Session 3: More Trains

20. How many trains of length 6 are there?

21. How many trains of length 6 are there that use exactly one rod? two rods? three rods?

22. Nagwa gives you *t*wo sets of all the trains of length 4. Try to figure out a way to use all these length 4 trains to generate all the trains of length 5.

23. How many trains of length 10 are there that use *only* cars of length 1 and 2?

24. Using *only* rods of lengths 2 and 3, how many trains of length 11 can be made?

25. Make a table of how many trains of length n can be made using *only* rods of length 2 and 3, for n from 1 to 11. Is there a rule you could use to continue the table?

26. At the world's largest ice cream store, you can order 3,464,840 bowls of ice cream with four different-flavored scoops. *Without* figuring out how many flavors there are, can you describe a way to find how many four-scoop cones of ice cream are available?

27. (a) Suppose you can make 156 different two-scoop cones at a certain ice cream shop. How many different flavors does this shop offer?

    (b) Suppose you can make 2730 different three-scoop cones at a certain ice cream shop. How many different flavors does this shop offer?

28. Let n and r be non-negative integers with $n \geq r$. Define

    $$_nP_r = \frac{n!}{(n-r)!}$$

    Explain why $_nP_r$ is the number of *permutations* (or, cones) of n things taken r at a time.

    (Technology note: On the TI-*n*spire, you can find this through the sequence "menu", "5:Probability", "2:Permutations". It's just as easy to use the green keys to type "npr" (no need to worry about capital letters). Since the calculators have a computer algebra system, you can type "npr(n,r)" to get the formula above.)

29. Using the definition above, find $_nP_0$. Explain using the ice cream analogy.

30. What's a simpler rule for $_nP_n$? (Explain and verify with the TI-*n*spire.)

31. Let n and r be non-negative integers with $n \geq r$. Define

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

(Sometimes this is written ${}_nC_r$, especially in TI-land, and one way of saying it is "n choose r." On the TI-$n$spire, type "ncr" or use the sequence above, replacing the last step with "3:Combinations".) Explain why $\binom{n}{r}$ is the number of r-scoop bowls you can make with n scoops of ice cream.

32. Using the definition, find $\binom{n}{0}$. Explain using the ice cream analogy.

33. What's a simpler rule for $\binom{n}{n}$? Explain using ice cream. Mmmm, fattening.

34. A pizza store offers 10 kinds of toppings.

    (a) Suppose you want exactly 2 toppings on your pizza. How many different pizzas can you make?

    (b) Suppose you want exactly 8 toppings on your pizza. How many different pizzas can you make?

    (c) What's going on here?

    (d) Explain why $\binom{n}{r} = \binom{n}{n-r}$.

35. Using the definition above, find $\binom{12}{0}$ and $\binom{12}{12}$. Explain the results using the concept of pizza toppings.

36. A 13-card hand is dealt from a standard 52-card deck. Find the probability that this hand contains exactly 3 aces and exactly 2 kings (which means exactly 8 of the rest. . .).

Tough Stuff

37. A binary string of length 12 has twelve digits: all are ones or zeros. One example is

    011010011100

    How many 12-digit binary strings...

    (a) ... do not start or end with a 1, *and*

    (b) ... do not include any two consecutive ones?

Session 4: Trains, Words, and Committees

39. (a) Make a table for $n = 2$ to 8 for the number of trains of length n that use *exactly* two rods.

    (b) Repeat for *exactly* three rods.

40. Continuing from the previous problem, complete this table where rows are train lengths and columns are the number of rods:

length \ # rods	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1		1				
4	1			1			
5	1				1		
6	1	5	10			1	
7	1						1

    Hmm, that's familiar.

41. Physically make all the trains of length 6 that use exactly two rods, and (separately, without destroying the two-rod trains) all the trains of length 6 that use exactly three rods.

42. Can you think of a way to use the trains you built in problem 41 to make all the trains of length 7 that use exactly three rods?

43. You've got an unlimited supply of length 2, 3, and 5 rods. How many different trains of length 12 can you make with all these rods?

44. How many trains of length 10 can you make with *no* rods of length 1? (I.e., using only rods of length 2, 3, 4, or more.)

45. Suppose you have an unlimited supply of length 3, 4, and 8 rods. Find the total number of ways to make a train of length 14, using any method you like.

46. Suppose you have an unlimited supply of length 4 and 7 rods. Are there any train lengths you *can't* make? Which ones?

47. If you were allowed to write the letters of the word MINIMUM in any order, how many total different seven-letter "words" could you make? (One such is INMUMIM, so don't worry about whether the words appear in any dictionary.)

48. One way to make a train of length 14 is to use three whites (1-length rods), two reds (2-length rods), one green (3-length rod), and one purple (4-length rod). How many different-looking trains of length 14 could you make using these specific rods?

49. Using either factorial or "choose" notation, write an expression for the number of ways to reorder the letters of 70's singing sensation ABBA. (Bonus points for singing one of their songs better than Pierce Brosnan.)

50. Repeat problem 49 for 80's singing sensation BANANARAMA. (Bonus points for remembering some of their songs.)

51. There are 12 students on a committee; 8 are juniors, and 4 are seniors. Determine the number of ways of forming the following subcommittees.

    (a) A subcommittee of 5 juniors.

    (b) A subcommittee of 3 seniors.

    (c) A subcommittee of 6 students.

    (d) A subcommittee of 3 juniors and 2 seniors.

52. (a) A committee of three is chosen from 3 Republicans and 4 Democrats. What is the probability that the committee consists of 2 Republicans and 1 Democrat?

    (b) Verify your result in (a) using a tree diagram, where the committee members are selected one at a time. (Hint: If you add some information to your tree, you can have two branches at the beginning rather than seven.)

Tough Stuff

53. With 9 people, committees can be picked with as few as 0 people and as many as 9. Give a convincing argument (or, to use the vernacular, *prove*) that the total number of committees that can be formed with an odd number of members is the same as the number of committees with an even number of members.

54. With 8 people, committees can be picked with as few as 0 people and as many as 8. Give a convincing argument that the total number of committees that can be formed with an odd number of members is the same as the number of committees with an even number of members.

55. How many odd numbers are there in the 100th row of Pascal's Triangle? (That's the row that starts 1, 100, 4950.)