) How many k-element subsets does S have? (Assume $n \geq k$.)

93. Let S be the set thing again from problem 92. How many *total* subsets are there? Here, k can be any number from 0 to n. See if you can do this problem two different ways.

You can *partition* a set of numbers into non-empty subsets. For example, the set $\{1,2,3\}$ can be partitioned into two subsets: $\{1,3\}$ and $\{2\}$ (which is the same as $\{2\}$ and $\{1,3\}$). Or, it can be partitioned into two other subsets: $\{1,2\}$ and $\{3\}$. It can even be partitioned into 1 or 3 subsets, though not in particularly exciting ways.

94. (a) How many total ways are there to partition $\{1, 2, 3\}$ into two subsets?
(b) How many total ways are there to partition $\{1, 2, 3, 4\}$ into two subsets?
(c) How many total ways are there to partition $\{1, 2, 3, 4, 5\}$ into two subsets?
(d) What's up with that?

95. Complete this table, with the number of elements as rows and the number of subsets as columns.

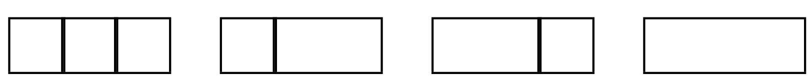
	1	2	3	4	5
1		–	–	–	–
2			–	–	–
3	1	3	1	–	–
4		7			–
5					

96. When you expand $(3x^2 + \frac{5}{x})^6$, there is a constant term. What is it?

Tough Stuff

97. Look at row 8 of Pascal's Triangle. Problem 90 was about adding the numbers together.
(a) Now alternate adding and subtracting, i.e., compute $1 - 8 + 28 - \cdots$.
(b) Try it for another few rows.
(c) Prove that what you've found happens for every row (see the comment about x and y in problem 90).
(d) Go back to problem 54 from Session 4 and have an epiphany.

98. Suppose you want to make all the trains of length 3, but not all at the same time. You want to make them one at a time. How many of each car do you need? Well, here are the trains:



You need three 1-cars, one 2-car (because any given train only uses one of them), and one 3-car.

How many cars (and which ones) do you need on your desk to make all the trains of length 4, doing it one train at a time? Now, suppose you want to make all the trains of length 5, one at a time. What do you need to *add* to the pile on your desk so you can do it? Then how many cars do you need to add to the pile in order to make all the trains of length 6? Generalize to length n: How many new cars do you need to add to a pile that lets you make all trains of length $n-1$ in order to get a pile that lets you make all trains of length n?